On the Combinatorics of Crystal Graphs

Cristian Lenart

State University of New York at Albany

FPSAC, June 19-23, 2006

Paper: `arXiv:math.RT/0509200`
Related papers: `arXiv:math.RT/0309207, 0502147`
Also available at: `http://math.albany.edu/math/pers/lenart`

1. Background on root systems

V finite-dimensional, real vector space;
$\langle \cdot, \cdot \rangle$ nondegenerate symmetric bilinear form.

1. Background on root systems

V finite-dimensional, real vector space;
$\langle \cdot\,, \cdot \rangle$ nondegenerate symmetric bilinear form.

Reflection $s_\alpha \ : \ \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \,,$
where $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle \,.$

1. Background on root systems

V finite-dimensional, real vector space;
$\langle \cdot\,, \cdot \rangle$ nondegenerate symmetric bilinear form.

Reflection $s_\alpha \;:\; \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\,,$
where $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle\,.$

Root system for a semisimple Lie algebra $\mathfrak{g}$:
$\Phi \subset V$ finite, invariant under s_α, $\alpha \in \Phi$.

1. Background on root systems

V finite-dimensional, real vector space;
$\langle \cdot, \cdot \rangle$ nondegenerate symmetric bilinear form.

Reflection $s_\alpha \ : \ \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$,
where $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$.

Root system for a semisimple Lie algebra $\mathfrak{g}$:
$\Phi \subset V$ finite, invariant under s_α, $\alpha \in \Phi$.

Positive and negative roots: $\Phi = \Phi^+ \cup \Phi^-$.
Simple roots: $\alpha_1, \ldots, \alpha_r$; form a basis of V.
Simple reflections: $s_i := s_{\alpha_i}$.

Weyl group:

$$W = \langle s_\alpha \; : \; \alpha \in \Phi \rangle = \langle s_i \; : \; i = 1, \ldots, r \rangle \, .$$

Length: $\ell(w) = \min \{ k \; : \; w = s_{i_1} \ldots s_{i_k} \} \, .$

Weyl group:

$$W = \langle s_\alpha \; : \; \alpha \in \Phi \rangle = \langle s_i \; : \; i = 1, \ldots, r \rangle \, .$$

Length: $\ell(w) = \min \{ k \; : \; w = s_{i_1} \ldots s_{i_k} \}$.

Bruhat order: $w \lessdot w s_\alpha$ if $\ell(w s_\alpha) = \ell(w) + 1$.

Weyl group:

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle \, .$$

Length: $\ell(w) = \min \{ k \ : \ w = s_{i_1} \ldots s_{i_k} \}$.

Bruhat order: $w \lessdot w s_\alpha$ if $\ell(w s_\alpha) = \ell(w) + 1$.

Example. Type A_{n-1}.

$$V = (\mathbb{R}\varepsilon_1 \oplus \ldots \oplus \mathbb{R}\varepsilon_n) / \mathbb{R}(\varepsilon_1 + \ldots + \varepsilon_n) \, .$$
$$\Phi = \{ \alpha_{ij} = \varepsilon_i - \varepsilon_j \ : \ 1 \le i \ne j \le n \} = \Phi^\vee \, .$$
$$W = S_n \, , \quad s_{\alpha_{ij}} \text{ is the transposition } t_{ij} \, .$$

Weights and dominant weights

$$\Lambda = \{\lambda \in V \ : \ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\,,\ \alpha \in \Phi\}\,.$$

$$\Lambda^+ = \{\lambda \in \Lambda \ : \ \langle \lambda, \alpha^\vee \rangle \geq 0\,,\ \alpha \in \Phi^+\}\,.$$

Weights and dominant weights

$$\Lambda = \{\lambda \in V \,:\, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\,,\ \alpha \in \Phi\}\,.$$

$$\Lambda^+ = \{\lambda \in \Lambda \,:\, \langle \lambda, \alpha^\vee \rangle \geq 0\,,\ \alpha \in \Phi^+\}\,.$$

Example. Type A_{n-1}:

$$\Lambda = \left\{ \sum_{i=1}^{n-1} \lambda_i \varepsilon_i = (\lambda_1, \ldots, \lambda_{n-1}) \,:\, \lambda_i \in \mathbb{Z} \right\}\,,$$

$$\Lambda^+ = \{(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0) \,:\, \lambda_i \in \mathbb{Z}\}\,.$$

Weights and dominant weights

$$\Lambda = \{\lambda \in V \ : \ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\,, \ \alpha \in \Phi\}\,.$$

$$\Lambda^+ = \{\lambda \in \Lambda \ : \ \langle \lambda, \alpha^\vee \rangle \geq 0\,, \ \alpha \in \Phi^+\}\,.$$

Example. Type A_{n-1}:

$$\Lambda = \left\{ \sum_{i=1}^{n-1} \lambda_i \varepsilon_i = (\lambda_1, \ldots, \lambda_{n-1}) \ : \ \lambda_i \in \mathbb{Z} \right\}\,,$$
$$\Lambda^+ = \{(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0) \ : \ \lambda_i \in \mathbb{Z}\}\,.$$

Alcoves

Hyperplanes $H_{\alpha,k} = \{\lambda \ : \ \langle \lambda, \alpha^\vee \rangle = k\}$ $(k \in \mathbb{Z})$.
Reflection in $H_{\alpha,k}$ denoted by $s_{\alpha,k}$.

Weights and dominant weights

$$\Lambda = \{\lambda \in V \;:\; \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\,,\; \alpha \in \Phi\}\,.$$

$$\Lambda^+ = \{\lambda \in \Lambda \;:\; \langle \lambda, \alpha^\vee \rangle \geq 0\,,\; \alpha \in \Phi^+\}\,.$$

Example. Type A_{n-1}:

$$\Lambda = \left\{ \sum_{i=1}^{n-1} \lambda_i \varepsilon_i = (\lambda_1, \ldots, \lambda_{n-1}) \;:\; \lambda_i \in \mathbb{Z} \right\}\,,$$

$$\Lambda^+ = \{(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0) \;:\; \lambda_i \in \mathbb{Z}\}\,.$$

Alcoves

Hyperplanes $H_{\alpha,k} = \{\lambda \;:\; \langle \lambda, \alpha^\vee \rangle = k\}$ $(k \in \mathbb{Z})$.
Reflection in $H_{\alpha,k}$ denoted by $s_{\alpha,k}$.

Alcoves: connected components of $V \setminus (\bigcup H_{\alpha,k})$.
Fundamental alcove:

$$A_\circ = \{\lambda \in V \;:\; 0 < \langle \lambda, \alpha^\vee \rangle < 1 \;\text{ for }\; \alpha \in \Phi^+\}\,.$$

2. New combinatorial model for characters (joint with A. Postnikov)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ. For $\lambda \in \Lambda^+$, we have an irreducible representation V_λ. Its character can be written

$$ch(V_\lambda) = \sum_{\mu \in \Lambda} c_\mu e^\mu \quad (c_\mu \in \mathbb{Z}_{\geq 0}),$$

2. New combinatorial model for characters (joint with A. Postnikov)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ. For $\lambda \in \Lambda^+$, we have an irreducible representation V_λ. Its character can be written

$$ch(V_\lambda) = \sum_{\mu \in \Lambda} c_\mu e^\mu \quad (c_\mu \in \mathbb{Z}_{\geq 0}) \,,$$

Given $\lambda \in \Lambda^+$, let

$$(A_\circ = A_0, A_1, \ldots, A_l = A_\circ - \lambda)$$

be a shortest sequence of adjacent alcoves.

2. New combinatorial model for characters (joint with A. Postnikov)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ. For $\lambda \in \Lambda^+$, we have an irreducible representation V_λ. Its character can be written

$$ch(V_\lambda) = \sum_{\mu \in \Lambda} c_\mu e^\mu \quad (c_\mu \in \mathbb{Z}_{\geq 0}),$$

Given $\lambda \in \Lambda^+$, let

$$(A_\circ = A_0, A_1, \ldots, A_l = A_\circ - \lambda)$$

be a shortest sequence of adjacent alcoves.

Let $F_i \subset H_{\beta_i,k_i}$: common wall of A_{i-1} and A_i.
Let $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i,k_i}$.

2. New combinatorial model for characters (joint with A. Postnikov)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ. For $\lambda \in \Lambda^+$, we have an irreducible representation V_λ. Its character can be written

$$ch(V_\lambda) = \sum_{\mu \in \Lambda} c_\mu e^\mu \quad (c_\mu \in \mathbb{Z}_{\geq 0}),$$

Given $\lambda \in \Lambda^+$, let

$$(A_\circ = A_0, A_1, \ldots, A_l = A_\circ - \lambda)$$

be a shortest sequence of adjacent alcoves.

Let $F_i \subset H_{\beta_i, k_i}$: common wall of A_{i-1} and A_i.
Let $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i, k_i}$.

λ-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_l)$.

Admissible subsets $\mathcal{A}(\lambda) = \mathcal{A}(\lambda, \Gamma)$:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I = \{1, \ldots, l\} \quad \text{such that}$$

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s} =: \kappa(J) \;\; \text{(key)}\,.$$

Admissible subsets $\mathcal{A}(\lambda) = \mathcal{A}(\lambda, \Gamma)$:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I = \{1, \ldots, l\} \quad \text{such that}$$

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s} =: \kappa(J) \;\; (\text{key})\,.$$

Weight of an admissible subset:

$$\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(-\lambda)\,.$$

Admissible subsets $\mathcal{A}(\lambda) = \mathcal{A}(\lambda, \Gamma)$:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I = \{1, \ldots, l\} \quad \text{such that}$$

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} \ldots r_{j_s} =: \kappa(J) \quad \text{(key)}\,.$$

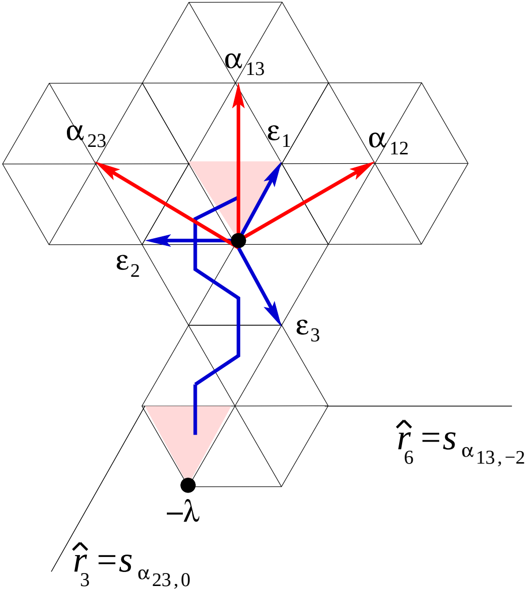
Weight of an admissible subset:

$$\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(-\lambda)\,.$$

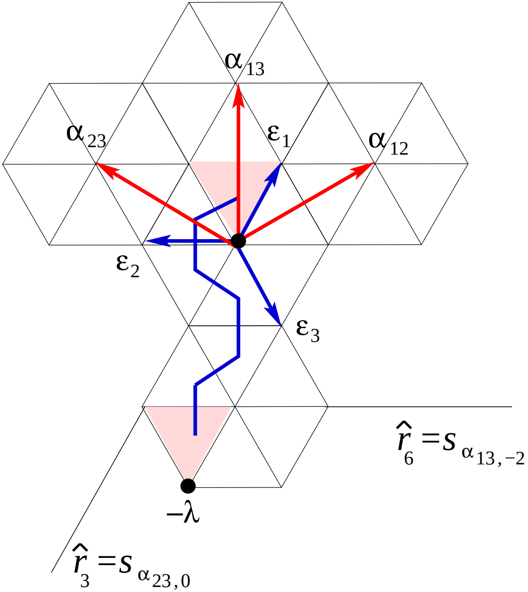
Theorem. (L. and Postnikov) *The irreducible character $ch(V_\lambda)$ of $\mathfrak{g}$ can be expressed as*

$$ch(V_\lambda) = \sum_{J \in \mathcal{A}(\lambda)} e^{\mu(J)}\,.$$

Example. Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$.

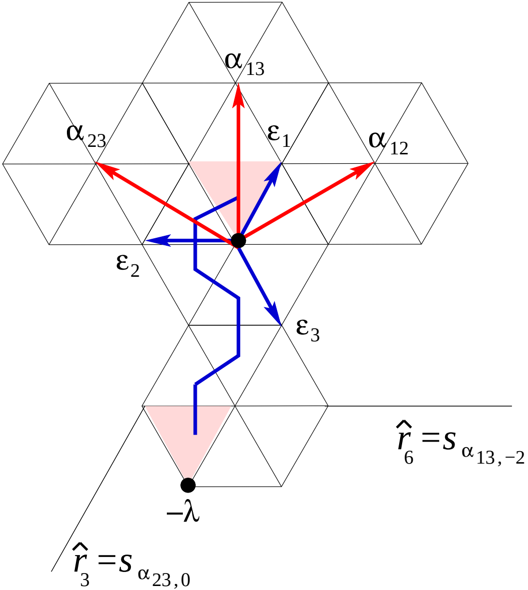


Example. Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$.



$J = \{3, 6\}$, saturated chain $e = 123 < t_{23} = 132 < t_{23}t_{13} = 231$.

Example. Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$.



$J = \{3, 6\}$, saturated chain $e = 123 < t_{23} = 132 < t_{23}t_{13} = 231$.

$J = \{6\}$ forbidden: $e < t_{13} = 321$.

3. Poset structure of $\mathcal{A}(\lambda)$

It is given by the crystal graph structure of the corresponding canonical basis. Minimum $J_{\min} = \emptyset$ and maximum $J_{\max}$.

3. Poset structure of $\mathcal{A}(\lambda)$

It is given by the crystal graph structure of the corresponding canonical basis. Minimum $J_{\text{min}} = \emptyset$ and maximum J_{max}.

Fact: $\mathcal{A}(\lambda)$ is a self-dual poset, i.e. there is a bijection $\eta : \mathcal{A}(\lambda) \to \mathcal{A}(\lambda)$ such that

$$J \leq J' \iff \eta(J) \geq \eta(J') .$$

In particular, $\eta : J_{\text{min}} \leftrightarrow J_{\text{max}}$.

3. Poset structure of $\mathcal{A}(\lambda)$

It is given by the crystal graph structure of the corresponding canonical basis. Minimum $J_{\min} = \emptyset$ and maximum $J_{\max}$.

Fact: $\mathcal{A}(\lambda)$ is a self-dual poset, i.e. there is a bijection $\eta : \mathcal{A}(\lambda) \to \mathcal{A}(\lambda)$ such that

$$J \leq J' \iff \eta(J) \geq \eta(J') \,.$$

In particular, $\eta : J_{\min} \leftrightarrow J_{\max}$.

The map η is given by the action of $w_\circ$ (longest element of W) on the canonical basis, so

$$\mu(\eta(J)) = w_\circ(\mu(J)) \,.$$

3. Poset structure of $\mathcal{A}(\lambda)$

It is given by the crystal graph structure of the corresponding canonical basis. Minimum $J_{\min} = \emptyset$ and maximum $J_{\max}$.

Fact: $\mathcal{A}(\lambda)$ is a self-dual poset, i.e. there is a bijection $\eta : \mathcal{A}(\lambda) \to \mathcal{A}(\lambda)$ such that

$$J \leq J' \iff \eta(J) \geq \eta(J') .$$

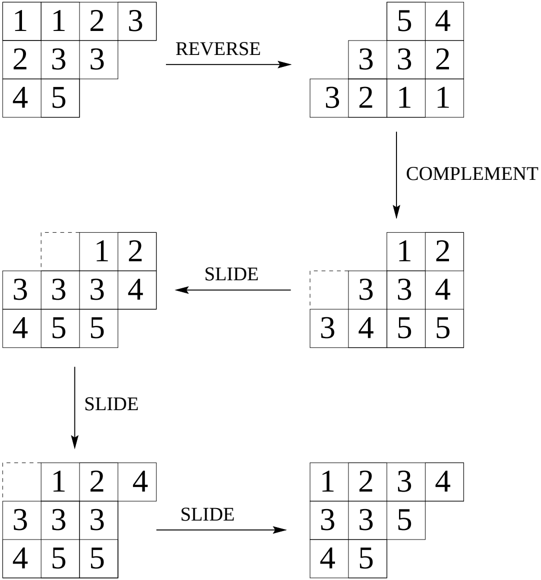
In particular, $\eta : J_{\min} \leftrightarrow J_{\max}$.

The map η is given by the action of $w_\circ$ (longest element of W) on the canonical basis, so

$$\mu(\eta(J)) = w_\circ(\mu(J)) .$$

Goal: describe η explicitly.
In type A, it is given by Schützenberger's evacuation on semistandard Young tableaux (Berenstein and Zelevinsky).

Schützenberger's evacuation



4. Generalizing Schützenberger's evacuation

Assume that λ is regular, for simplicity (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$).

4. Generalizing Schützenberger's evacuation

Assume that λ is regular, for simplicity (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$).

Consider the λ-chain

$$\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}, \beta_1, \ldots, \beta_l)\,,$$

where $\{\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}\} = \Phi^+\,.$

4. Generalizing Schützenberger's evacuation

Assume that λ is regular, for simplicity (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$).

Consider the λ-chain

$$\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}, \beta_1, \ldots, \beta_l)\,,$$

where $\{\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}\} = \Phi^+$.

Fact:

$$\Gamma^{\mathrm{rev}} := (\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}, \beta_l, \beta_{l-1}, \ldots, \beta_1)$$

is also a λ-chain.

4. Generalizing Schützenberger's evacuation

Assume that λ is regular, for simplicity (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$).

Consider the λ-chain

$$\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}, \beta_1, \ldots, \beta_l)\,,$$

where $\{\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}\} = \Phi^+$.

Fact:

$$\Gamma^{\mathrm{rev}} := (\beta_{\overline{1}}, \ldots, \beta_{\overline{m}}, \beta_l, \beta_{l-1}, \ldots, \beta_1)$$

is also a λ-chain.

STEP 1 (REVERSE-COMPLEMENT)
Define a bijection

$$J \in \mathcal{A}(\lambda, \Gamma) \;\mapsto\; J^{\mathrm{rev}} \in \mathcal{A}(\lambda, \Gamma^{\mathrm{rev}})\,,$$

such that

$$\mu(J^{\mathrm{rev}}) = w_\circ(\mu(J))\,.$$

Example.

Type A_2, $\lambda = 4\varepsilon_1 + 2\varepsilon_2$, $J = \{2, 4\}$,

$$\begin{array}{cccccccc} \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (\alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}) \end{array}$$

Example.

Type A_2, $\lambda = 4\varepsilon_1 + 2\varepsilon_2$, $J = \{2, 4\}$,

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (& \alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12},} & \alpha_{13}, & \underline{\alpha_{23},} & \alpha_{13}) \end{array}$$

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\text{rev}} = (& \underline{\alpha_{12},} & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{23},} & \alpha_{13}, & \underline{\alpha_{12},} & \alpha_{13}) \end{array}$$

$w_\circ \kappa(J) = (t_{12} t_{23} t_{12})(t_{12} t_{23}) = t_{12}$.

Example.

Type A_2, $\lambda = 4\varepsilon_1 + 2\varepsilon_2$, $J = \{2, 4\}$,

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (& \alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}) \end{array}$$

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\mathrm{rev}} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}) \end{array}$$

$w_\circ \kappa(J) = (t_{12} t_{23} t_{12})(t_{12} t_{23}) = t_{12}$.

$J^{\mathrm{rev}} = \{\overline{1}, 2, 4\}$

STEP 2 (SLIDE)

Yang-Baxter moves. Let Γ, Γ' be λ-chains related as follows:

$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \;\mapsto$$
$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l)\,,$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \overline{\Phi}^+$ of rank 2.

STEP 2 (SLIDE)

Yang-Baxter moves. Let Γ, Γ' be λ-chains related as follows:

$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto$$
$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l)\,,$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \overline{\Phi}^+$ of rank 2.

Theorem. (L.) *There is a bijection*

$$J \in \mathcal{A}(\lambda, \Gamma) \overset{YB}{\mapsto} J' \in \mathcal{A}(\lambda, \Gamma')$$

$$\textit{such that} \quad J \setminus [i,j] = J' \setminus [i,j]\,, \quad \kappa(J) = \kappa(J')\,, \quad \mu(J) = \mu(J')\,.$$

STEP 2 (SLIDE)

Yang-Baxter moves. Let Γ, Γ' be λ-chains related as follows:

$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto$$
$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l),$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \overline{\Phi}^+$ of rank 2.

Theorem. (L.) *There is a bijection*

$$J \in \mathcal{A}(\lambda, \Gamma) \stackrel{YB}{\mapsto} J' \in \mathcal{A}(\lambda, \Gamma')$$

such that $J \setminus [i,j] = J' \setminus [i,j]$, $\kappa(J) = \kappa(J')$, $\mu(J) = \mu(J')$.

Let $\Gamma^{\mathrm{rev}} = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma$ be related as above. We have

$$J \in \mathcal{A}(\lambda, \Gamma) \mapsto J^{\mathrm{rev}} = J_1 \in \mathcal{A}(\lambda, \Gamma_1) \stackrel{YB}{\mapsto}$$
$$\stackrel{YB}{\mapsto} J_2 \in \mathcal{A}(\lambda, \Gamma_2) \stackrel{YB}{\mapsto} \ldots \stackrel{YB}{\mapsto} J_k = J^* \in \mathcal{A}(\lambda, \Gamma).$$

STEP 2 (SLIDE)

Yang-Baxter moves. Let Γ, Γ' be λ-chains related as follows:

$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto$$
$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l),$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \overline{\Phi}^+$ of rank 2.

Theorem. (L.) *There is a bijection*

$$J \in \mathcal{A}(\lambda, \Gamma) \overset{YB}{\mapsto} J' \in \mathcal{A}(\lambda, \Gamma')$$

$$\textit{such that} \quad J \setminus [i,j] = J' \setminus [i,j], \quad \kappa(J) = \kappa(J'), \quad \mu(J) = \mu(J').$$

Let $\Gamma^{\mathrm{rev}} = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma$ be related as above. We have

$$J \in \mathcal{A}(\lambda, \Gamma) \mapsto J^{\mathrm{rev}} = J_1 \in \mathcal{A}(\lambda, \Gamma_1) \overset{YB}{\mapsto}$$
$$\overset{YB}{\mapsto} J_2 \in \mathcal{A}(\lambda, \Gamma_2) \overset{YB}{\mapsto} \ldots \overset{YB}{\mapsto} J_k = J^* \in \mathcal{A}(\lambda, \Gamma).$$

Theorem. (L.) *We have* $J^* = \eta(J)$.

Example.

$J^{\mathrm{rev}} = \{\overline{1}, 2, 4\}$

$$\Gamma^{\mathrm{rev}} = \begin{array}{cccccccc} \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ (\underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) \end{array}$$

$$\Gamma = \begin{array}{cccccccc} \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ (\ \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \end{array}$$

Example.

$J^{\mathrm{rev}} = \{\overline{1}, 2, 4\}$

$$\begin{array}{ccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\mathrm{rev}} = & (\underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) \end{array}$$

$$\begin{array}{ccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = & (\ \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \end{array}$$

$v = t_{12}, \quad \kappa(J^{\mathrm{rev}}) = vt_{23}t_{12} = 321 = vt_{13}t_{23}$

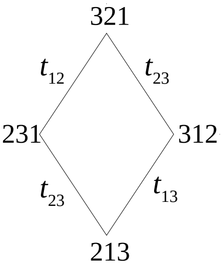
Example.

$J^{\mathrm{rev}} = \{\bar{1}, 2, 4\}$

$$\begin{array}{rcccccccc} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\mathrm{rev}} = & (\underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) \end{array}$$

$$\begin{array}{rcccccccc} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = & (\ \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \end{array}$$

$v = t_{12}, \quad \kappa(J^{\mathrm{rev}}) = vt_{23}t_{12} = 321 = vt_{13}t_{23}$



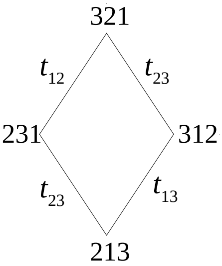
Example.

$J^{\mathrm{rev}} = \{\bar{1}, 2, 4\}$

$$\begin{array}{rcccccccc} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\mathrm{rev}} = & (\underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) \end{array}$$

$$\begin{array}{rcccccccc} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = & (\ \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \end{array}$$

$v = t_{12}, \quad \kappa(J^{\mathrm{rev}}) = vt_{23}t_{12} = 321 = vt_{13}t_{23}$



$J = \{2, 4\} \mapsto J^{*} = \{\bar{1}, 3, 4\}$

5. Related results

1. Chevalley-type multiplication formulas in $K_T(G/B)$;
   symmetries of coefficients in these formulas;

5. Related results

1. Chevalley-type multiplication formulas in $K_T(G/B)$; symmetries of coefficients in these formulas;
2. generalization of the model to complex symmetrizable Kac-Moody algebras, i.e., infinite root systems (character formula, Littlewood-Richardson rule, and crystal graph structure);

(1) and (2) are joint with A. Postnikov.

5. Related results

1. Chevalley-type multiplication formulas in $K_T(G/B)$; symmetries of coefficients in these formulas;
2. generalization of the model to complex symmetrizable Kac-Moody algebras, i.e., infinite root systems (character formula, Littlewood-Richardson rule, and crystal graph structure);
3. combinatorial construction of the commutator in the category of crystals, i.e., the isomorphism between the crystals $A \otimes B$ and $B \otimes A$.

(1) and (2) are joint with A. Postnikov.