Beyond Cell Transfer

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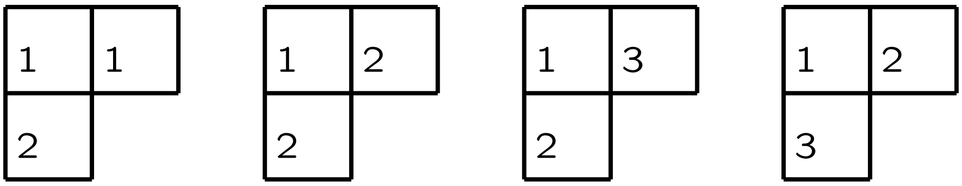
Schur functions s_λ, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$.



$s_{(2,1)} = \sum_{i<j}(x_i^2 x_j + x_i x_j^2) + 2\sum_{i<j<k} x_i x_j x_k$.

$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$, $c_{\lambda\mu}^\nu \geq 0$.

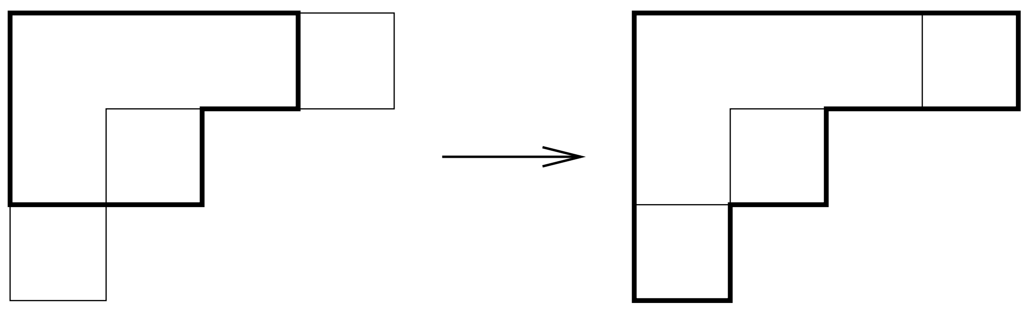
$f \geq_s g$ means $f - g = \sum c_\lambda s_\lambda$, $c_\lambda \geq 0$.

$$\lambda \vee \mu := (\mathrm{max}(\lambda_1, \mu_1), \mathrm{max}(\lambda_2, \mu_2), \dots),$$

$$\lambda \wedge \mu := (\mathrm{min}(\lambda_1, \mu_1), \mathrm{min}(\lambda_2, \mu_2), \dots).$$

$$(\lambda/\mu) \vee (\nu/\rho) := \lambda \vee \nu / \mu \vee \rho,$$

$$(\lambda/\mu) \wedge (\nu/\rho) := \lambda \wedge \nu / \mu \wedge \rho.$$



$$(3,2) \vee (4,1,1) = (4,2,1),$$

$$(3,2) \wedge (4,1,1) = (3,1).$$

Theorem 1. *(Cell transfer)[LPP]*

$$s_{\lambda/\mu}\, s_{\nu/\rho} \leq_s s_{(\lambda/\mu)\vee(\nu/\rho)}\, s_{(\lambda/\mu)\wedge(\nu/\rho)}.$$

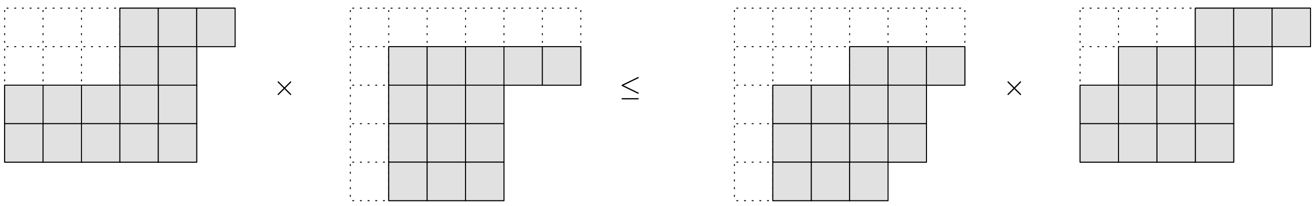
Example 2.

$\lambda/\mu = (6,5,5,5)/(3,3)$,

$\nu/\rho = (6,6,4,4,4)/(6,1,1,1,1)$,

$(\lambda/\mu) \vee (\nu/\rho) = (6,6,5,5,4)/(6,3,1,1,1)$,

$(\lambda/\mu) \wedge (\nu/\rho) = (6,5,4,4)/(3,1)$.



Consequences:

- Okounkov's conjecture ('97)

- Fomin-Fulton-Li-Poon conjecture ('05)

- $q = 1$ case of Lascoux-Leclerc-Thibon conjecture ('97)

Example: FFLP conjecture.

$\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \dots)$ - weakly decreasing rearrangement of λ and μ.

$\mathrm{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \dots)$

$\mathrm{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \dots)$.

Conjecture 3. *(Fomin-Fulton-Li-Poon)*

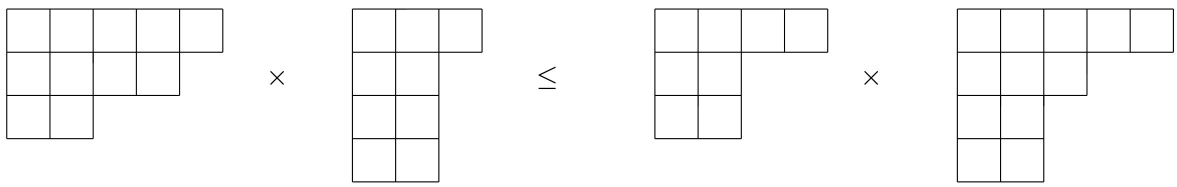
$$s_\lambda s_\mu \leq_s s_{\mathrm{sort}_1(\lambda,\mu)} s_{\mathrm{sort}_2(\lambda,\mu)}.$$

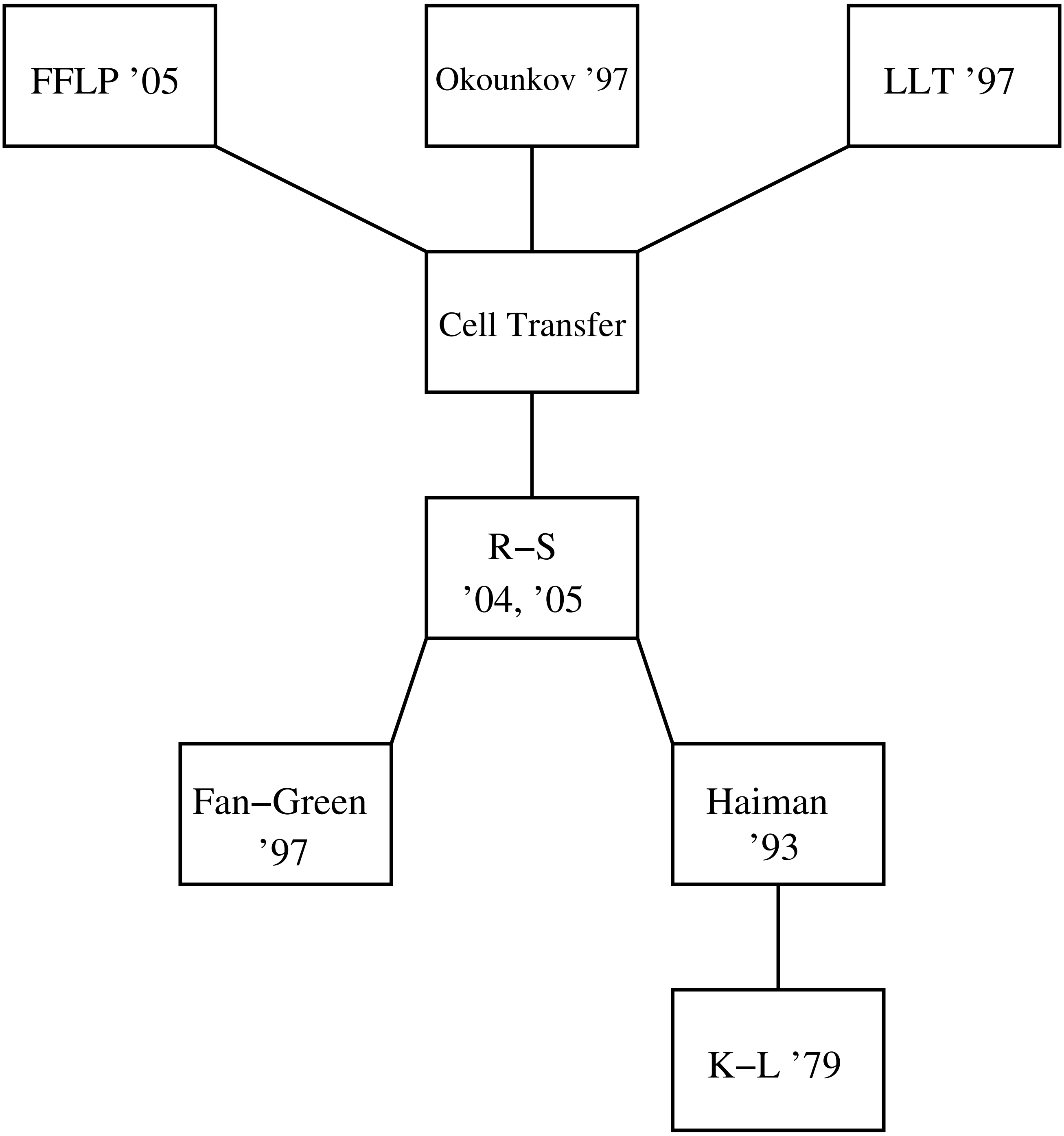
$\lambda = (5, 4, 2),\ \nu = (3, 2, 2, 2).$

$\lambda \cup \nu = (5, 4, 3, 2, 2, 2, 2).$

$\mathrm{sort}_1(\lambda, \nu) = (5, 3, 2, 2),$

$\mathrm{sort}_2(\lambda, \nu) = (4, 2, 2).$




FFLP '05
Okounkov '97
LLT '97
Cell Transfer
R−S
'04, '05
Fan−Green
'97
Haiman
'93
K−L '79

Proof using Rhoades-Skandera work:

Generalized Jacobi-Trudi matrix:

$\left(h_{\mu_i-\nu_j}\right)_{i,j=1}^{n}$, for partitions

$$\mu = (\mu_1 \geq \mu_2 \cdots \geq \mu_n \geq 0),$$

$$\nu = (\nu_1 \geq \nu_2 \cdots \geq \nu_n \geq 0).$$

For matrix $X = (x_{ij})$ **Temperley-Lieb immanant**

$$\mathrm{Imm}_w^{\mathsf{TL}}(X) := \sum_{v \in S_n} f_w(v)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

w - non-crossing matching on $2n$ vertices.

$f_w(v)$ - some function.

Theorem 4 (Rhoades-Skandera)**.** *Temperley-Lieb immanants of generalised Jacobi-Trudi matrices are Schur-nonnegative.*

$I, J \subset [n]$, $|I| = |J|$, $\Delta_{I,J}(X)$: minor of X.

Theorem 5 (Rhoades-Skandera)**.**

$$\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X) = \sum_{w \in \Theta(S)} \mathrm{Imm}_w^{\mathsf{TL}}(X).$$

Monomial Cell Transfer theorem.

P-poset, I - order ideal.

K_I - associated generating function of labellings.

Theorem 6. *[LP] $K_{I\wedge J}K_{I\vee J} - K_I K_J$ is monomial non-negative.*

Strange phenomenon: K-non-negative!

0-Hecke algebra: generators $T_i, 1 \le i \le n-1$ and relations

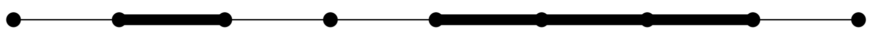
$$T_i^2 = T_i;$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1};$$

$$T_i T_j = T_j T_i, |i-j| > 1.$$

Characters: fundamental quasisymmetric functions

$$L_\alpha = \sum_{i_1 \le \cdots \le i_n;\ i_j < i_{j+1}, j \in S_\alpha} x_{i_1} \cdots x_{i_n}.$$

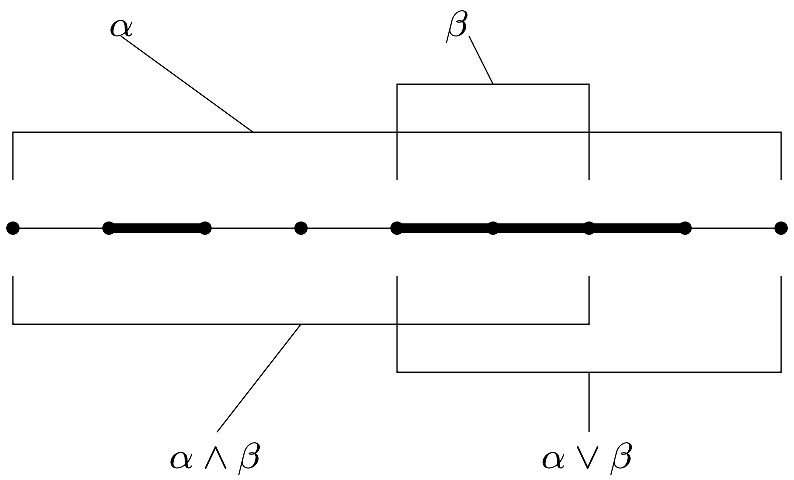


Here $\alpha = (1,2,1,4,1)$, $S_\alpha = \{1,3,4,8\}$.

Cell Transfer for chains:

Corollary 7. $L_{\alpha\vee\beta}L_{\alpha\wedge\beta}-L_\alpha L_\beta$ *is monomial non-negative.*

Example:

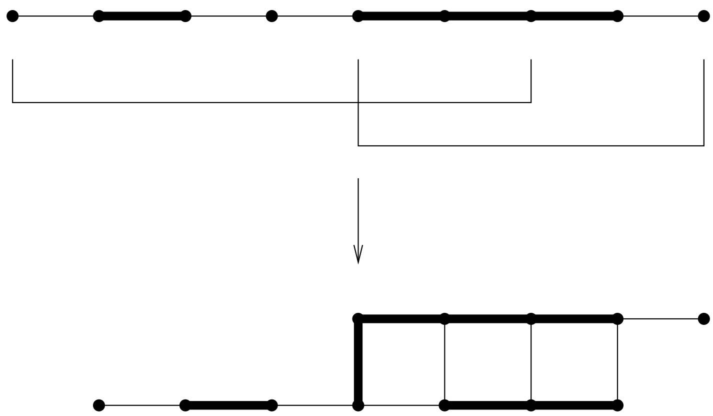


Thus, $L_{(1,2,1,3)}L_{(4,1)} - L_{(1,2,1,4,1)}L_{(3)} \geq_M 0$.

In fact:

Theorem 8. *$L_{\alpha\vee\beta}L_{\alpha\wedge\beta}-L_\alpha L_\beta$ is L-non-negative.*

Form *oriented poset* P:

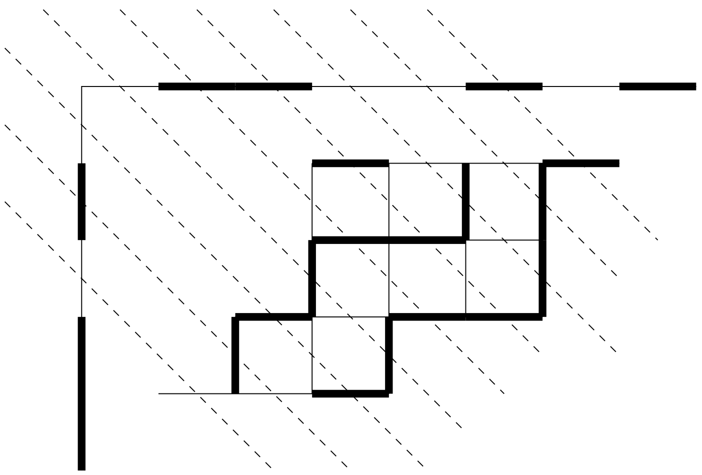


K_P - quasisymmetric function associated with P.

Proposition 9.

$$K_P = \begin{vmatrix} L_{\alpha\vee\beta} & L_\alpha \\ L_\beta & L_{\alpha\wedge\beta} \end{vmatrix}.$$

Diagonal-alternating labelled grid:



Jacobi-Trudi like formula: $K_P =$

$$\begin{vmatrix} L_{(2,1,2)} & L_{(3,1,2)} & L_{(2,3,1,2)} & L_{(1,1,2,3,1,2)} \\ L_{(2,1)} & L_{(3,1)} & L_{(2,3,1)} & L_{(1,1,2,3,1)} \\ L_{(2)} & L_{(2,1)} & L_{(2,3)} & L_{(1,1,2,3)} \\ 0 & 1 & L_{(2)} & L_{(1,1,2)} \end{vmatrix}.$$

λ - strict partition.

Marked Shifted Tableaux of shape λ: filled with

$$1' < 1 < 2' < 2 < \cdots$$

so that

- labels in rows and columns weekly increase,

- each row contains at most one k',

- each column contains at most one k.

$1'$	1	1	$3'$	3
	$2'$	2	$3'$	
		3	4	

Example: MST of shape $(5, 3, 2)$ and weight $(3, 2, 4, 1)$.

Schur Q-function: generating function

$$Q_\lambda = \sum_T x^{wt(T)}$$

over all MST of shape λ.

Example:

$$Q_{(2,1)} = 4\sum_{i<j}(x_i^2 x_j + x_i x_j^2) + 8\sum_{i<j<k} x_i x_j x_k.$$

Multiply non-negatively:

$$Q_\lambda Q_\mu = \sum_\nu d^\nu_{\lambda,\mu} Q_\nu,$$

where $d^\nu_{\lambda,\mu} \geq 0$.

Cell transfer for strict partitions:

$$\lambda \vee \mu := (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \dots),$$

$$\lambda \wedge \mu := (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \dots).$$

Corollary 10. $Q_{\lambda\vee\mu}Q_{\lambda\wedge\mu} - Q_\lambda Q_\mu$ *is monomial non-negative.*

Conjecture 11. $Q_{\lambda\vee\mu}Q_{\lambda\wedge\mu} - Q_\lambda Q_\mu$ *is Q-non-negative.*

Example:

$$Q[4,2]Q[3,1] - Q[4,1]Q[3,2] =$$

$$4Q[6,4] + 2Q[6,3,1] + 2Q[5,4,1] + 2Q[5,3,2].$$

In fact:

$$HL[4,2]HL[3,1] - HL[4,1]HL[3,2] =$$

$$(1-t)(HL[5,3,2]t^2 + HL[6,4]t^2 -$$

$$HL[5,5]t - HL[5,3,1,1]t - tHL[4,4,2] -$$

$$HL[6,4]t - tHL[6,3,1] + HL[4,4,1,1] +$$

$$HL[5,4,1] + HL[5,5]).$$

Pfaffian (definition by example):

$$\begin{matrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{matrix}$$

$Pf = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$

Pfaffian formula for Q-s: if we form matrix

$$\begin{matrix} 0 & Q_{2,1} & Q_{3,1} & Q_{4,1} \\ -Q_{2,1} & 0 & Q_{3,2} & Q_{4,2} \\ -Q_{3,1} & -Q_{3,2} & 0 & Q_{4,3} \\ -Q_{4,1} & -Q_{4,2} & -Q_{4,3} & 0 \end{matrix}$$

then $Pf = Q[4,3,2,1]$.

Pfaffinants? Case $n = 4$.

$$\begin{matrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{matrix}$$

$$P_a = x_{13}x_{24} - x_{12}x_{34}$$

$$P_b = x_{13}x_{24} - x_{14}x_{23}$$

$$P_c = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$$

$$Pf_{12}Pf_{34} = x_{12}x_{34} = P_c + P_b$$

$$Pf_{14}Pf_{23} = x_{14}x_{23} = P_c + P_a$$

$$Pf_{13}Pf_{24} = x_{13}x_{24} = P_c + P_b + P_a$$

When applied to:

$$\begin{matrix} 0 & Q_{2,1} & Q_{3,1} & Q_{4,1} \\ -Q_{2,1} & 0 & Q_{3,2} & Q_{4,2} \\ -Q_{3,1} & -Q_{3,2} & 0 & Q_{4,3} \\ -Q_{4,1} & -Q_{4,2} & -Q_{4,3} & 0 \end{matrix}$$

$P_a = 4Q[7,3]+2Q[7,2,1]+4Q[6,4]+6Q[6,3,1]+$
$4Q[5,4,1] + 4Q[5,3,2]$

$P_b = 4Q[6,4]+2Q[6,3,1]+2Q[5,4,1]+2Q[5,3,2]$

$P_c = Q[4,3,2,1]$