Ehrhart polynomials of lattice-face polytopes

by *Fu Liu*

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Preliminaries

Definition 1. A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

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Definition 2. For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \ : \ \mathbf{x} \in P\}$. We denote by

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Example: When $d = 1$, P is an interval $[a, b]$, where $a, b \in \mathbb{Z}$. Then $mP = [ma, mb]$ and

$$i(P, m) = (b - a)m + 1.$$

Theorem of Ehrhart

Theorem 3. *(Ehrhart) Let P be a d-dimensional integral polytope, then $i(P, m)$ is a polynomial in m of degree d.*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

Coefficients of Ehrhart polynomials

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- The second coefficient equals $1/2$ times the sum of volumes of each facet, each normalized with respect to the sublattice in the hyperplane spanned by the facet.
- The constant term of $i(P, m)$ is always 1.
- No results for other coefficients for general polytopes.

Motivation

De Loera conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $T = \{t_1, \dots, t_n\}_<$ a linearly ordered set, a d-dimensional *cyclic polytope* $C_d(T) = C_d(t_1, \dots, t_n)$ is the convex hull $\mathrm{conv}\{v_d(t_1), v_d(t_2), \dots, v_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i), 1 \leq i \leq n,$ on the moment curve.

The *moment curve* in $\mathbb{R}^d$ is defined by

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Example: $T = \{1, 2, 3, 4\}, d = 3:$

$C_d(T)$ is the convex polytope whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}.$

Theorem 4. *For any d-dimensional integral cyclic polytope $C_d(T)$,*

$$i(C_d(T), m) = \mathrm{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$\begin{aligned} i(C_d(T), m) &= \sum_{k=0}^{d} \mathrm{Vol}_k(mC_k(T)) \\ &= \sum_{k=0}^{d} \mathrm{Vol}_k(C_k(T)) m^k, \end{aligned}$$

where $\mathrm{Vol}_k(mC_k(T))$ is the volume of $mC_k(T)$ in k-dimensional space, and by convention we let $\mathrm{Vol}_0(mC_0(T)) = 1$.

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➠ $2, 4, 3$ and 1 are the volumes of $C_3(T)$, $C_2(T)$, $C_1(T)$ and $C_0(T)$, respectively.

Note that if we define $\pi^k : \mathbb{R}^d \to \mathbb{R}^{d-k}$ to be the map which ignores the last k coordinates of a point, then $\pi^k(C_d(T)) = C_{d-k}(T)$. So when $P = C_d(T)$ is an integral cyclic polytope, we have that

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P))m^k, \tag{5}$$

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Question: Are there other integral polytopes which have the same form of Ehrhart polynomials as cyclic polytopes? In other words, what kind of integral d-polytopes P are there whose Ehrhart polynomials will be in the form of (5)?

Properties of integral cyclic polytopes

What are some key properties of an integral cyclic polytope $C_d(T)$?

When $d = 1$, $C_d(T)$ is just an integral polytope.

For $d \geq 2$, for any d-subset $T' \subset T$, let $U = \nu_d(T')$ be the corresponding d-subset of the vertex set $V = \nu_d(T)$ of $C_d(T)$. Then:

a) $\pi(\text{conv}(U)) = \pi(C_d(T')) = C_{d-1}(T')$ is an integral cyclic polytope, and

b) $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{d-1}$, where H_U is the affine space spanned by U. In other words, after dropping the last coordinate of the lattice of H_U, we get the $(d-1)$-dimensional lattice.

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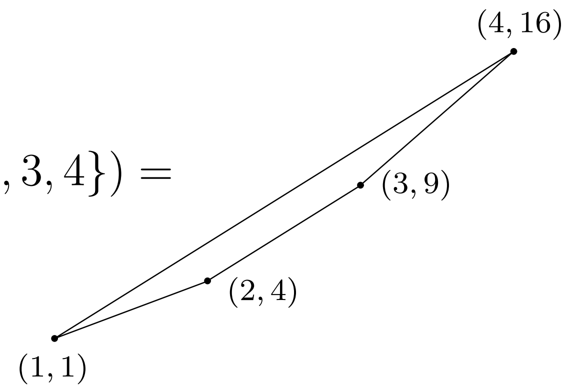

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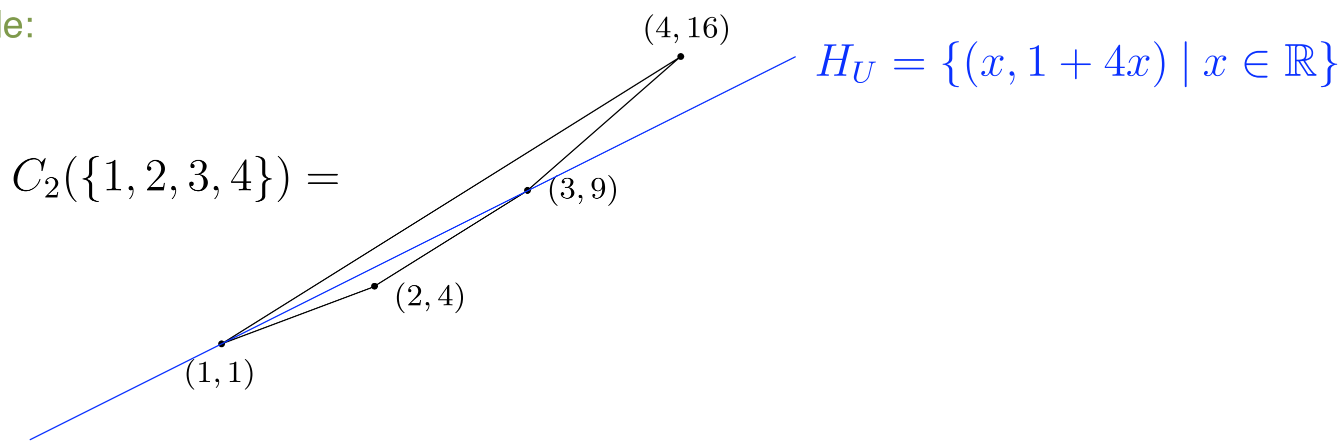

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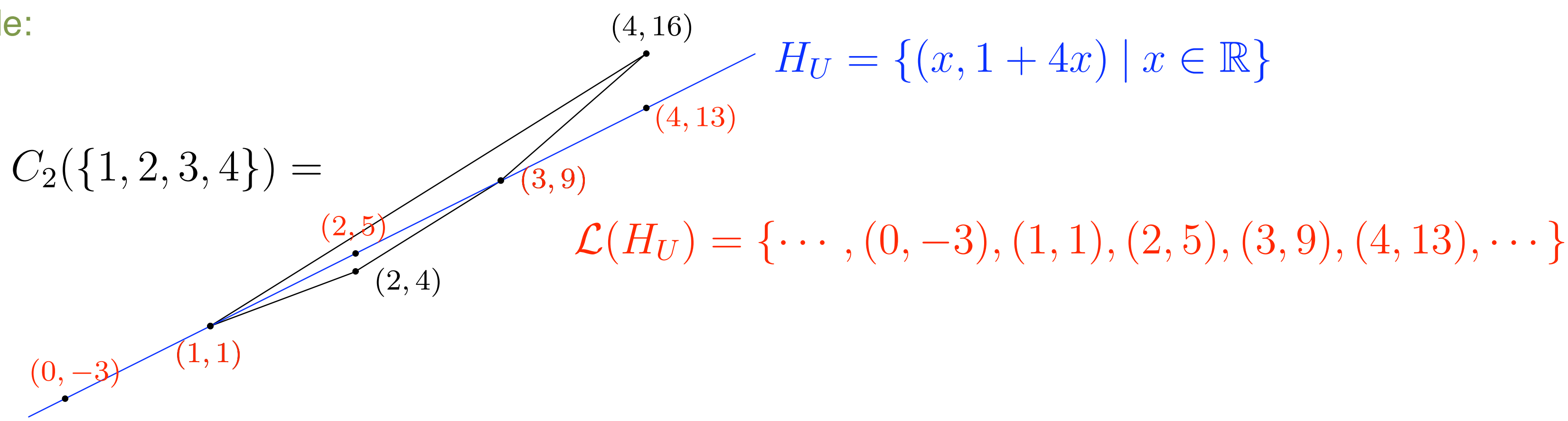

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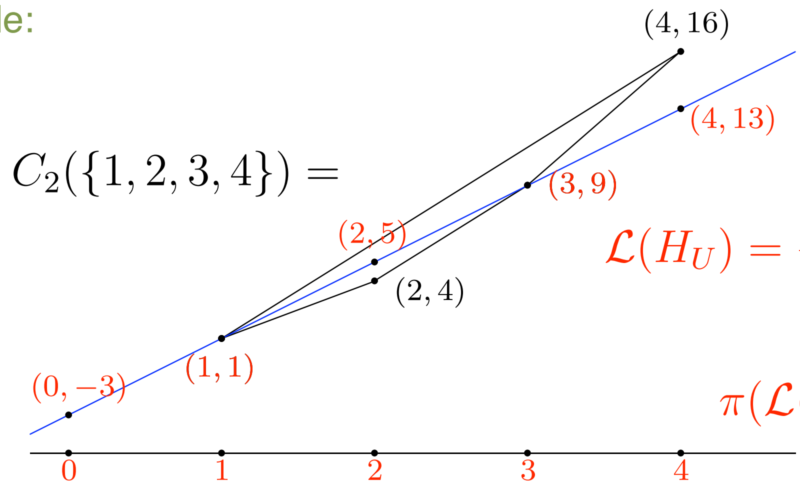

$$H_U = \{(x, 1 + 4x) \mid x \in \mathbb{R}\}$$

$$\mathcal{L}(H_U) = \{\cdots, (0, -3), (1, 1), (2, 5), (3, 9), (4, 13), \cdots\}$$

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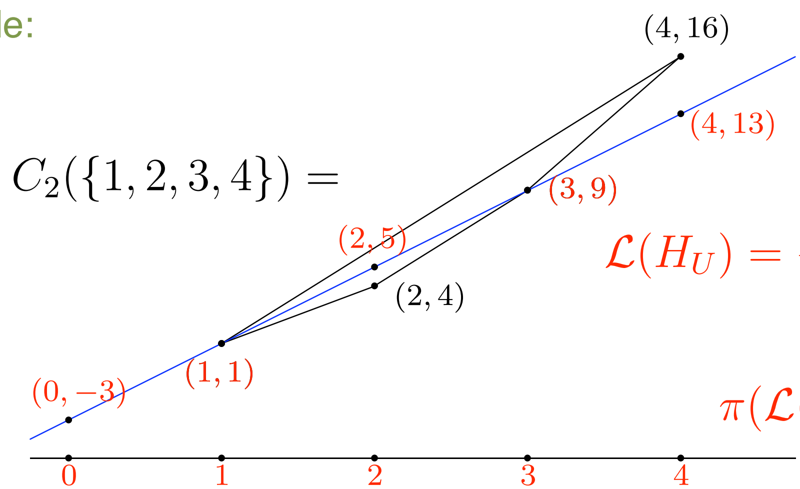

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Remark: Condition b) is equivalent to say that for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\pi^{-1}(y) \cap H_U$, the intersection of H_U with the inverse image of y under π, is a lattice point.

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For $d \geq 2$, we call a d-dimensional polytope P with vertex set V a *lattice-face* polytope if for any d-subset $U \subset V$,

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The Main Theorem

Theorem 8. *Let P be a lattice-face d-polytope, then*

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P))m^k.$$

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2. mP is a lattice-face d-polytope $\Rightarrow$ it's enough to show that

$$|\mathcal{L}(P)| = \mathrm{Vol}(P) + |\mathcal{L}(\pi(P))|.$$

More Notation

1. For any polytope $P \subset \mathbb{R}^d$ and any point $y \in \pi(P)$, let $n(y, P)$ be the point of $\pi^{-1}(y) \cap P$ having the smallest last coordinate.

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Example:

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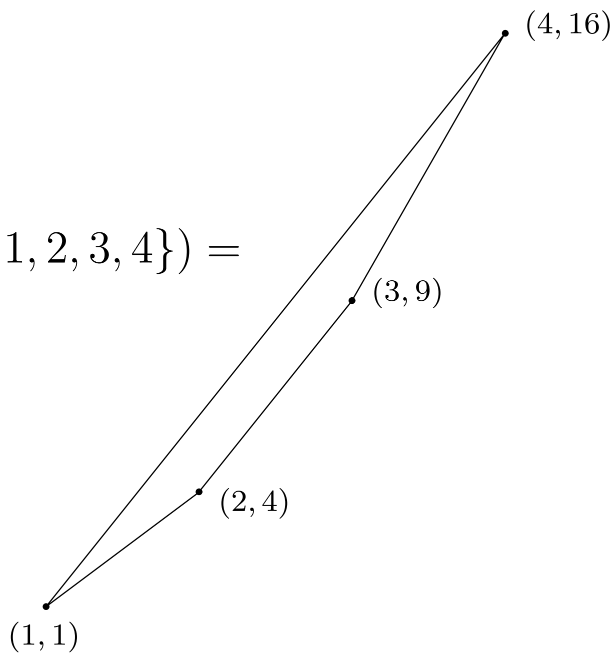

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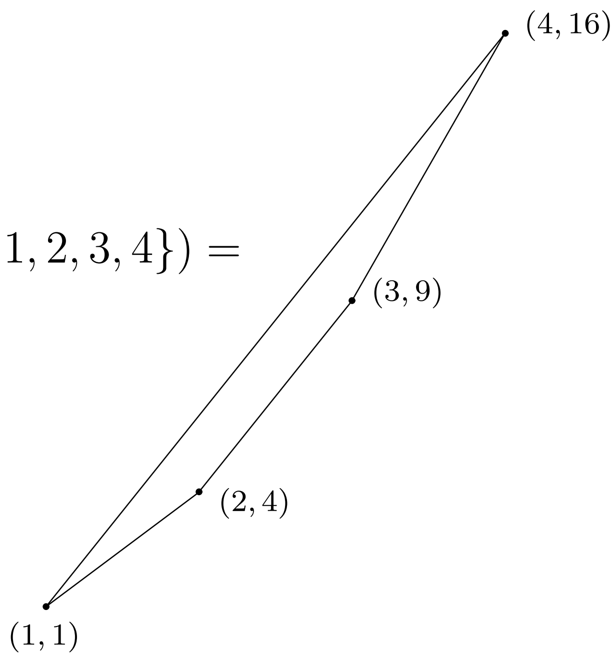

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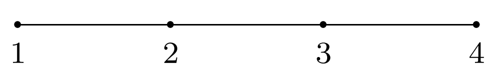

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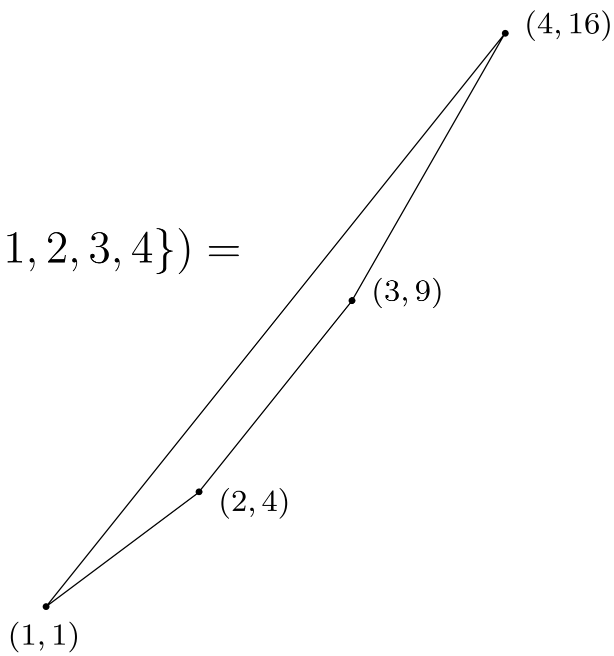

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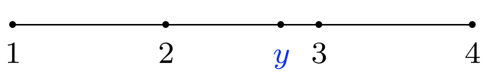

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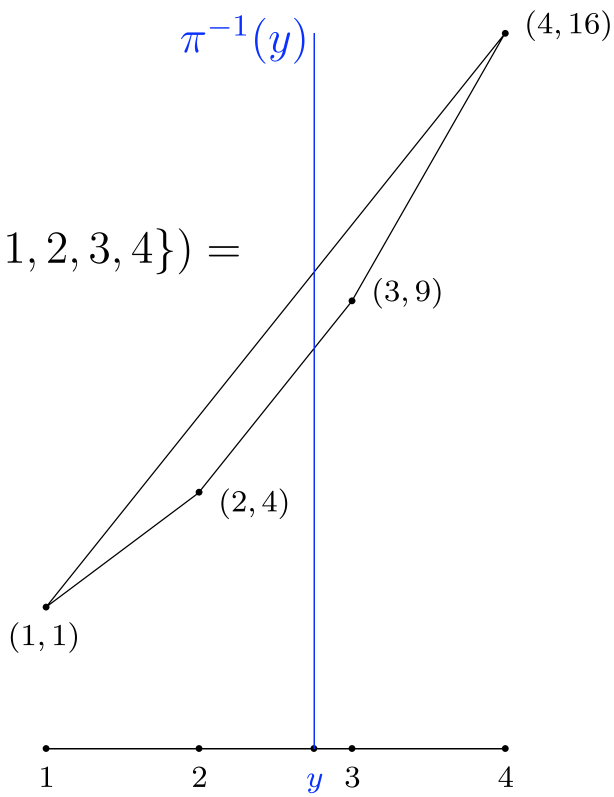

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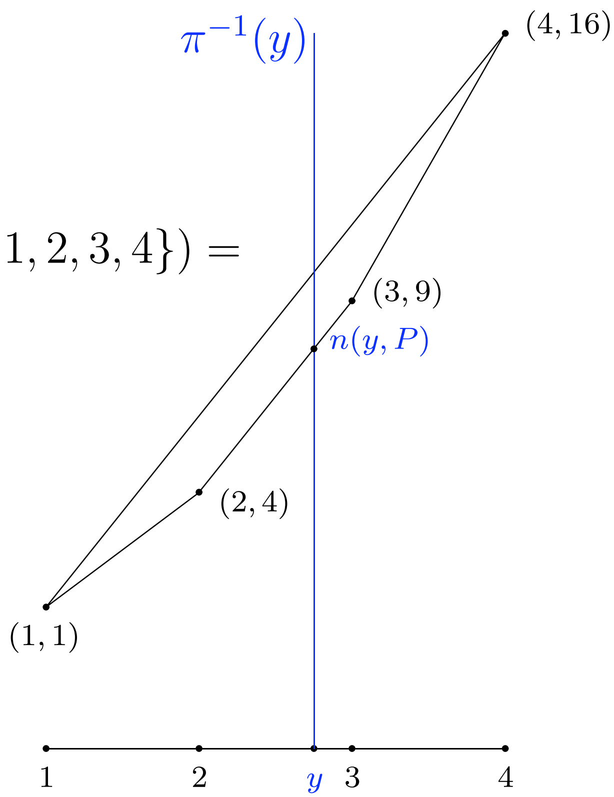

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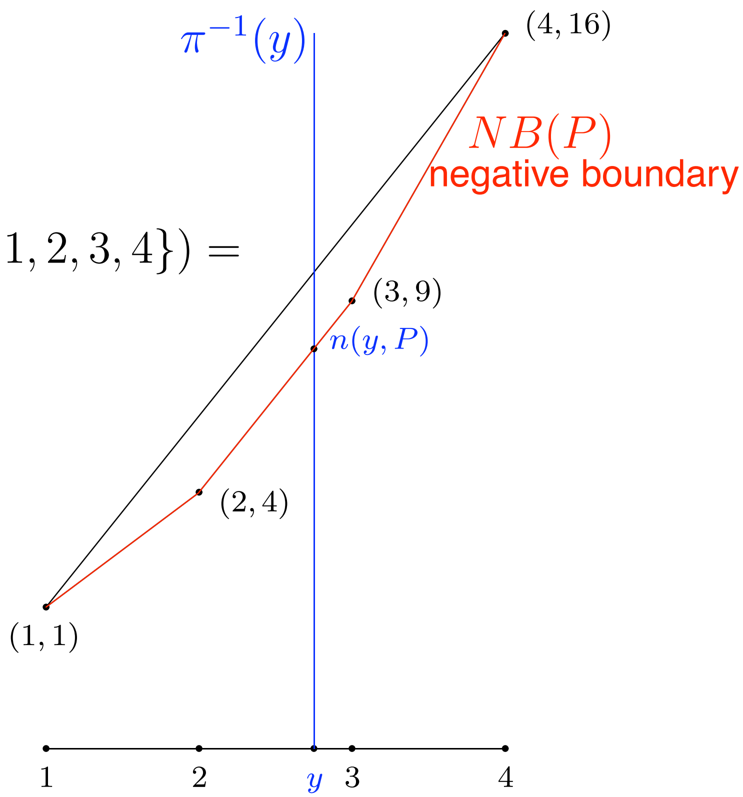

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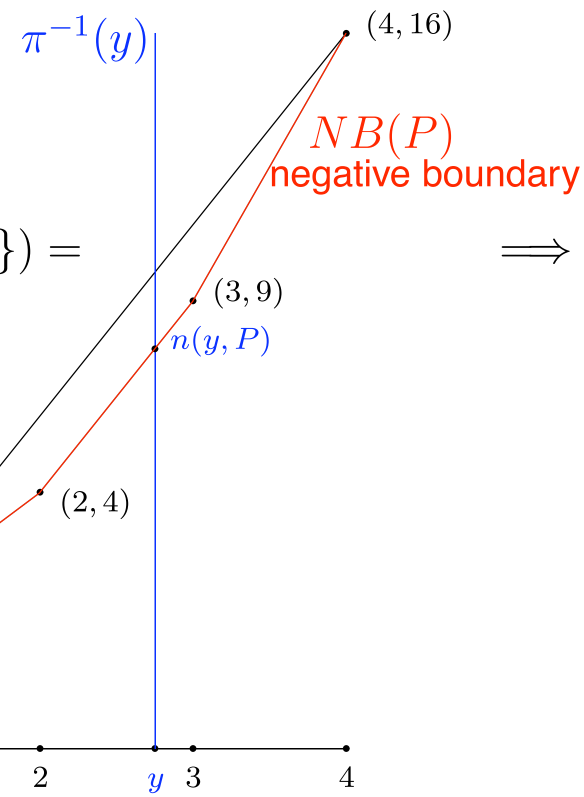
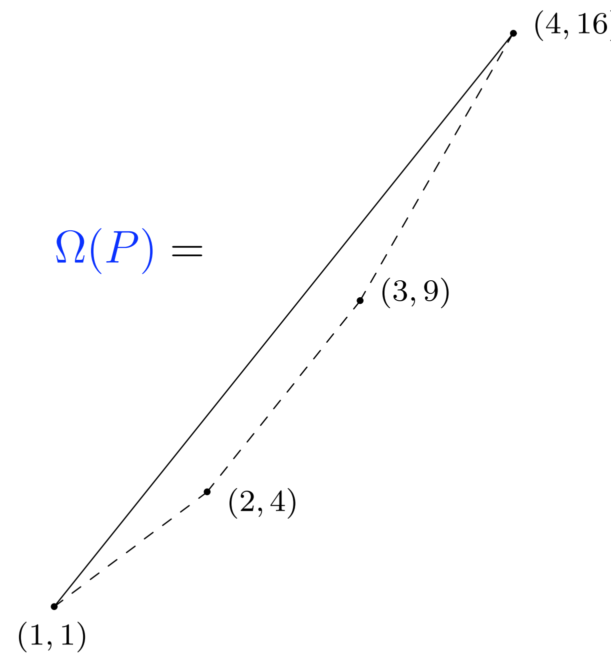


$\Longrightarrow$ $\Omega(P) =$



Clearly, π induces a bijection between $\mathcal{L}(NB(P))$ and $\mathcal{L}(\pi(P))$. Therefore,

$$\begin{aligned} & |\mathcal{L}(P)| \\ = \ & |\mathcal{L}(\Omega(P))| + |\mathcal{L}(NB(P))| \\ = \ & |\mathcal{L}(\Omega(P))| + |\mathcal{L}(\pi(P))|. \end{aligned}$$

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Comparing with the formula we want to show:

$$|\mathcal{L}(P)| = \mathrm{Vol}(P) + |\mathcal{L}(\pi(P))|,$$

one see that to prove Theorem 8 it is sufficient to prove the following theorem.

Theorem 9. *For any P a lattice-face d-polytope,*

$$\mathrm{Vol}(P) = |\mathcal{L}(\Omega(P))|.$$

For any triangulation (without introducing new vertices) $P_1 \cup \cdots \cup P_k$ of a lattice-face polytope P, (note that (iv) implies that all P_i are lattice-face polytopes,) we have that

$$\Omega(P) = \bigoplus_{i=1}^{k} \Omega(P_i), \text{ which implies that } |\mathcal{L}(\Omega(P))| = \sum_{i=1}^{k} |\mathcal{L}(\Omega(P_i))|.$$

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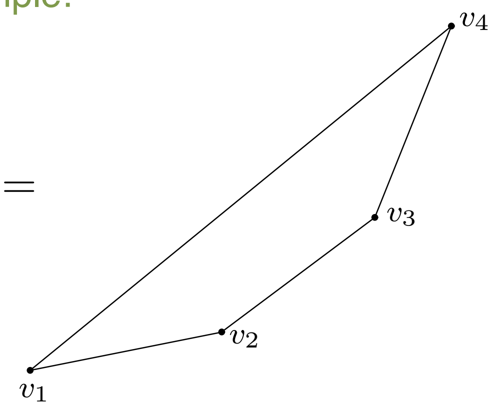

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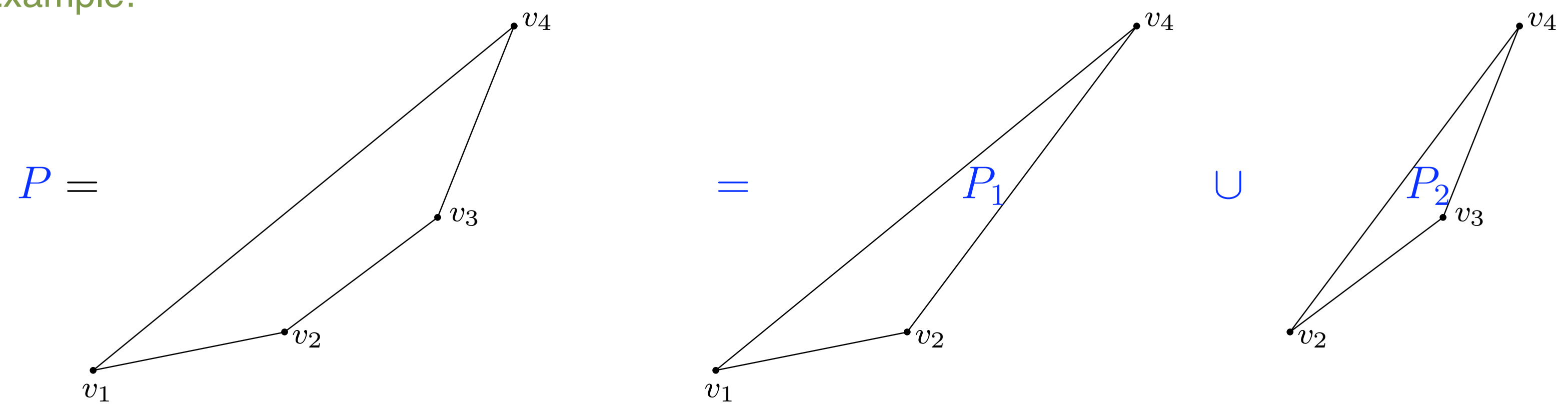

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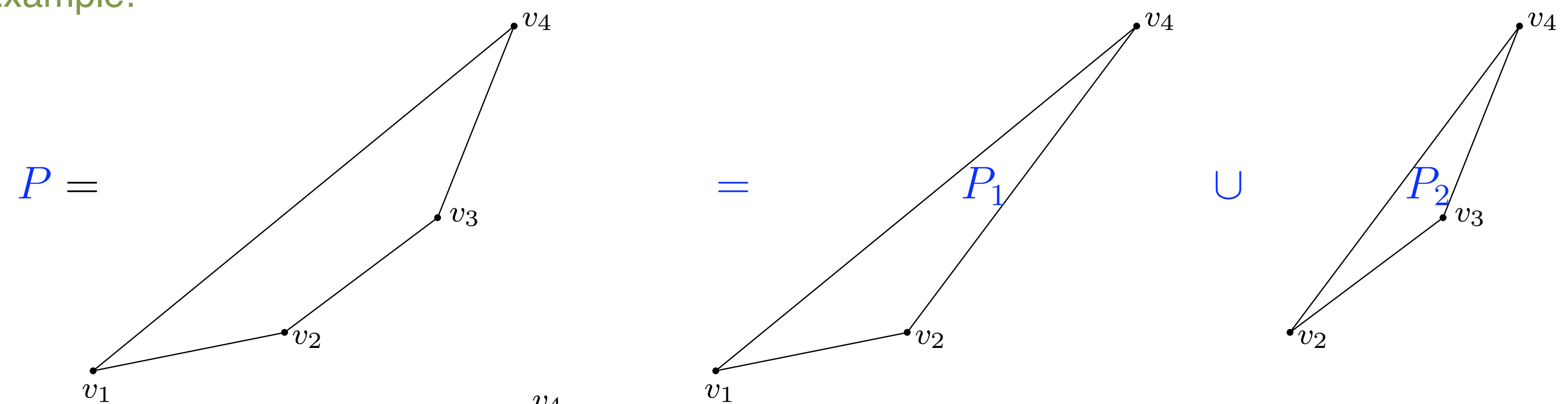

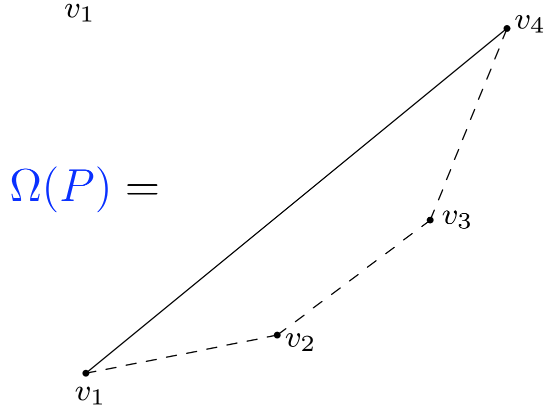

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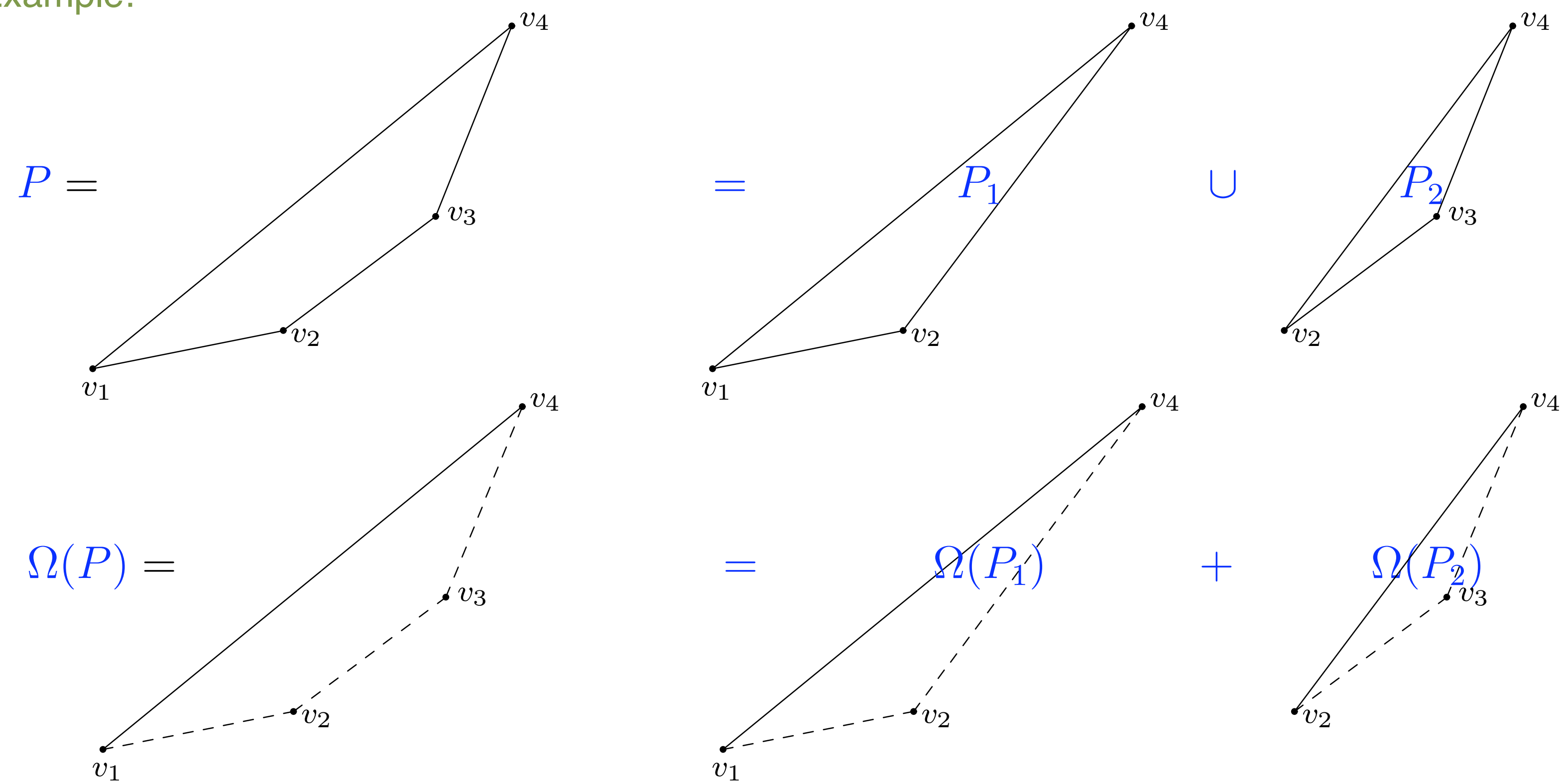

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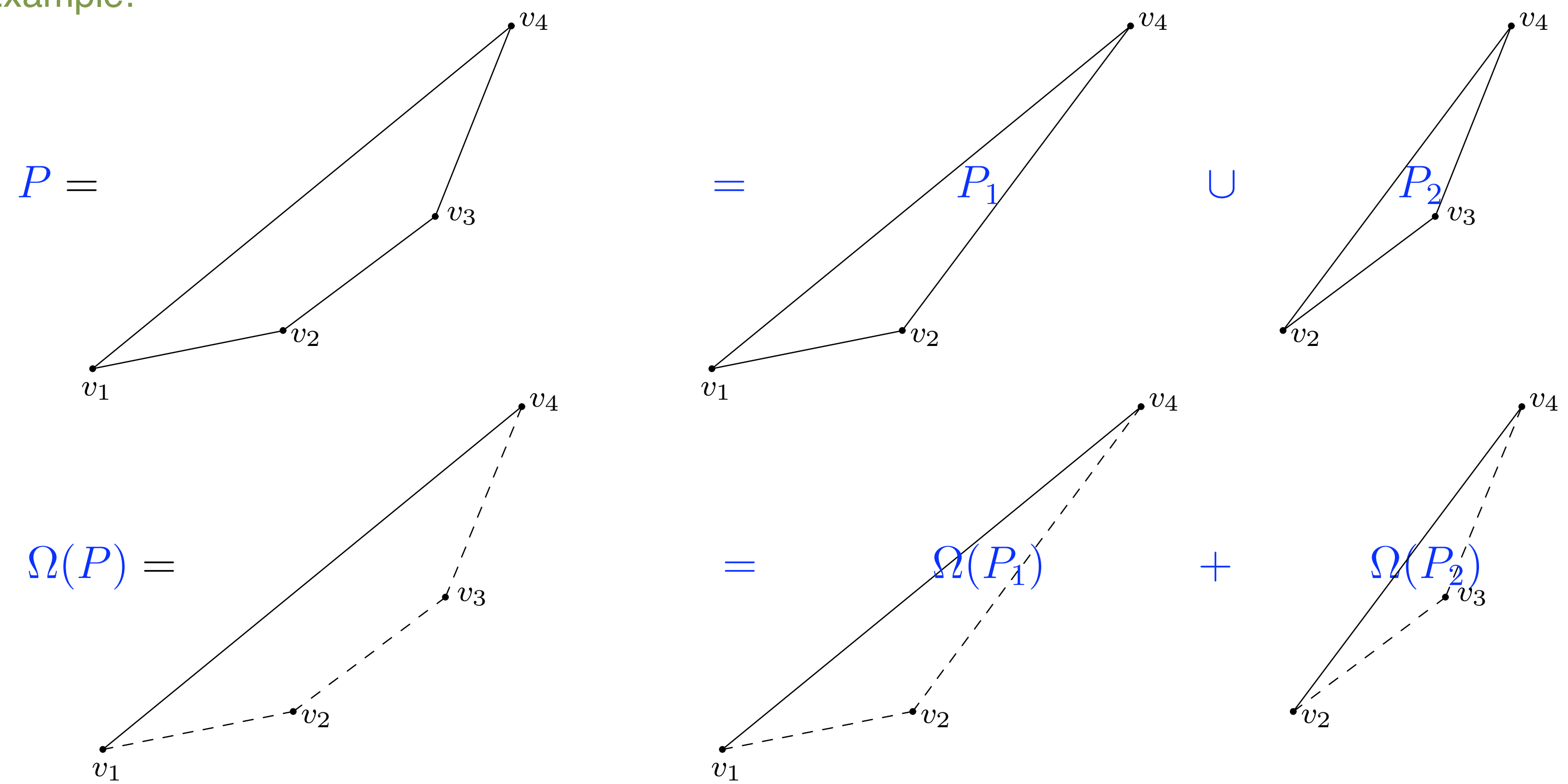

$$\Longrightarrow |\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P_1))| + |\mathcal{L}(\Omega(P_2))|$$

However, for any triangulation (without introducing new vertices) $P_1 \cup \cdots \cup P_k$, we have that

$$\mathrm{Vol}(P) = \sum_{i=1}^{k} \mathrm{Vol}(P_i).$$

Comparing this with

$$|\mathcal{L}(\Omega(P))| = \sum_{i=1}^{k} |\mathcal{L}(\Omega(P_i))|,$$

we conclude that, to prove Theorem 9 ($\mathrm{Vol}(P) = |\mathcal{L}(\Omega(P))|$), it is enough to prove the the case when P is a lattice-face d-simplex, i.e., P has $d+1$ vertices which are affinely independent.

Idea of the Proof

We will use two dimensional lattice-face simplices to illustrate the idea of our proof.

Assume P is a 2-dimensional lattice-face simplex with vertex set $V = \{v_1, v_2, v_3\}$, where the coordinates of v_i are $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

WLOG, we assume that v_1, v_2, v_3 are in an order such that both $\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$ and $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$ are positive. In other words, v_1, v_2, v_3 are in counterclockwise order and $x_1 < x_2$.

We can define an affine transformation T which maps $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ \dfrac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} \end{pmatrix}$. One can check that

1. T gives a bijection between the lattice points in $\Omega(P)$ and the lattice points in $\Omega(T(P))$. Therefore, we want to show that

$$\mathrm{Vol}(P) = |\mathcal{L}(\Omega(T(P)))|.$$

2. $T(P)$ is a lattice-face polytope, as well.

Let $P' = T(P)$. Then its vertex set is $V' = \{v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}\}$,

where $y_3' = \dfrac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}$.

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By our assumption, $y_3' > 0$. There are 3 cases for the position of the vertices of P' :
(i) $x_1 < x_2 < x_3$; (ii) $x_1 < x_3 < x_2$; (iii) $x_3 < x_1 < x_2$.

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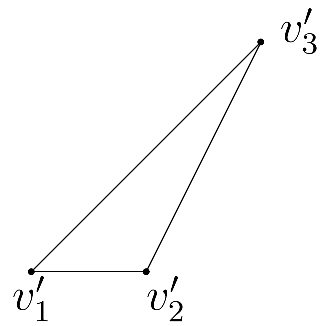
$P' =$ 

$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

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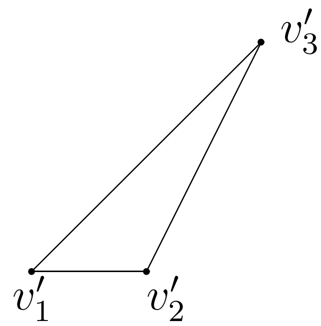
$P' \quad =$ 

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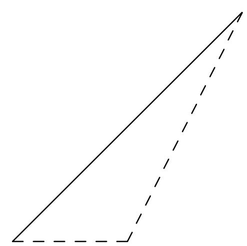
$\Omega(P') \quad =$ 

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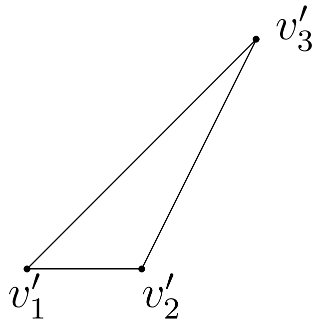

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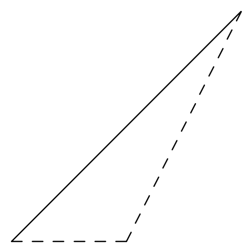
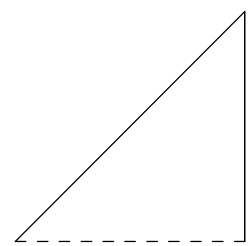
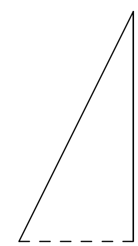
$\Omega(P') =$  $=$  $-$ 

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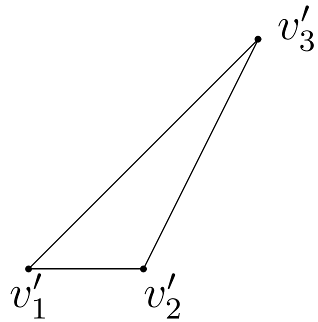

$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

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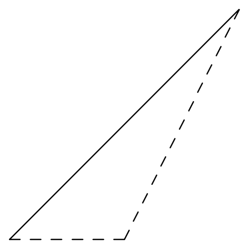

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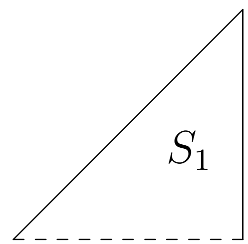

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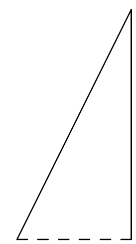

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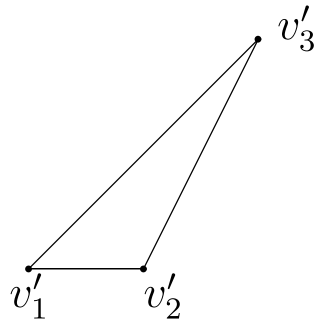

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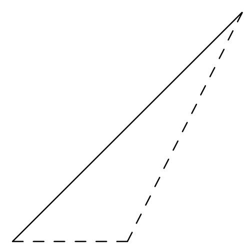

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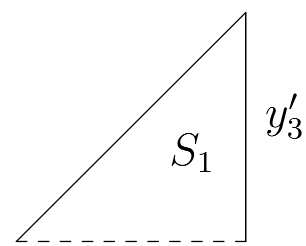

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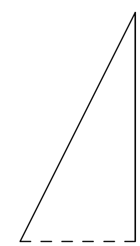

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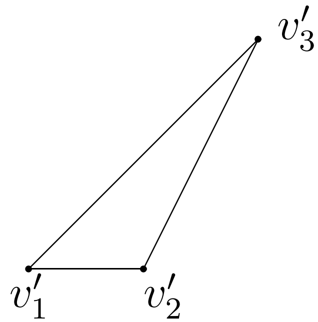
$P' =$ 

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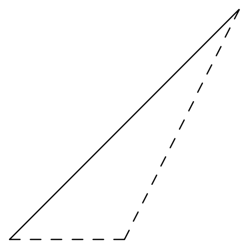
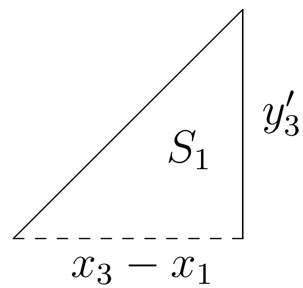
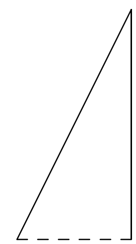
$\Omega(P') =$  $=$ 

$-$ 

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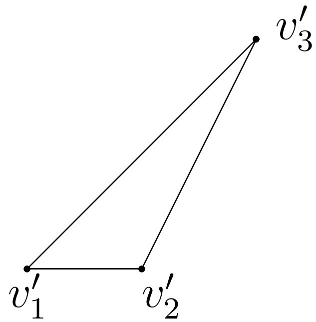
$P' =$ 

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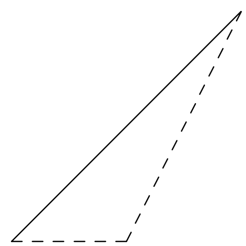
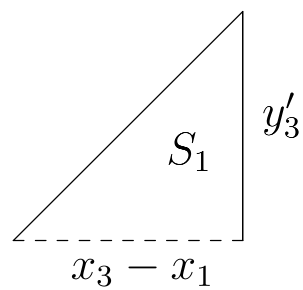
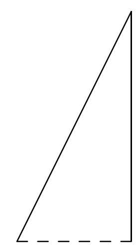
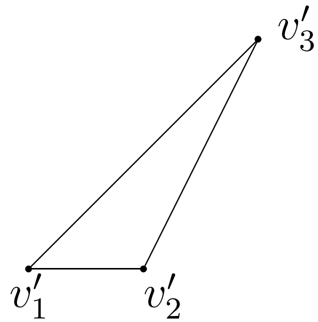
$\Omega(P') =$  $=$ 

$-$ 

$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

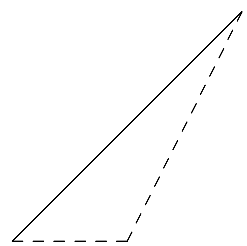
(i) $x_1 < x_2 < x_3$:

$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$\Omega(P') =$



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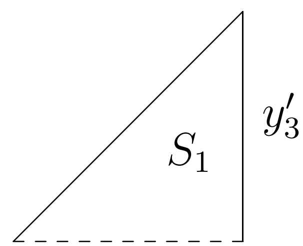

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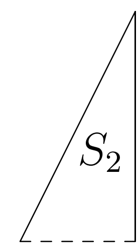
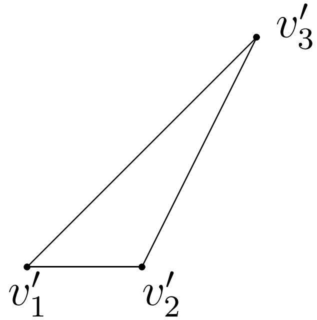


$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

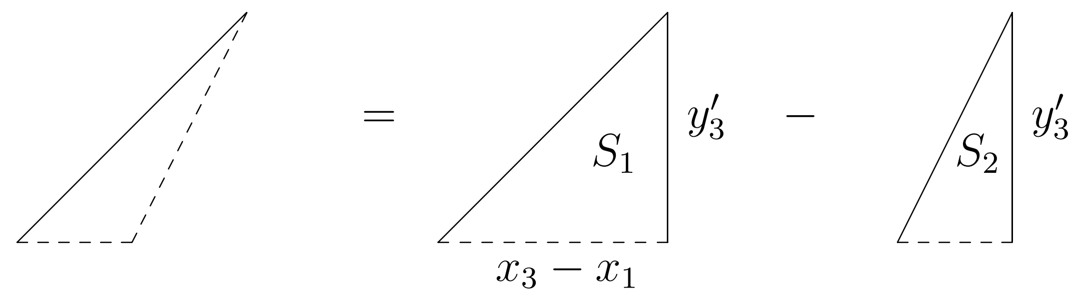
(i) $x_1 < x_2 < x_3$:

$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

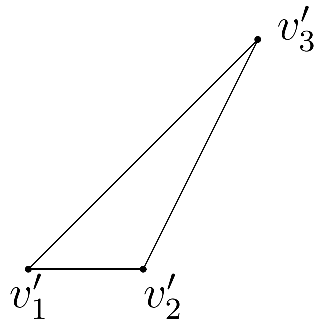
$\Omega(P') =$



$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

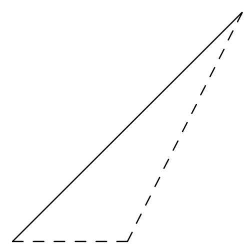
(i) $x_1 < x_2 < x_3$:

$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$\Omega(P') =$



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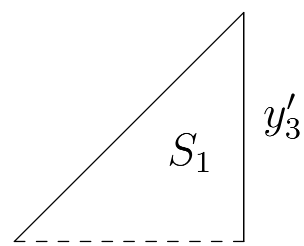

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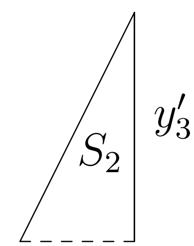
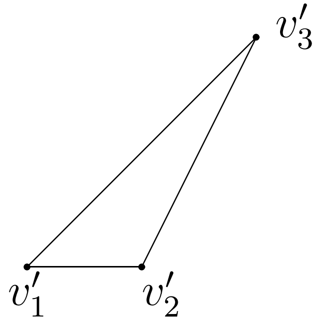


$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \qquad x_3 - x_2$$

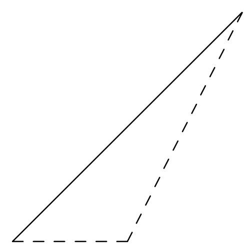
(i) $x_1 < x_2 < x_3$:

$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$\Omega(P') =$



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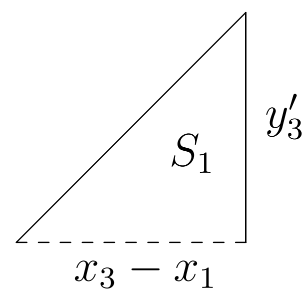

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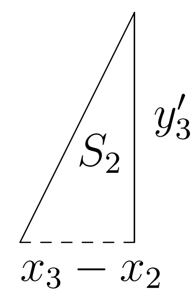
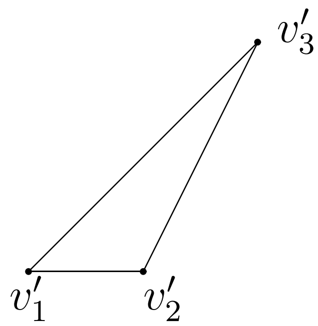


$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, \qquad x_3 - x_2 = \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

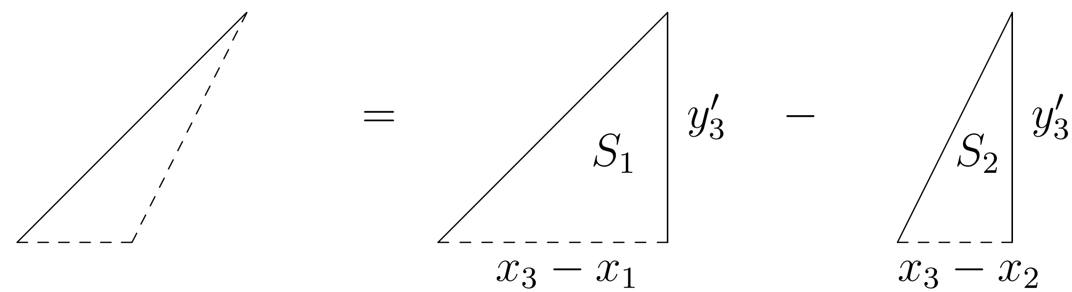
(i) $x_1 < x_2 < x_3$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

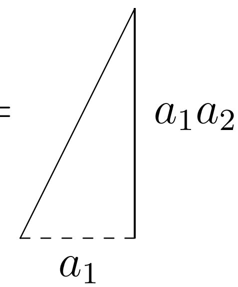
$\Omega(P') =$



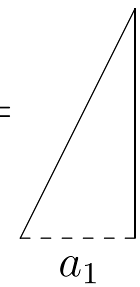
$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \qquad x_3 - x_2 = \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

Because P' is a lattice-face polytope, it is not hard to show that y_3' is a multiple of both $\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$ and $\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$.

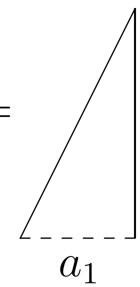
For any positive integers a_1, a_2, if $S =$



,

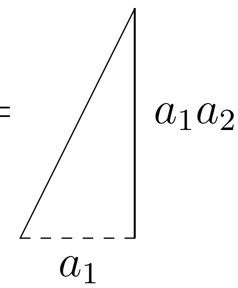
For any positive integers $a_1, a_2,$ if $S =$ 

$a_1 a_2$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

For any positive integers a_1, a_2, if $S =$ 

$a_1 a_2$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

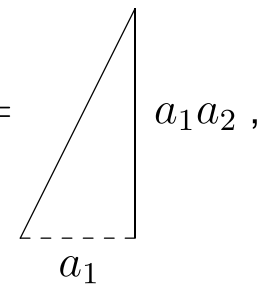
Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$, for any positive integers a_1, a_2.

For any positive integers a_1, a_2, if $S =$ 

, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1,$ for any positive integers a_1, a_2. Thus,

$$|\mathcal{L}(S_1)| = \sum_{s_1=1}^{\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}} \sum_{s_2=1}^{\left(y_3' / \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = f_2 \left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \right),$$ and

For any positive integers a_1, a_2, if $S =$ 

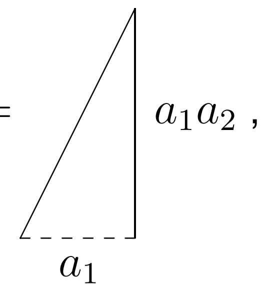
, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1,$ for any positive integers a_1, a_2. Thus,

$$|\mathcal{L}(S_1)| = \sum_{s_1=1}^{\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}} \sum_{s_2=1}^{\left(y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right), \text{ and}$$

$$|\mathcal{L}(S_2)| = \sum_{s_1=1}^{\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}} \sum_{s_2=1}^{\left(y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right).$$

For any positive integers a_1, a_2, if $S =$



, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1,$ for any positive integers a_1, a_2. Thus,

$$|\mathcal{L}(S_1)| = \sum_{s_1=1}^{\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}} \sum_{s_2=1}^{\left(y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right), \text{ and}$$

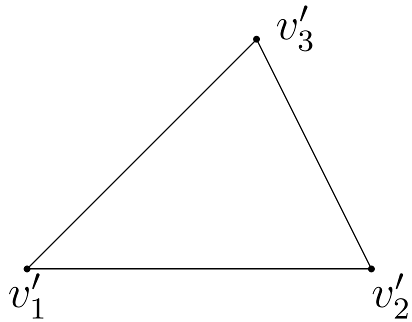
$$|\mathcal{L}(S_2)| = \sum_{s_1=1}^{\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}} \sum_{s_2=1}^{\left(y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right).$$

Hence, for case (i) $x_1 < x_2 < x_3$: $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = |\mathcal{L}(S_1)| - |\mathcal{L}(S_2)|$

$$= f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) - f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right).$$

(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

(ii) $x_1 < x_3 < x_2$:

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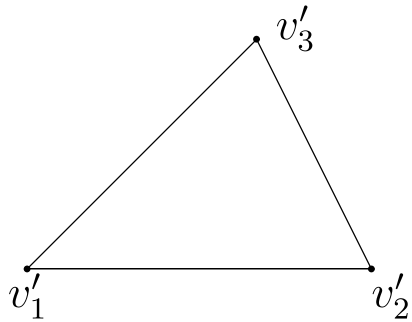
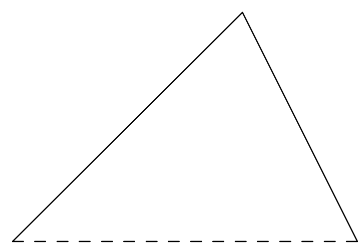


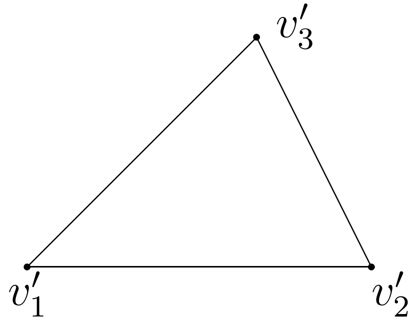
$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$



(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

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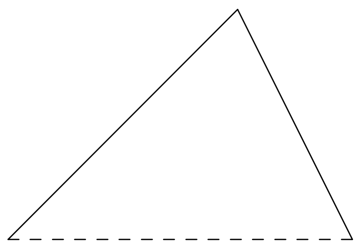

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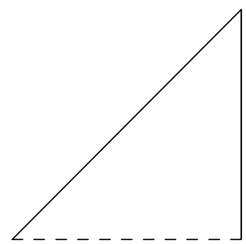

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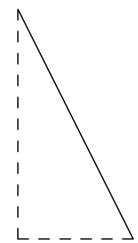
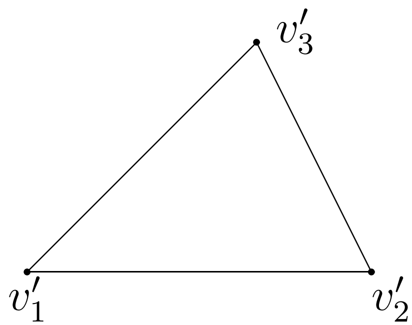


(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

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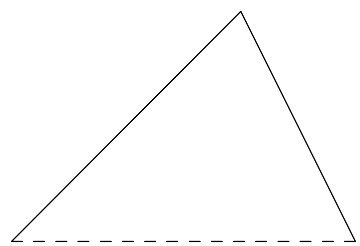

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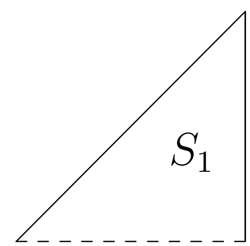

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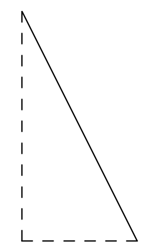
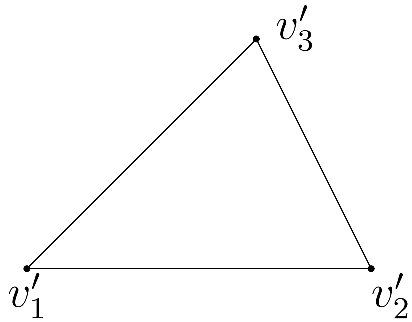


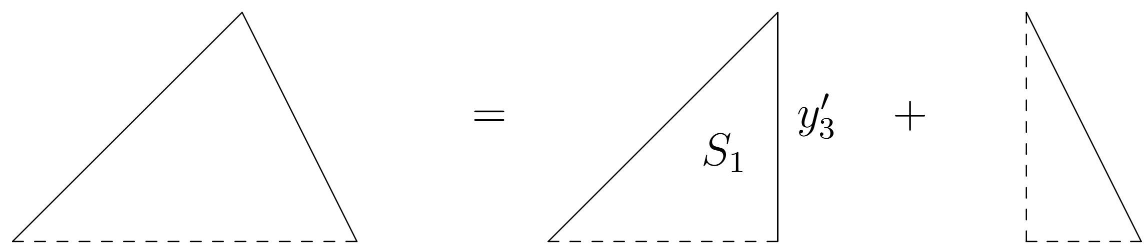
(ii) $x_1 < x_3 < x_2$:

$P' =$



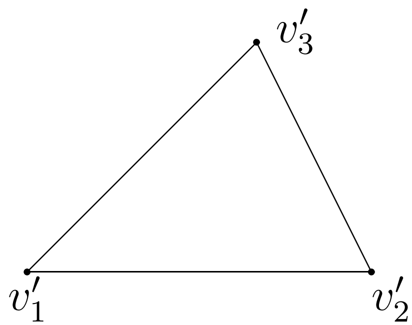
$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$



(ii) $x_1 < x_3 < x_2$:

$P' =$



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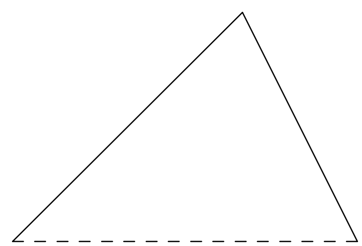
$\Omega(P') =$



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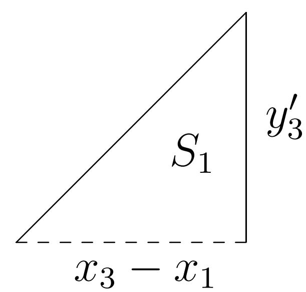

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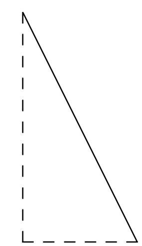
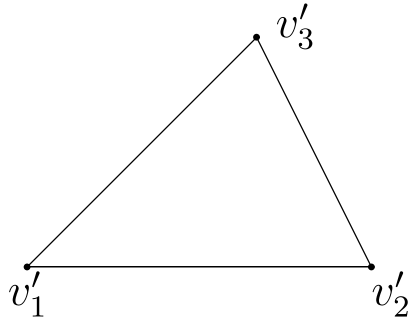


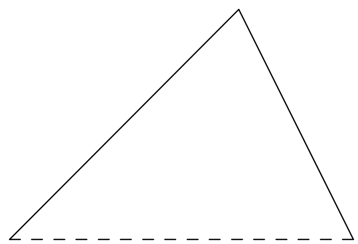
(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$\Omega(P') =$



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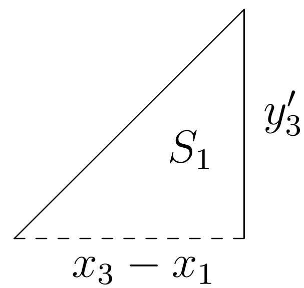

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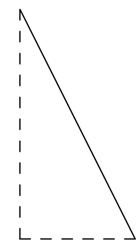

$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

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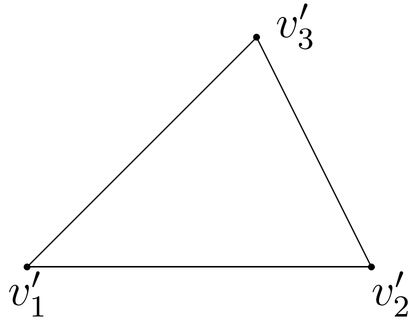

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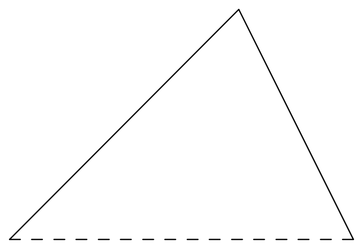

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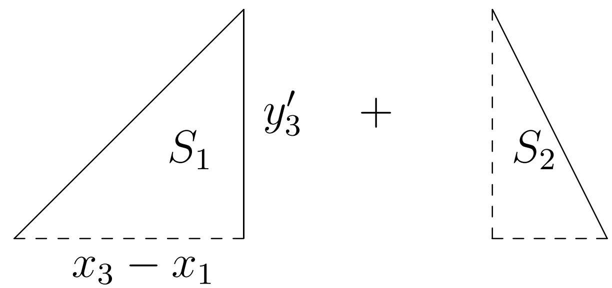

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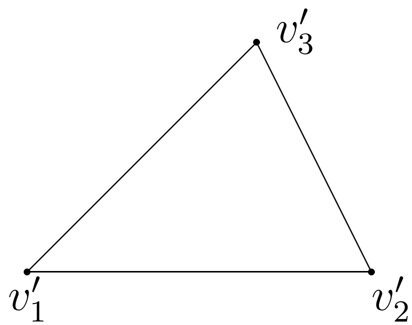
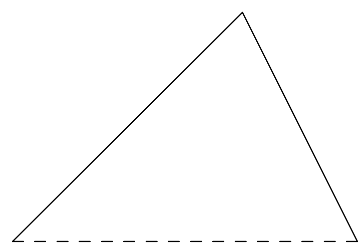
$P' =$



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$\Omega(P') =$



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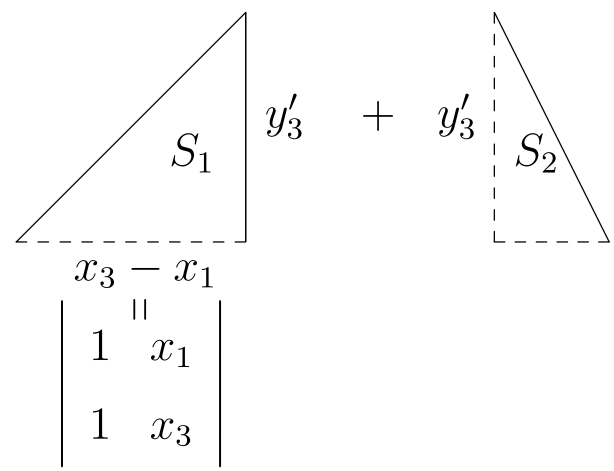
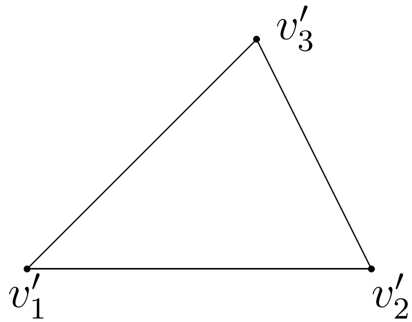


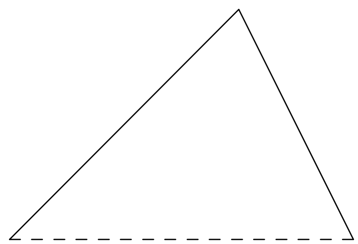
(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$



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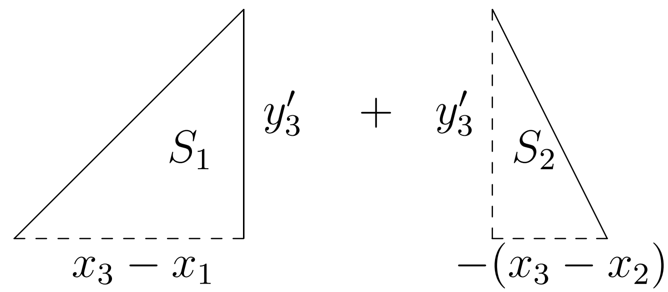
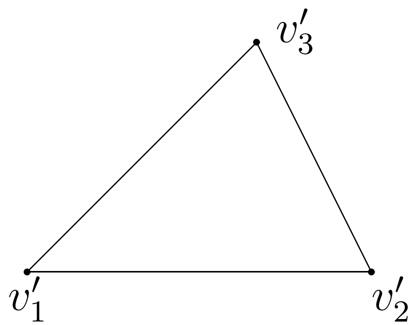


$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

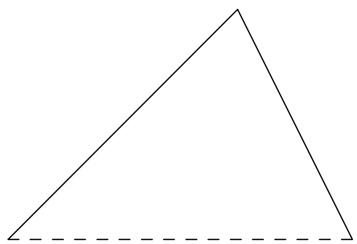
(ii) $x_1 < x_3 < x_2$:

$P' =$



$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$



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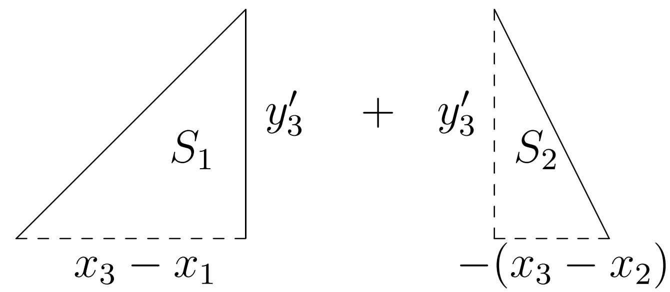
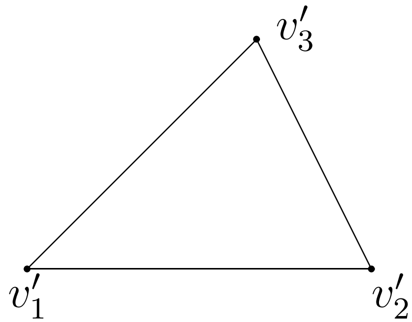
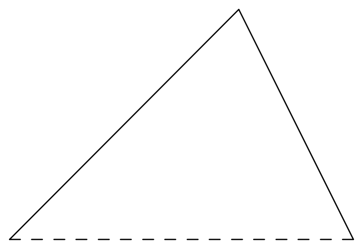
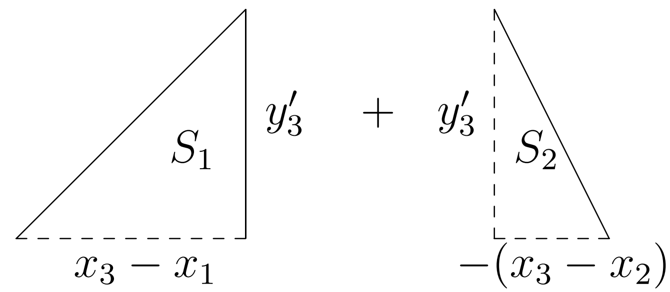


$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \qquad -(x_3 - x_2) = -\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

(ii) $x_1 < x_3 < x_2$:

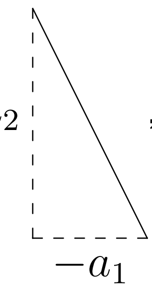
$P' =$ 

$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

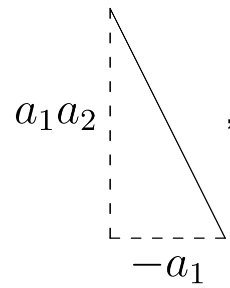
$\Omega(P') =$  $=$ 

$$x_3 - x_1 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, \qquad -(x_3 - x_2) = - \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

Clearly $|\mathcal{L}(S_1)| = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y'_3 / \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \right)$, but what is $|\mathcal{L}(S_2)|$?

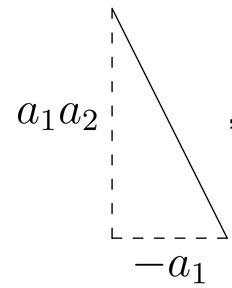
For any negative integers a_1, a_2, if $S = a_1 a_2$  ,

For any negative integers a_1, a_2, if $S =$



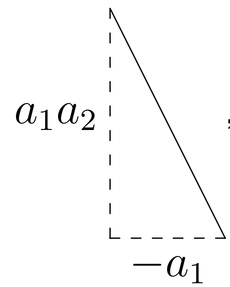
, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$

For any negative integers a_1, a_2, if $S =$



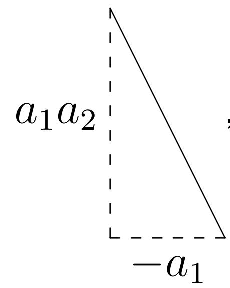
, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$

$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$

For any negative integers $a_1, a_2,$ if $S =$ 

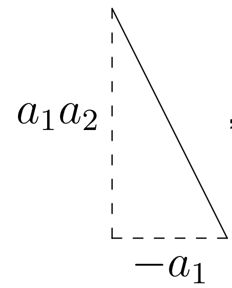
, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$$
$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$
$$= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1)$$

For any negative integers a_1, a_2, if $S =$ 

, then

$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \tfrac{1}{2}(-a_1 - 1)(-a_1) \\ &= -a_2 \tfrac{a_1}{2}(a_1 + 1) \end{aligned}$$

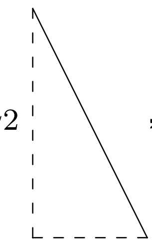
For any negative integers a_1, a_2, if $S =$ 

, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$$
$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$
$$= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1)$$
$$= -a_2 \frac{a_1}{2} (a_1 + 1)$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$

For any negative integers a_1, a_2, if $S =$



, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$$
$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$
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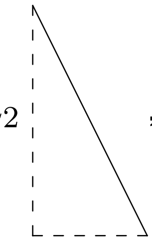
Recall that for any $a_1, a_2 \in \mathbb{N}$, $\quad f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 \; = \sum_{s_1}^{a_1} a_2 s_1$

For any negative integers a_1, a_2, if $S =$ 

, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$$
$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$
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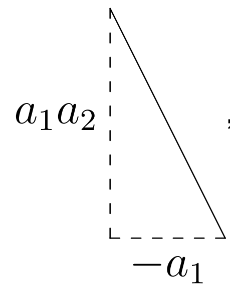
For any negative integers a_1, a_2, if $S =$



, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$

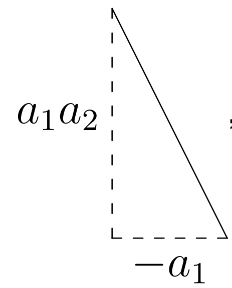
$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$

$$= -a_2 \frac{1}{2}(-a_1 - 1)(-a_1)$$

$$= -a_2 \frac{a_1}{2}(a_1 + 1)$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2}(a_1 + 1)$.

Because $a_2 \frac{a_1}{2}(a_1 + 1)$ is a polynomial in a_1, a_2, we can extend the domain of f_2 from $\mathbb{N}^2$ to $\mathbb{Z}^2$ or even $\mathbb{R}^2$.

For any negative integers a_1, a_2, if $S =$ 

, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 = \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) = -a_2 \frac{1}{2}(-a_1 - 1)(-a_1) = -a_2 \frac{a_1}{2}(a_1 + 1)$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2}(a_1 + 1)$.

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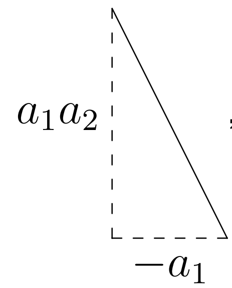
Then

$$|\mathcal{L}(S)| = -f_2(a_1, a_2).$$

For any negative integers a_1, a_2, if $S =$ 

, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$

$$= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1)$$

$$= -a_2 \tfrac{1}{2}(-a_1 - 1)(-a_1)$$

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Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2}(a_1 + 1)$.

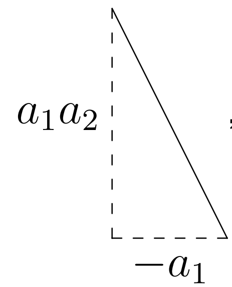
Because $a_2 \frac{a_1}{2}(a_1 + 1)$ is a polynomial in a_1, a_2, we can extend the domain of f_2 from $\mathbb{N}^2$ to $\mathbb{Z}^2$ or even $\mathbb{R}^2$.

Then

$$|\mathcal{L}(S)| = -f_2(a_1, a_2).$$

Thus,

$$|\mathcal{L}(S_2)| = \sum_{s_1=1}^{-\left|\begin{smallmatrix} 1 & x_2 \\ 1 & x_3 \end{smallmatrix}\right| - 1} \sum_{s_2=1}^{\left(-y_3' / \left|\begin{smallmatrix} 1 & x_2 \\ 1 & x_3 \end{smallmatrix}\right|\right) s_1} 1 = -f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \right).$$

For any negative integers a_1, a_2, if $S =$ 

, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 = \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) = -a_2 \frac{1}{2}(-a_1-1)(-a_1) = -a_2 \frac{a_1}{2}(a_1+1)$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $\quad f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2}(a_1+1)$.

Because $a_2 \frac{a_1}{2}(a_1+1)$ is a polynomial in a_1, a_2, we can extend the domain of f_2 from $\mathbb{N}^2$ to $\mathbb{Z}^2$ or even $\mathbb{R}^2$.

Then

$$|\mathcal{L}(S)| = -f_2(a_1, a_2).$$

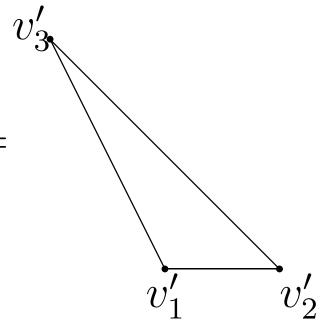
Thus,

$$|\mathcal{L}(S_2)| = \sum_{s_1=1}^{-\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} - 1} \sum_{s_2=1}^{\left(-y_3' / \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right) s_1} 1 = -f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \right).$$

Hence, for case (ii) $x_1 < x_3 < x_2$: $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = |\mathcal{L}(S_1)| + |\mathcal{L}(S_2)| = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \right) - f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \right).$

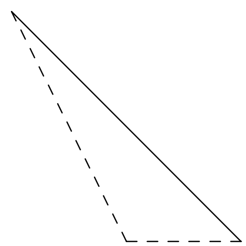
(iii) $x_3 < x_1 < x_2$:

$P' =$

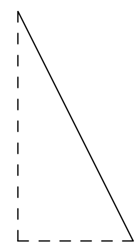


$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

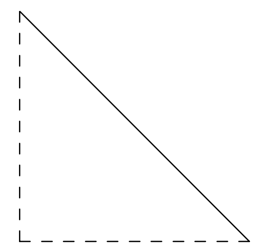
$\Omega(P') =$



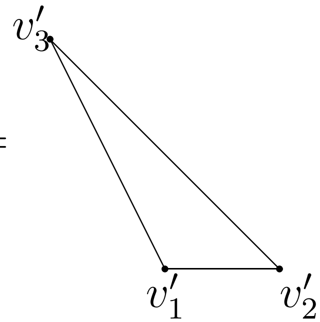
$= -$



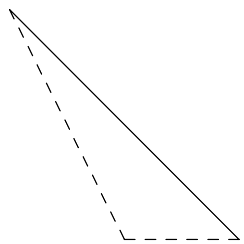
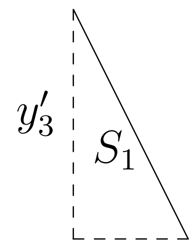
$+$

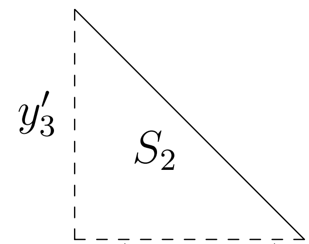


(iii) $x_3 < x_1 < x_2$:

$P' =$ 

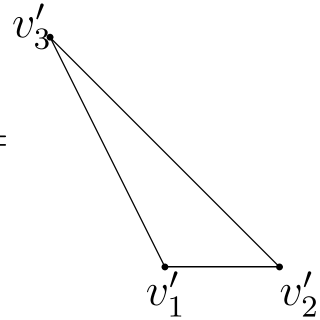
$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$  $= -$ 

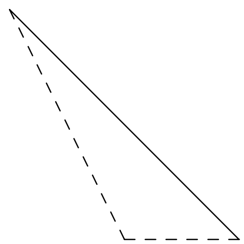
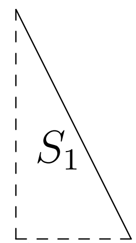
$+$ 

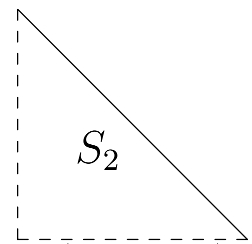
$$-(x_3 - x_1) = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \qquad -(x_3 - x_2) = -\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

(iii) $x_3 < x_1 < x_2$:

$P' =$ 

$$v_1' = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v_3' = \begin{pmatrix} x_3 \\ y_3' \end{pmatrix}$$

$\Omega(P') =$  $= - \; y_3'$ 

$+ \; y_3'$ 

$$-(x_3 - x_1) = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, \qquad -(x_3 - x_2) = -\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

As before, we have that $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = -|\mathcal{L}(S_1)| + |\mathcal{L}(S_2)|$

$$= f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) - f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3' / \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right).$$

Therefore, for any of the three cases,

$$|\mathcal{L}(\Omega(P))| = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3' \Big/ \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) - f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3' \Big/ \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right),$$

where $y_3' = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \Big/ \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}.$

Therefore, for any of the three cases,

$$|\mathcal{L}(\Omega(P))| = f_2\left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}\right) - f_2\left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y_3'/\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}\right),$$

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Recall that $f_2(a_1, a_2) = a_2 \frac{a_1}{2}(a_1 + 1)$, we can calculate that

$$|\mathcal{L}(\Omega(P))| = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \mathrm{Vol}(P).$$

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This completes the proof of Theorem 9 for dimension 2.

Further Discussion

We have an alternative definition of lattice-face polytopes, which is equivalent to the original definition we gave earlier. Indeed, a d-polytope on a vertex set V is a lattice-face polytope if and only if for all $k : 0 \leq k \leq d-1$,

$$(\star) \quad \text{for any } (k+1)\text{-subset } U \subset V, \quad \pi^{d-k}(\mathcal{L}(H_U)) = \mathbb{Z}^k,$$

where H_U is the affine space spanned by U. In other words, after dropping the last $d-k$ coordinates of the lattice of H_U, we get the k-dimensional lattice.

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Therefore, one may ask

Question: If P is a polytope that satisfies ($\star$) for all $k \in K$, where K is a fixed subset of $\{0, 1, \dots, d-1\}$, can we say something about the Ehrhart polynomial of P?

A conjecture

A special set K can be chosen as the set of consecutive integers from 0 to d', where d' is an integer no greater than $d-1$. Based on some examples in this case, the Ehrhart polynomials seems to follow a certain pattern, so we conjecture the following:

Conjecture 10. *Given $d' \leq d-1$, if P is a d-polytope with vertex set V such that $\forall k : 0 \leq k \leq d'$, ($\star$) is satisfied, then for $0 \leq k \leq d'$, the coefficient of m^k in $i(P,m)$ is the same as in $i(\pi^{d-d'}(P), m)$. In other words,*

$$i(P,m) = i(\pi^{d-d'}(P), m) + \sum_{i=d'+1}^{d} c_i m^i.$$

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Example: $P = \operatorname{conv}\{(0,0,0),(4,0,0),(3,6,0),(2,2,2)\}$. One can check that P satisfies ($\star$) for $k = 0, 1$ but not for $k = 2$.

$$i(P, m) = 8m^3 + 10m^2 + 4m + 1,$$

where $4m + 1$ is the Ehrhart polynomial of $\pi^2(P) = [0, 4]$.