Matrix compositions

Emanuele Munarini
Dipartimento di Matematica
Politecnico di Milano
emanuele.munarini@polimi.it

Joint work with
Maddalena Poneti and Simone Rinaldi

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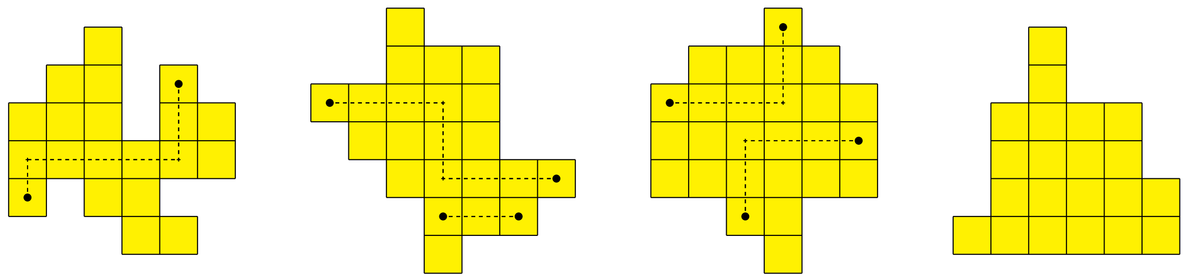
Motivation: L-convex polyominoes

A polyomino is a connected finite set of cells in the plane $\mathbb{Z} \times \mathbb{Z}$. Some examples are



A polyomino is L-convex when every pair of cells can be connected by an internal path with at most one change of direction. Using formal power series techniques, in

> G. Castiglione, A. Frosini, A. Restivo, S. Rinaldi, *Enumeration of L-convex polyominoes*, 2005

it was proved that the numbers f_n of all L-convex polyominoes with perimeter $2(n+2)$ satisfy the recurrence

$$f_{n+2} = 4f_{n+1} - 2f_n \qquad (n \geq 1).$$

Specifically $f_n = 1, 2, 7, 24, 42, 120, \ldots$

Compositions and 2-compositions

A composition of a number $n \in \mathbb{N}$ is any k-tuple $(x_1, \ldots, x_k)$ of positive integers such that

$$x_1 + \cdots + x_k = n.$$

A 2-composition of length k of a number $n \in \mathbb{N}$ is a $2 \times k$ matrix

$$M = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \end{bmatrix}$$

with nonnegative integer entries, without zero columns, such that the sum of all the entries is n.

For instance the 2-compositions of $n = 2$ are:

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The number of all 2-compositions of n is f_n

Polyominoes and 2-compositions

Since the number of all L-convex polyominoes with perimeter $2(n+2)$ is equal to the number all 2-compositions of n, there exists a bijection between this two combinatorial classes.

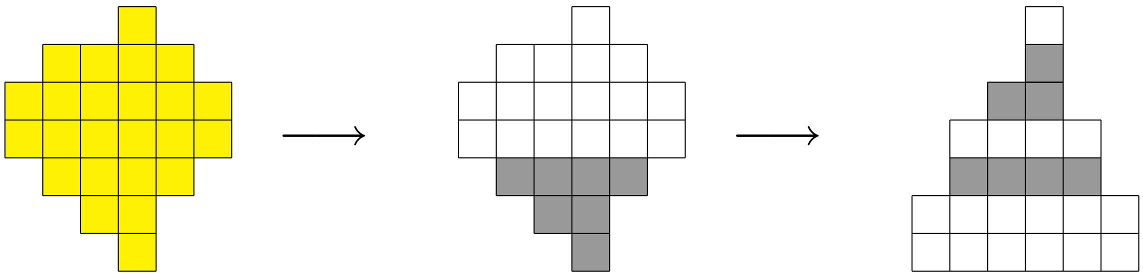
In

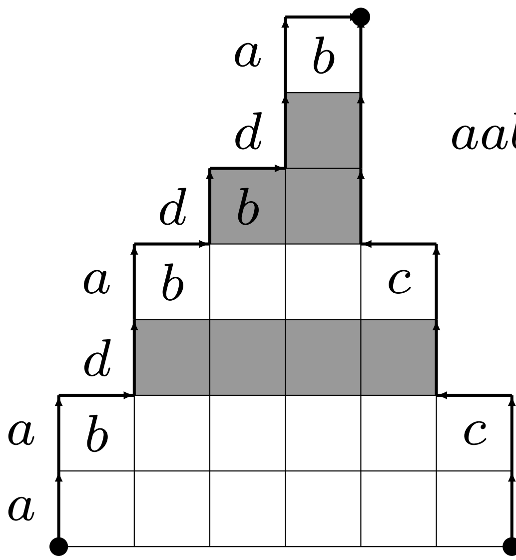
> G. Castiglione, A. Frosini, E. Munarini, A. Restivo, S. Rinaldi, *Enumeration of L-convex polyominoes. II. Bijection and area*, FPSAC'05

it was given an explicit bijection.

Here follows a sketch.

The bijection





$aabcdabcdbdab$

The factorization in $c^h a$, bd^k $(h, k \geq 0)$, $c^r d^s$ $(r, s \geq 1)$ implies the bijection:

$$c^h a \to \begin{bmatrix} h+1 \\ 0 \end{bmatrix}, \; bd^k \to \begin{bmatrix} 0 \\ k+1 \end{bmatrix}, \; c^r d^s \to \begin{bmatrix} r \\ s \end{bmatrix}$$

$$a(c^0 a)(bd^0)(cd)(c^0 a)(bd^0)(cd)(bd)(c^0 a)b \rightsquigarrow$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

Aim of the talk: to give an elementary introduction to matrix compositions.

Structure of the talk:

◇ definition of matrix compositions

◇ recurrences and explicit form

◇ Cassini-like identity and combinatorial interpretation

◇ some encoding

◇ other results.

Definition of m-compositions

Let $m \in \mathbb{N}$, $m > 0$. An m-row matrix composition (m-composition for short) of *length* k of a number $n \in \mathbb{N}$ is an $m \times k$ matrix

$$M = \begin{bmatrix} x_{11} & \dots & x_{1k} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mk} \end{bmatrix}$$

such that

- $x_{ij} \in \mathbb{N} \qquad \forall i, j$
- $(x_{1j}, \dots, x_{kj}) \neq (0, \dots, 0) \qquad \forall j$
- $\sigma(M) = \sum_{i=1}^{m} \sum_{j=1}^{k} x_{ij} = n$

$$\mathcal{C}_n^{(m)} = \{\ m\text{-compositions of } n\ \}$$

$$\mathcal{C}_{n,k}^{(m)} = \{\ m\text{-compositions of } n \text{ of length } k\ \}$$

$$c_n^{(m)} = |\mathcal{C}_n^{(m)}| \qquad c_{n,k}^{(m)} = |\mathcal{C}_{n,k}^{(m)}|$$

Multisets

A multiset on a set X is a function $\mu : X \to \mathbb{N}$
The multiplicity of an element $x \in X$ is $\mu(x)$
The order of μ is $\text{ord}(\mu) = \sum_{x \in X} \mu(x)$
The number of all multisets of order k on a set of size n is the multiset coefficient

$$\left(\!\!\binom{n}{k}\!\!\right) = \frac{n^{\overline{k}}}{k!} = \frac{n(n+1)\dots(n+k-1)}{k!}$$

The columns of an m-composition are multisets with positive order on an m-set.

$$M = \begin{bmatrix} \vdots & c_1 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & c_m & \vdots \end{bmatrix}$$

Each column $(c_1, \dots, c_k)$ is equivalent to the multiset $\mu : [m] \to \mathbb{N}$ defined by

$$\mu = \begin{pmatrix} 1 & 2 & \dots & m \\ c_1 & c_2 & \dots & c_k \end{pmatrix}$$

with positive order: $\mu(1) + \dots + \mu(m) > 0$.

Sum recursions

$$M = \left[\begin{array}{c|ccc} x_{11} & x_{12} & \cdots & x_{1k+1} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mk+1} \end{array}\right]$$

$$c^{(m)}_{n+m,k+1} = \sum_{i=1}^{n+m-k} \left(\!\!\binom{m}{i}\!\!\right) c^{(m)}_{n+m-i,k}$$

$$c^{(m)}_{n+m} = \sum_{i=1}^{n+m} \left(\!\!\binom{m}{i}\!\!\right) c^{(m)}_{n+m-i}$$

Main recurrence

Let A_i be the set of all m-compositions

$$M = \left[\begin{array}{c|c} \vdots & \cdots \\ x_{i1} & \cdots \\ \vdots & \cdots \end{array} \qquad \right]$$

of $n+m$ with $x_{i1} \neq 0$. Then

$$\mathcal{C}_{n+m}^{(m)} = A_1 \cup \ldots \cup A_m$$

and for the Principle of Inclusion-Exclusion

$$c_{n+m}^{(m)} = \sum_{\substack{S \subseteq [m] \\ S \neq \emptyset}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|.$$

$\bigcap_{i \in S} A_i$ is formed by all the m-compositions

$$M = \left[\begin{array}{c|c} \vdots & \cdots \\ x_{i1} & \cdots \\ \vdots & \cdots \end{array} \qquad \right] \sim \left[\begin{array}{c|c} \vdots & \cdots \\ x_{i1} - 1 & \cdots \\ \vdots & \cdots \end{array} \qquad \right]$$

for every $i \in S$. Then

$$\left| \bigcap_{i \in S} A_i \right| = 2 c_{n+m-|S|}^{(m)}.$$

Therefore we have the main recurrence

$$c_{n+m}^{(m)} = 2\sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i}^{(m)}$$

For $m = 2, 3, 4$ we have the recurrences:

$$c_{n+2}^{(2)} = 4c_{n+1}^{(2)} - 2c_n^{(2)}$$

$$c_{n+3}^{(3)} = 6c_{n+2}^{(3)} - 6c_{n+1}^{(3)} + 2c_n^{(3)}$$

$$c_{n+4}^{(4)} = 8c_{n+3}^{(4)} - 12c_{n+2}^{(4)} + 8c_{n+1}^{(4)} - 2c_n^{(4)}.$$

Similarly

$$c_{n+m,k+1}^{(m)} = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i,k}^{(m)} + \\ + \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i,k+1}^{(m)}$$

Explicit form

Let A_i be the set of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ where the i-th column is the zero vector and $\sigma(M) = n$. Then

$$\mathcal{C}_{n,k}^{(m)} = A_1' \cap \ldots \cap A_k'$$

and for the Principle of Inclusion-Exclusion

$$c_{n,k}^{(m)} = |A_1' \cap \ldots \cap A_k'| = \sum_{S \subseteq [k]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|.$$

An element in $\bigcap_{i \in S} A_i$ is a matrix $M \in \mathcal{M}_{m,k}(\mathbb{N})$ with a zero vector in each column indexed by the elements of S. Removing such columns we just have a multiset of order n on a set of size $mk - m|S|$. Then

$$\left| \bigcap_{i \in S} A_i \right| = \left(\!\!\binom{m(k - |S|)}{n}\!\!\right)$$

and consequently

$$c_{n,k}^{(m)} = \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right) (-1)^i$$

Cassini-like identities

The numbers $f_n = c_n^{(2)}$ of all 2-compositions of n satisfy a Cassini-like identity:

$$f_n f_{n+2} - f_{n+1}^2 = -2^{n-1}$$

for every $n \geq 1$.

Equivalently

$$\begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix} = -2^{n-1}.$$

For the numbers $c_n^{(m)}$ such an identity becomes

$$\begin{vmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-1}^{(m)} & c_{n+m}^{(m)} & \cdots & c_{n+2m-2}^{(m)} \end{vmatrix} = (-1)^{\lfloor \frac{m}{2} \rfloor} \, 2^{n-1}$$

for every $m \geq 1$ and $n \geq 1$.

First reduction

Let

$$C_n^{(m)} = \begin{bmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-2}^{(m)} & c_{n+m-1}^{(m)} & \cdots & c_{n+2m-3}^{(m)} \\ c_{n+m-1}^{(m)} & c_{n+m}^{(m)} & \cdots & c_{n+2m-2}^{(m)} \end{bmatrix}.$$

By the main recurrence

$$c_{n+m}^{(m)} = \alpha_{m-1} c_{n+m-1}^{(m)} + \cdots + \alpha_1 c_{n+1}^{(m)} + \alpha_0 c_n^{(m)}$$

where

$$\alpha_k = (-1)^{m-k-1} 2 \binom{m}{k}.$$

The last row can be simplified subtracting a suitable linear combination of the first $m-1$ rows.

We obtain the determinant

$$\begin{vmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-2}^{(m)} & c_{n+m-1}^{(m)} & \cdots & c_{n+2m-3}^{(m)} \\ \alpha_0 c_{n-1}^{(m)} & \alpha_0 c_n^{(m)} & \cdots & \alpha_0 c_{n+m-2}^{(m)} \end{vmatrix}$$

Extracting $\alpha_0 = (-1)^{m-1}2$ from the last row and shifting cyclically all rows downward we have

$$\det C_n^{(m)} = 2 \det C_{n-1}^{(m)}.$$

Then, for every $n \geq 1$, it follows that:

$$\det C_n^{(m)} = 2^{n-1} \det C_1^{(m)}$$

where

$$C_1^{(m)} = [\ c_{i+j+1}^{(m)}\ ]_{i,j=0}^{m-1}.$$

To compute the determinant of this matrix it is useful to consider a combinatorial interpretation of m-compositions.

Combinatorial interpretation

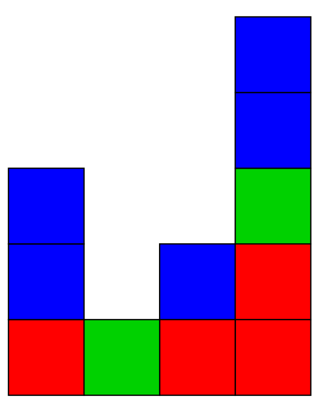
Given the 3-composition

$$M = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

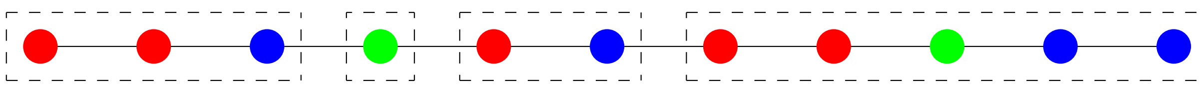
consider its entries as the number of occurrences of three given colors, for instance red, green and blue, linearly ordered red $<$ green $<$ blue. Then

$$M = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

is equivalent to a 3-colored bargraph of area 11 with 4 columns



Now, writing the columns horizontally, from the bottom to the top, we have a 3-colored linear partition of the set $\{1, 2, \ldots, 11\}$ with 4 blocks:



In general, the m-compositions of n of length k are equivalent to the

◇ m-colored bargraphs of area n with k columns.

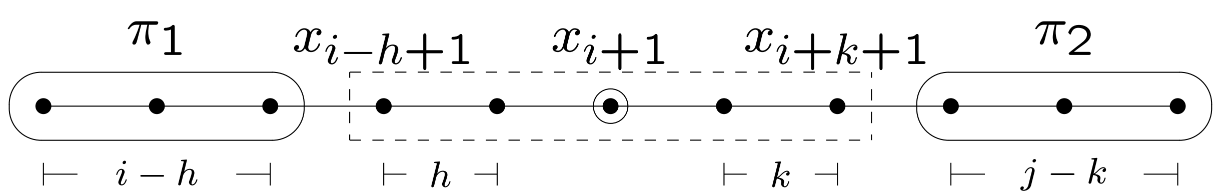
◇ m-colored linear partitions of $[n]$ with k blocks;

A useful sum

Let π be an m-colored linear partition of

$$L = \{x_1, \ldots, x_{i+1}, \ldots, x_{i+j+1}\}.$$

The element x_{i+1} belongs to a block of the form $\{x_{i-h+1}, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k+2}\}$:



Removing this block, π splits into two m-colored linear partitions π_1 and π_2. Then it follows that

$$c_{i+j+1}^{(m)} = \sum_{h,k \geq 0} \left(\!\!\binom{m}{h+k+1}\!\!\right) c_{i-h}^{(m)} c_{j-k}^{(m)}$$

Second reduction

The previous identity implies the decomposition

$$C_1^{(m)} = L^{(m)} M^{(m)} L_T^{(m)}$$

where

$$L^{(m)} = [\, c_{i-j}^{(m)} \,]_{i,j=0}^{m-1} = \begin{bmatrix} c_0 & & & \\ c_1 & c_0 & & \\ \vdots & \ddots & \ddots & \\ c_{m-1} & \cdots & c_1 & c_0 \end{bmatrix}$$

$$M^{(m)} = \left[\left(\!\!\binom{m}{i+j+1}\!\!\right) \right]_{i,j=0}^{m-1}.$$

Then

$$\det C_1^{(m)} = \det M^{(m)}.$$

To calculate $\det M^{(m)}$ we will use another identity coming form our combinatorial interpretation.

Another useful sum

Recall that $\left(\!\!\binom{m}{i+j+1}\!\!\right)$ gives the number of all the order maps $f : [i+j+1] \to [m]$.

Suppose that $f(i+1) = k$, with $k \in [m]$. Since f is order preserving, it follows that

- $1 \le x \le i$ then $1 \le f(x) \le k$
- $i+2 \le x \le i+j+1$ then $k \le f(x) \le m$.

Then

$$\left(\!\!\binom{m}{i+j+1}\!\!\right) = \sum_{k=1}^{m} \left(\!\!\binom{k}{i}\!\!\right)\left(\!\!\binom{m-k+1}{j}\!\!\right)$$
$$= \sum_{k=0}^{m-1} \binom{i+k}{i}\left(\!\!\binom{m-k}{j}\!\!\right)$$

Final step

The previous identity implies that

$$M^{(m)} = B^{(m)} \widetilde{B}^{(m)}$$

where

$$B^{(m)} = \left[\binom{i+j}{i} \right]_{i,j=0}^{m-1}$$

and

$$\widetilde{B}^{(m)} = \left[\left(\!\!\binom{m-i}{j}\!\!\right) \right]_{i,j=0}^{m-1}.$$

Since $\widetilde{B}^{(m)} = J^{(m)} B^{(m)}$ where

$$J^{(m)} = [\delta_{i+j,m-1}]_{i,j=0}^{m-1} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix},$$

we have

$$M^{(m)} = B^{(m)} J^{(m)} B^{(m)}$$

and

$$\det M^{(m)} = \det J^{(m)} (\det B^{(m)})^2.$$

Since, as well known,

$$\det J^{(m)} = (-1)^{\lfloor m/2 \rfloor}$$

and

$$\det B^{(m)} = 1$$

we have

$$\det M^{(m)} = (-1)^{\lfloor m/2 \rfloor}$$

and consequently

$$\det C_1^{(m)} = \det M^{(m)} = (-1)^{\lfloor m/2 \rfloor}.$$

Finally, since for every $n \geq 1$ we have

$$\det C_n^{(m)} = 2^{n-1} \det C_1^{(m)},$$

then

$$\det C_n^{(m)} = (-1)^{\lfloor m/2 \rfloor} 2^{n-1}.$$

m-compositions as words

Consider each m-composition as the concatenation of its columns:

$$M = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Each column $(c_1, \ldots, c_k)$ is equivalent to the multiset $\mu : [m] \to \mathbb{N}$ defined by

$$\mu = \begin{pmatrix} 1 & 2 & \ldots & m \\ c_1 & c_2 & \ldots & c_k \end{pmatrix}$$

with positive order: $\mu(1) + \ldots + \mu(m) > 0$.

This means that

$$\mathcal{C}^{(m)} = \{a_\mu \ : \ \mu \in \mathcal{M}^{(m)}_{\neq 0}\}^*$$

where $\mathcal{M}^{(m)}_{\neq 0}$ is the set of all multisets $\mu : [m] \to \mathbb{N}$ with positive order.

Let $X = \{x_\mu \ : \ \mu \in \mathcal{M}^{(m)}_{\neq 0}\}$. Then, for $a_\mu \rightsquigarrow x_\mu$, we have the generating series for $\mathcal{C}^{(m)}$

$$c(X) = \frac{1}{1 - \sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} x_\mu}$$

In particular, for $x_\mu = x^{\mathrm{ord}(\mu)}$ we get

$$c^{(m)}(x) = \sum_{n\geq 0} c_n^{(m)}\, x^n = \frac{1}{1-h(x)}$$

where

$$h(x) = \sum_{\mu\in\mathcal{M}^{(m)}_{\neq 0}} x_\mu = \sum_{k\geq 1} \left(\!\binom{m}{k}\!\right) x^k$$

that is

$$h(x) = \frac{1}{(1-x)^m} - 1.$$

Then

$$c^{(m)}(x) = \frac{(1-x)^m}{2(1-x)^m - 1}$$

From here we have the previous results and a the following new recurrence

$$c_{n+1}^{(m)} = -\delta_{n,0} + 2c_n^{(m)} + \sum_{k=0}^{n} \binom{m+k-1}{k+1} c_{n-k}^{(m)}$$

m-compositions of Carlitz type

A Carlitz composition is a composition without two equal consecutive elements.

We say that an m-composition is of Carlitz type when no two adjacent columns are equal.

For instance

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 \\ 4 & x & 3 & 3 \end{bmatrix}$$

is of Carlitz type only when $x \neq 4$.

Let $\mathcal{Z}$ be the set of all words corresponding to the m-compositions of Carlitz type and let $\mathcal{Z}_\mu$ be the subset of $\mathcal{Z}$ formed by the words ending with a_μ, for every $\mu \in \mathcal{M}^{(m)}_{\neq 0}$.

Then

$$\mathcal{Z} = 1 + \sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} \mathcal{Z}_\mu$$

$$\mathcal{Z}_\mu = (\mathcal{Z} - \mathcal{Z}_\mu) a_\mu \qquad \forall\, \mu \in \mathcal{M}^{(m)}_{\neq 0}\,.$$

To obtain the generating series for $\mathcal{Z}$ and $\mathcal{Z}_\mu$ substitute each letter a_μ with the indeterminate x_μ:

$$z(X) = 1 + \sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} z_\mu(X)$$

$$z_\mu(X) = (z(X) - z_\mu(X)) x_\mu \qquad \forall\, \mu \in \mathcal{M}^{(m)}_{\neq 0}\,.$$

Then

$$z_\mu(X) = \frac{x_\mu}{1 + x_\mu}\, z(X)$$

and finally

$$z(X) = \frac{1}{1 - \displaystyle\sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} \frac{x_\mu}{1 + x_\mu}}.$$

For $x_\mu = x^{\mathrm{ord}(\mu)}$ we have

$$\sum_{n\geq 0} z_n^{(m)}\, x^n = \frac{1}{1 - \sum_{k\geq 1} \left(\!\binom{m}{k}\!\right) \frac{x^k}{1+x^k}}$$

where $z_n^{(m)}$ is the number of all m-compositions of Carlitz type of n.

Expanding this series we can obtain:

$$z_n^{(m)} = \sum_{k\geq 0} \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^k \\ \alpha\cdot\beta=n}} \left(\!\binom{m}{\alpha}\!\right) (-1)^{|\beta|-k}.$$

where

$$\alpha\cdot\beta = a_1 b_1 + \cdots + a_k b_k$$

$$|\beta| = b_1 + \cdots + b_k$$

$$\left(\!\binom{m}{\alpha}\!\right) = \left(\!\binom{m}{a_1}\!\right) \cdots \left(\!\binom{m}{a_k}\!\right)$$

for every $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$.

A regular language for m-compositions

The encoding used in

A. Björner, R. Stanley, *An analogue of Young's lattice for compositions*, FP-SAC'05

for ordinary compositions can be extended to m-compositions as follows. For instance

$$\begin{aligned}
M &= \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \\
&= \begin{matrix} 2 \\ 0 \\ 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} + \begin{matrix} 2 \\ 1 \\ 2 \end{matrix} \\
&= \begin{matrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} + \begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \\
&= \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} + \begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{matrix} \\
&= a_1 a_1 a_3 b_2 b_1 a_3 b_1 a_1 a_2 a_3 a_3.
\end{aligned}$$

More precisely, given the finite alphabet

$$\mathcal{A}_m = \{a_1, \cdots, a_m, b_1, \ldots, b_m\},$$

we can define a map $\ell : \mathcal{C}^{(m)} \to \mathcal{A}_m^*$ setting

$$\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \stackrel{\ell}{\longmapsto} a_1\,, \qquad \cdots \qquad \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \stackrel{\ell}{\longmapsto} a_m\,,$$

$$+\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \stackrel{\ell}{\longmapsto} b_1\,, \qquad \cdots \qquad +\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \stackrel{\ell}{\longmapsto} b_m$$

and proceeding as in the example.

The words of the language $\mathcal{L}_m = \ell(\mathcal{C}^{(m)}) \subseteq \mathcal{A}_m^*$, corresponding to the m-compositions, are characterized by the following conditions:

i) the first letter is a_1 or ... or a_m;

ii) each letter a_i or b_i can be followed by any b_j, while it can be followed by a letter a_j only when $i \leq j$.

This characterization implies a unique factorization of the form xy, where:

- $x = a_1^{i_1} \dots a_m^{i_m}$ ($\neq \varepsilon$), with $i_1, \dots, i_m \geq 0$;
- $y = y_1 \dots y_k$ (possibly empty), where

$$y_r = b_j a_j^{q_j} \dots a_m^{q_m}, \qquad q_j, \dots, q_m \geq 0.$$

Then $\mathcal{L}_m$ is a regular language defined by the unambiguous regular expression:

$$\varepsilon + \mathcal{L}'_m \mathcal{L}''_m$$

where ε is the empty word and

$$\mathcal{L}'_m = a_1^+ a_2^* \dots a_m^* + a_2^+ a_3^* \dots a_m^* + \dots + a_m^+,$$

$$\mathcal{L}''_m = (\, b_1 a_1^* a_2^* \dots a_m^* + b_2 a_2^* \dots a_m^* + \dots + b_m a_m^* \,)^*$$

This is the basis for an efficient algorithm for the exhaustive generation of m-compositions (Gray code), as described in:

E. Grazzini, E. Munarini, M. Poneti, S. Rinaldi, *On the generation of m-compositions and m-partitions*, 2006

Other results:

◇ Asymptotic expansion:

$$c_n^{(m)} \sim_n \frac{1}{2m(\sqrt[m]{2}-1)} \left(\frac{\sqrt[m]{2}}{\sqrt[m]{2}-1} \right)^n$$

◇ m-compositions without zero rows:

$$f_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{n-k} c_n^{(k)}$$

$$f^{(m)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{n-k} \frac{(1-x)^k}{2(1-x)^k - 1}$$

◇ m-compositions with palindromic rows:

$$\sum_{n\geq 0} p_n^{(m)} x^n = \frac{(1+x)^m}{2(1-x^2)^m - 1}$$

◇ enumeration of various kind of m-colored bargraphs

◇ . . .