Bounds on the number of inference functions of a graphical model

Sergi Elizalde

Dartmouth College

Kevin Woods

UC Berkeley

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Overview

- Background
  - Sequence alignment
  - Hidden Markov model
  - Graphical models
  - Inference functions
- Upper bound on the number of inference functions

  Sketch of the proof:
  - From inference functions to vertices of a polytope
  - The number of vertices of a Minkowski sum of polytopes
- Lower bound on the number of inference functions
- Application to sequence alignment

Sequence alignment

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4	$\longleftarrow$	$z = \#$matches	$\longrightarrow$	5
4	$\longleftarrow$	$x = \#$mismatches	$\longrightarrow$	1
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$$4 \longleftarrow z = \#\text{matches} \longrightarrow 5$$
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$$1 \longleftarrow y = \#\text{insertions} \longrightarrow 3$$

One possible model: matches are rewarded by 1, mismatches are penalized by α, and insertions are penalized by β.

$$4 - 4\alpha - \beta \longleftarrow \text{score} := z - x\alpha - y\beta \longrightarrow 5 - \alpha - 3\beta$$

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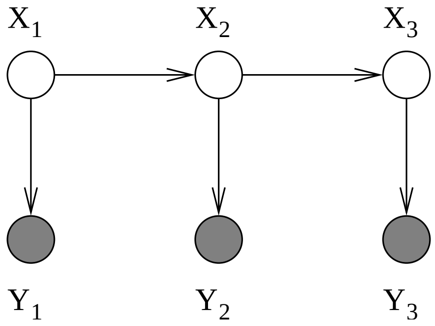

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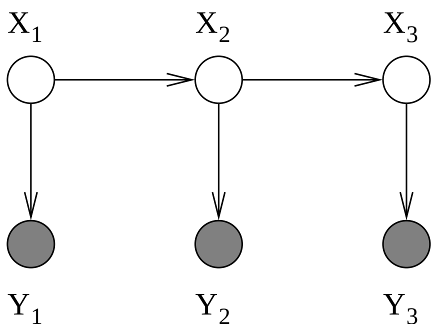

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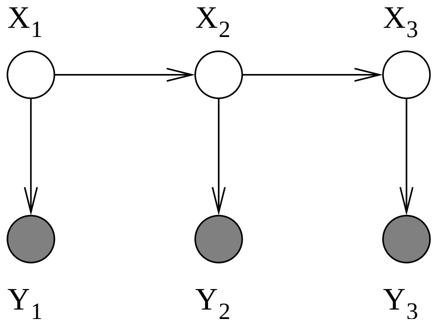

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} 12 parameters

This model can be used to determine what parts of the genome are introns and what parts are exons.

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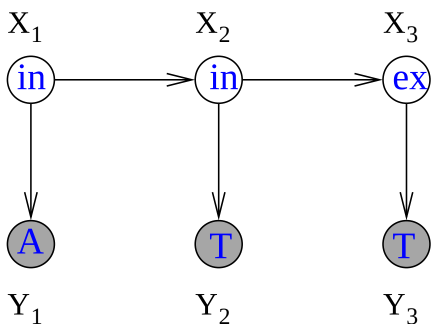

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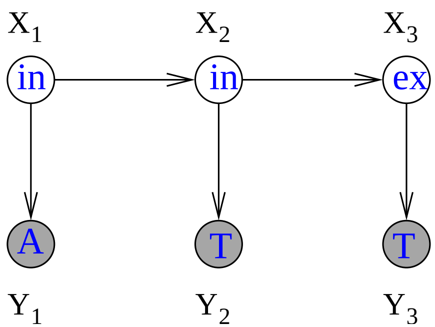

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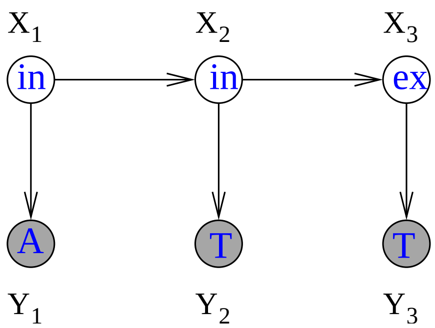

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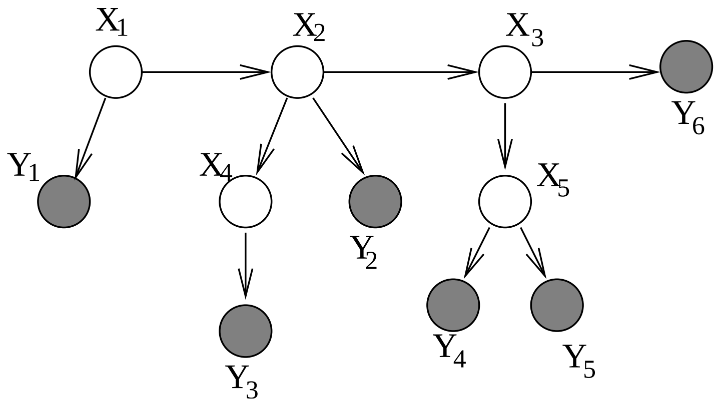
Given an observation (e.g. ATT), one wants to find the most likely values of X_1, X_2, X_3.

More generally:



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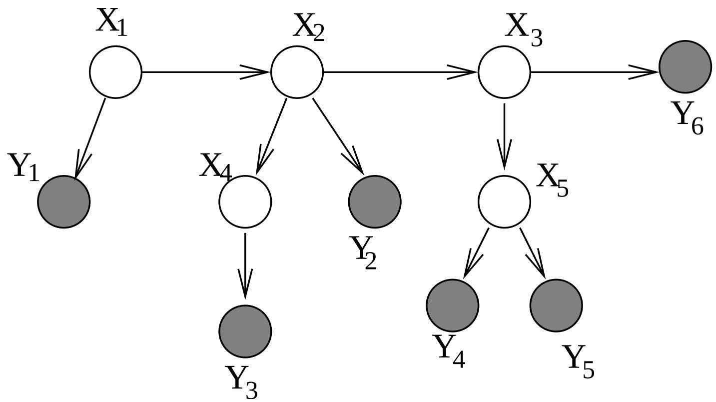

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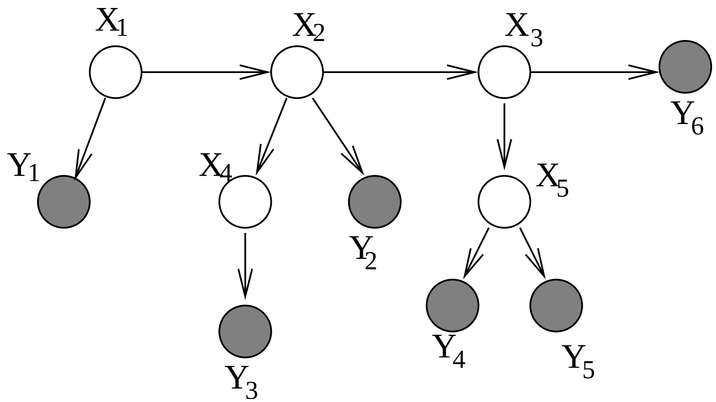

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For each observation $\tau \in (\Sigma')^n$,

$$f_\tau = \mathrm{Prob}(\mathbf{Y} = \tau) = \sum_{\mathbf{h} \in \Sigma^q} \underbrace{\mathrm{Prob}(\mathbf{X} = \mathbf{h}, \mathbf{Y} = \tau)}_{\text{monomial in } \theta_1, \theta_2, \ldots, \theta_d}$$

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In the previous example, an inference function is a map $\{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{T}\}^n \longrightarrow \{\mathrm{in}, \mathrm{ex}\}^n$. These are called *gene finding functions*.

The number of inference functions

In general, if $l = |\Sigma|$, $l' = |\Sigma'|$, there are in total $l^{q(l')^n}$ functions

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Main Theorem. Fix $d > 0$. Consider a graphical model with d parameters, and let E be the number of edges of the underlying graph. Then, the number of inference functions of the model is at most

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Usually, E is a linear in n, so the number of inference functions is in fact $O(n^{d(d-1)})$.

Geometric interpretation of the inference problem

Definition. The *Newton polytope* of

$$f_{\tau}(\theta_1, \theta_2, \ldots, \theta_d) = \sum_i \theta_1^{a_{1,i}} \theta_2^{a_{2,i}} \cdots \theta_d^{a_{d,i}},$$

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Example.

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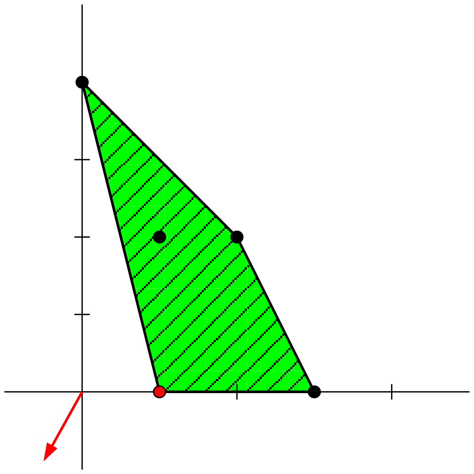

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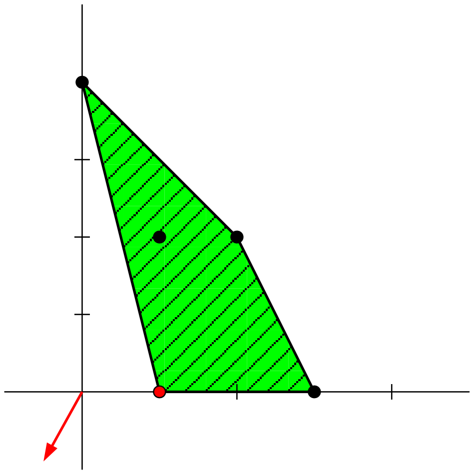

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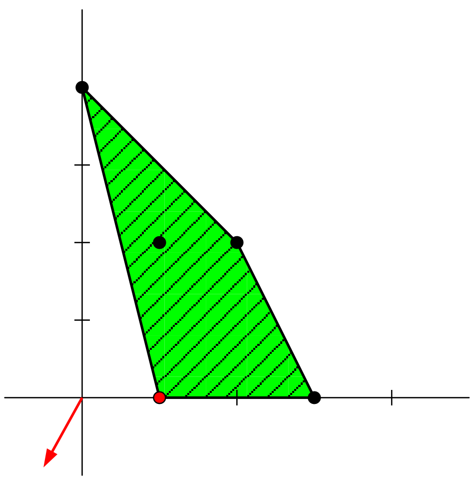
$$\theta_1 = 0.6, \quad \theta_2 = 0.4,$$
$$\mathbf{v} = (\log\theta_1, \log\theta_2) = (-0.51, -0.92)$$



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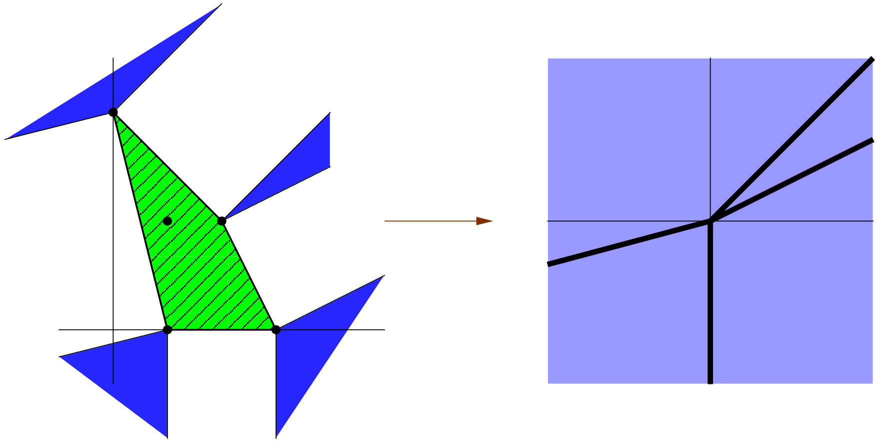

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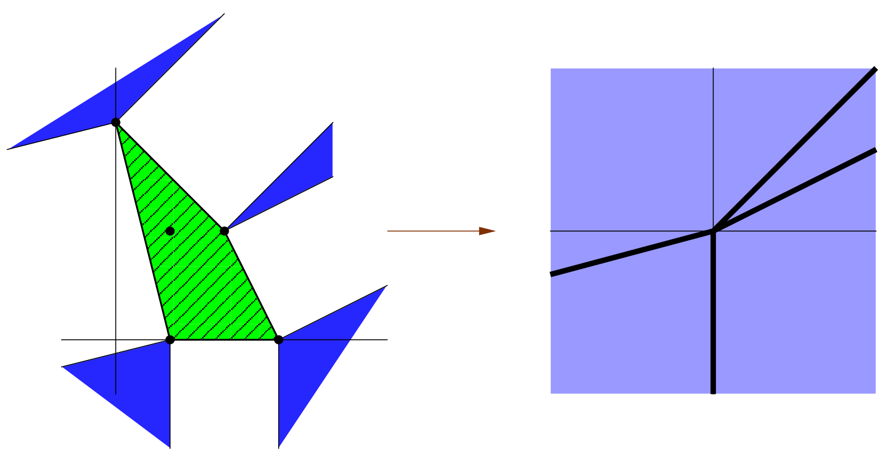

$\Phi_{\mathbf{v}}(\tau) = \Phi_{\mathbf{v}'}(\tau)$ iff $\mathbf{v}$ and $\mathbf{v}'$ belong to the same cone of $\mathcal{F}(\mathrm{NP}(f_\tau))$.

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The number of inference functions equals the **number of cones in the common refinement of fans**.

Minkowski sum of polytopes

The common refinement of the normal fans is the normal fan of the Minkowski sum of polytopes:

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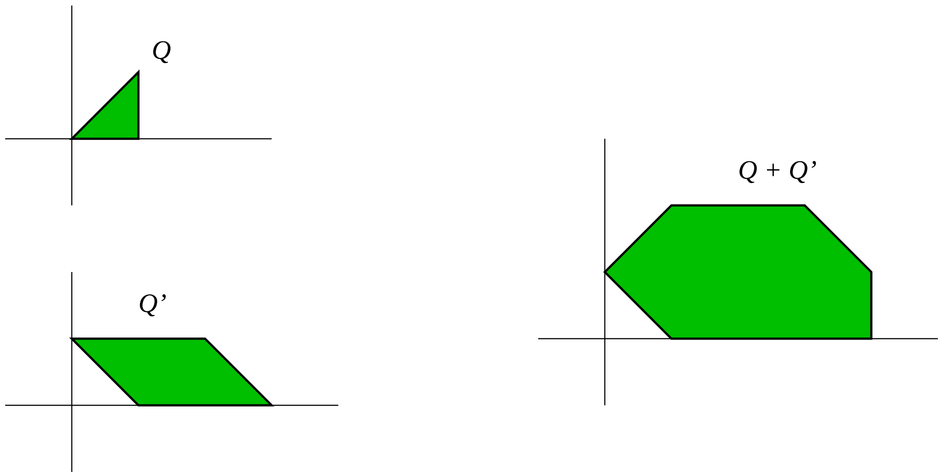

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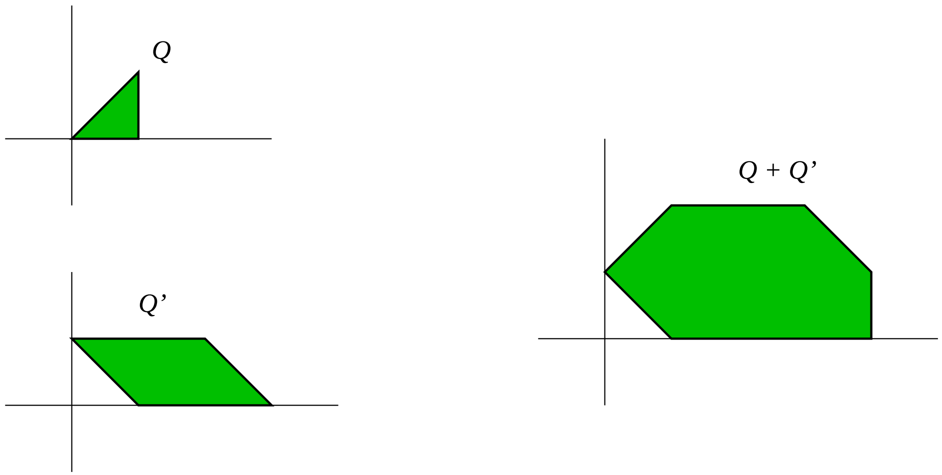

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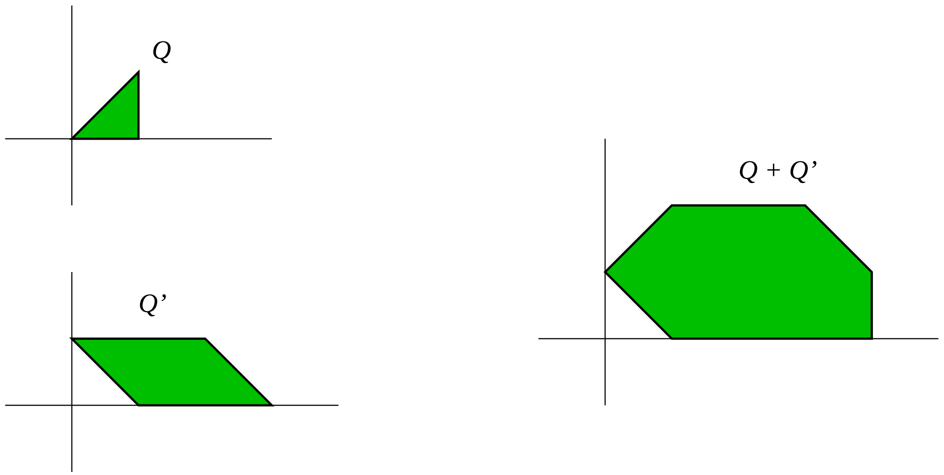

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The number of inference functions equals the **number of vertices of** P.

Edges of $\mathrm{NP}(f_\tau)$

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- Besides, $\mathrm{NP}(f_\tau)$ is a lattice polytope (i.e., its vertices have integral coordinates).

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since each θ_i can appear as a factor of a monomial of f_τ at most E times.

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The number of inference functions is $O(E^{d(d-1)})$. $\square$

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Idea of the proof:

- Construct an HMM $\mathcal{M}_n$ of length n with d parameters s.t. for any $a = (a_1, \ldots, a_d) \in \mathbb{Z}_+^d$ with $\sum_i a_i < n$, there is an observed sequence which has one explanation if
  $a_1 v_1 + \cdots + a_d v_d > 0$ and another explanation if
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- Show that the hyperplane arrangement consisting of the hyperplanes of the form $\{x : \langle a, x \rangle = 0\}$ with $a \in \mathbb{Z}_+^d$ and $\sum_i a_i < n$ has at least $\Omega(n^{d(d-1)})$ chambers.

- So, $\mathcal{M}_n$ has $\Omega(n^{d(d-1)}) = \Omega(E^{d(d-1)})$ distinct inference functions.

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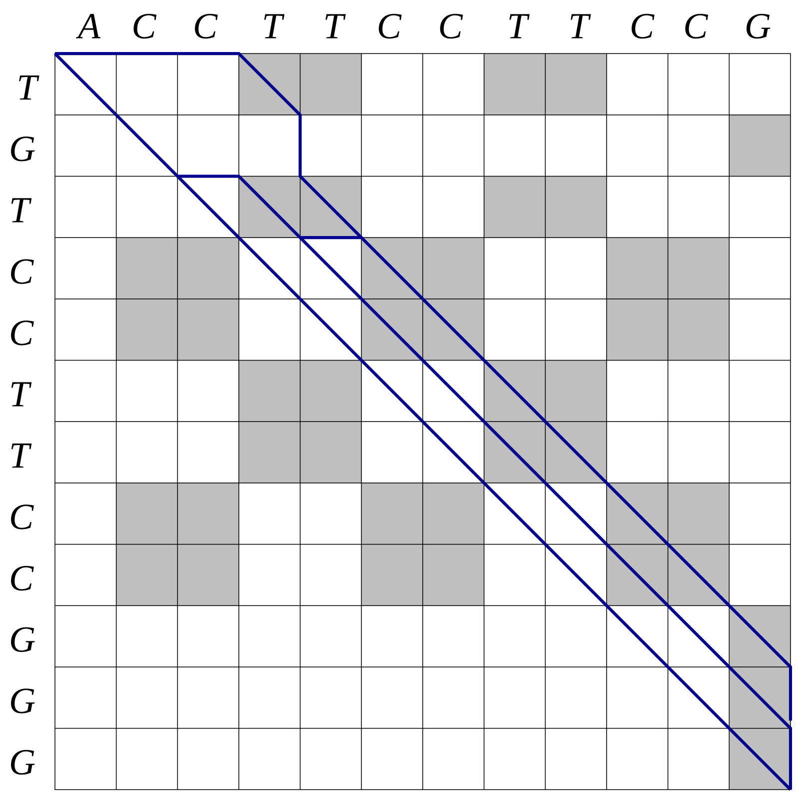
By the Main Theorem, the number of inference functions of this model is

$$O(n^{d(d-1)}) = O(n^2).$$

In fact, it is $\Theta(n^2)$.

Alignment graphs

- Alignments can be represented as paths from the upper-left to the lower-right corner in the alignment graph:



x	y	z

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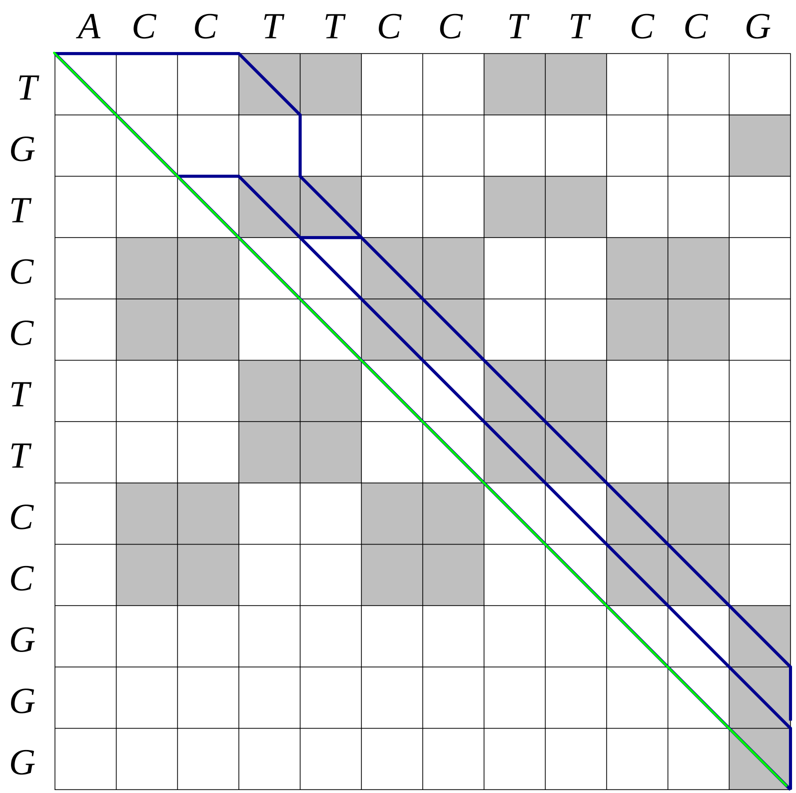

x	y	z
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TGTCCTTCCGGG
ACCTTCCTTCCG
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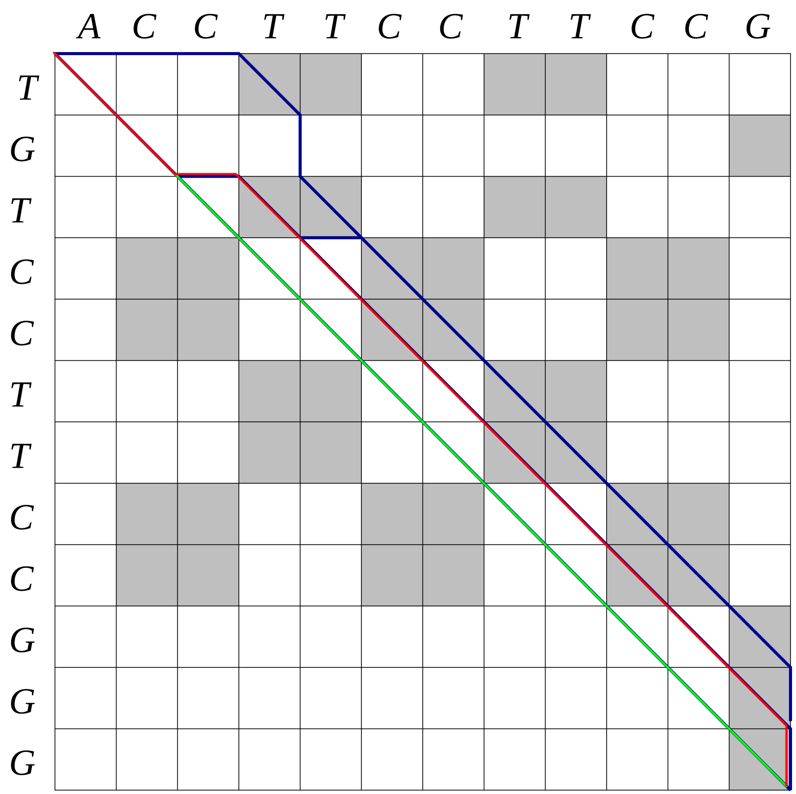

x	y	z
11	0	1
6	1	5

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TGTCCTTCCGGG   TG-TCCTTCCGGG
ACCTTCCTTCCG   ACCTTCCTTCCG-
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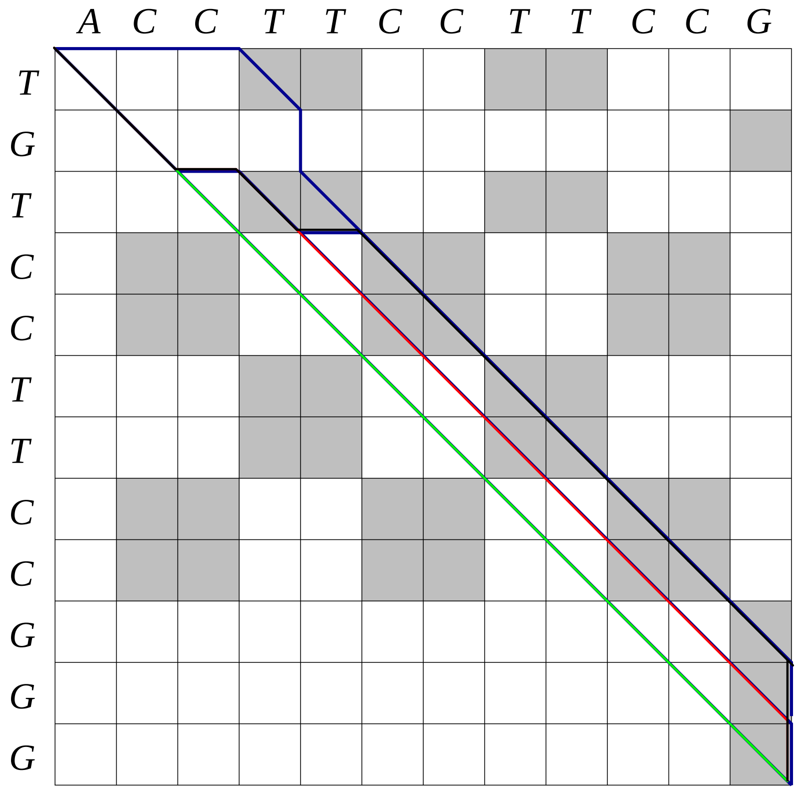

x	y	z
11	0	1
6	1	5
2	2	8

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TGTCCTTCCGGG    TG-TCCTTCCGGG    TG-T-CCTTCCGGG
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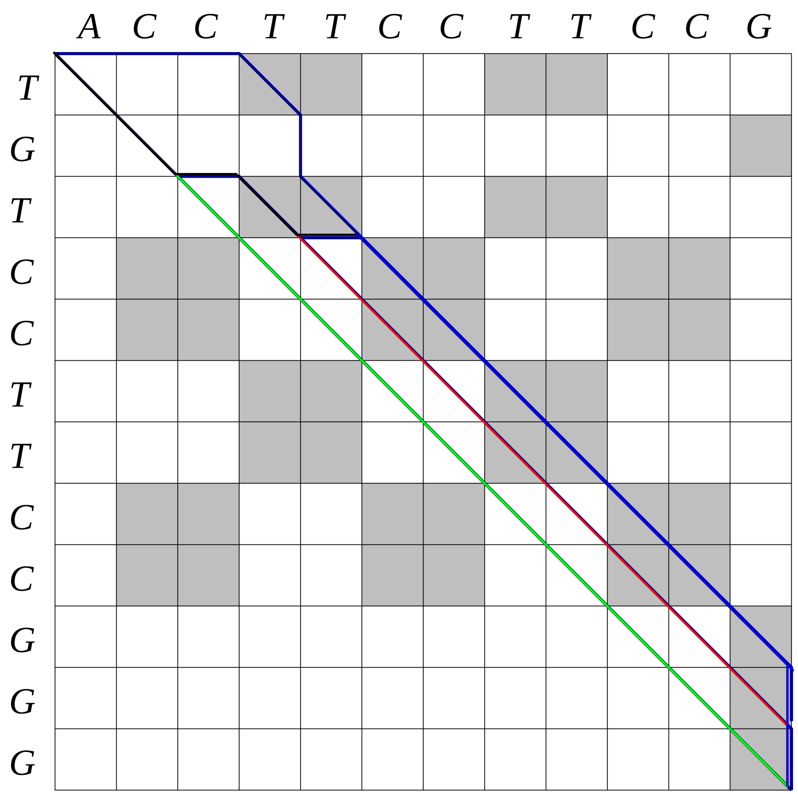
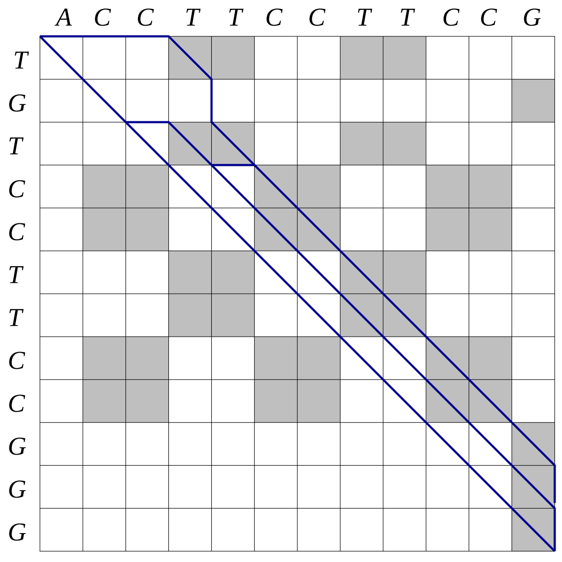


x	y	z
11	0	1
6	1	5
2	2	8
0	3	9
⋮	⋮	⋮

```
TGTCCTTCCGGG     TG-TCCTTCCGGG     TG-T-CCTTCCGGG     ---TGTCCTTCCGGG
ACCTTCCTTCCG     ACCTTCCTTCCG-     ACCTTCCTTCCG--     ACCT-TCCTTCCG--
```

The Newton polytope of a pair of sequences



x	y	z
11	0	1
6	1	5
2	2	8
0	3	9
11	1	0
0	12	0

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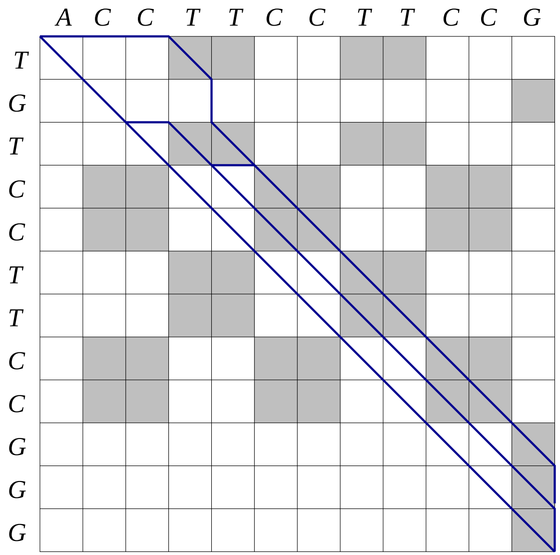

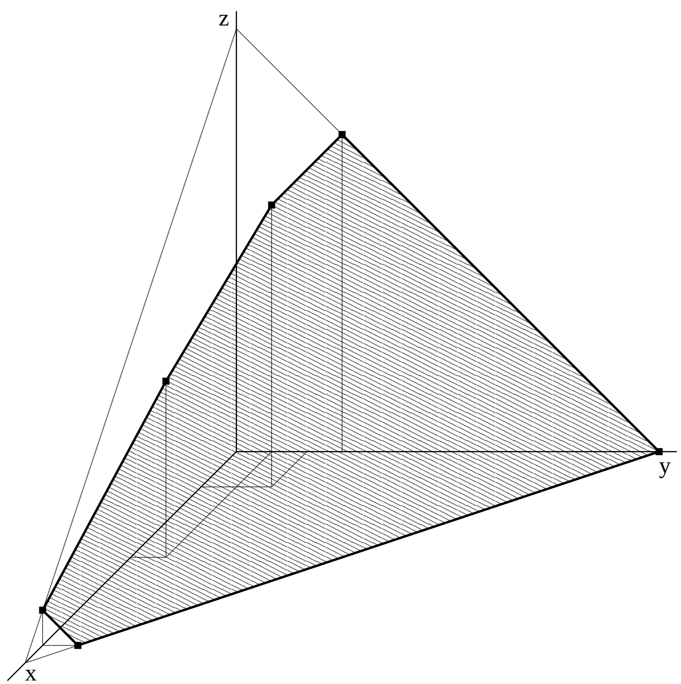

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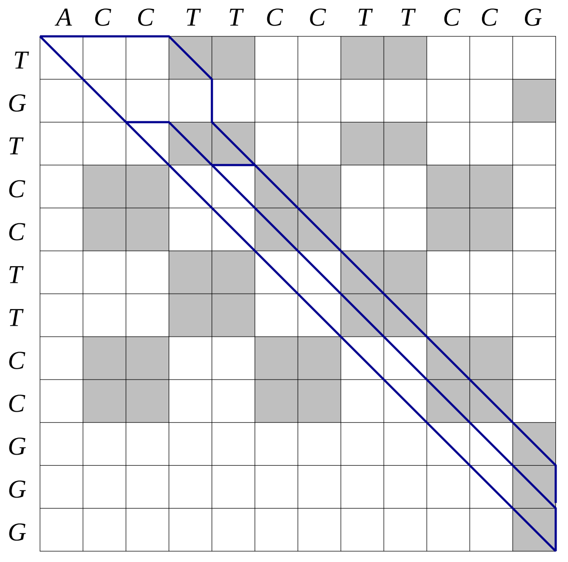

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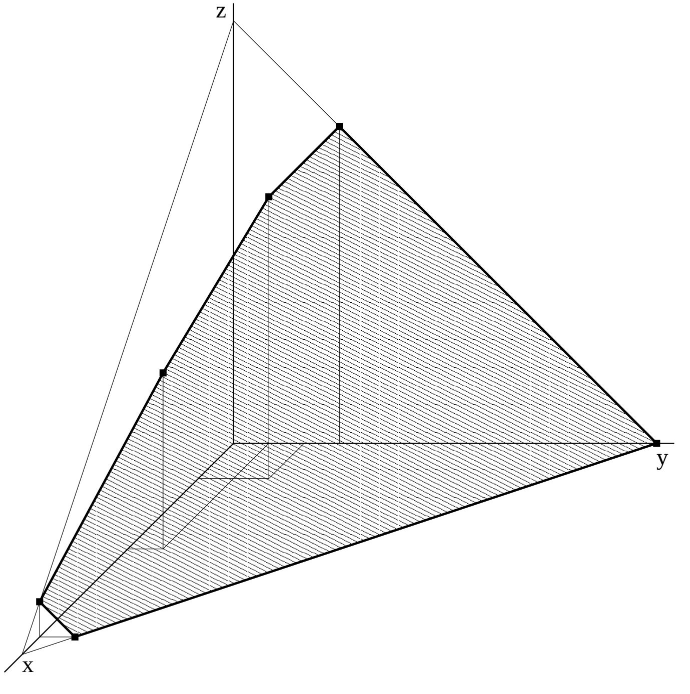

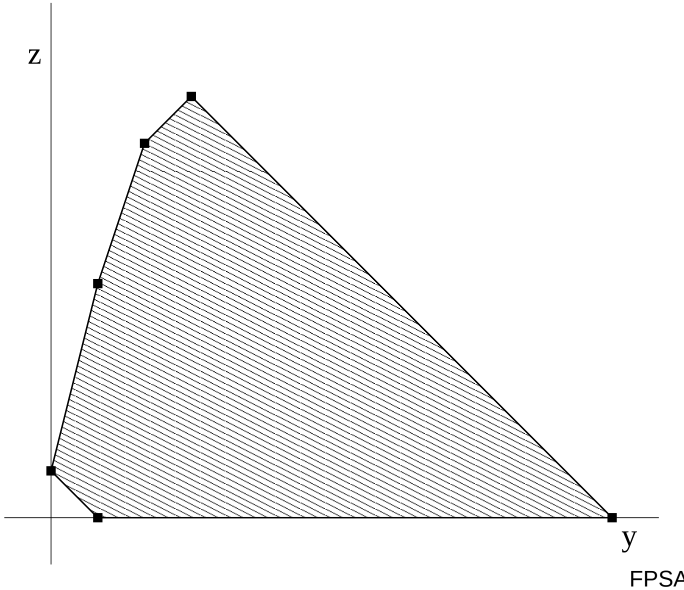

Open problem

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$$P = \bigoplus_{\tau} \mathrm{NP}(f_\tau) \text{ has } \Theta(n^2) \text{ vertices } (= \# \text{ inference functions}).$$

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Conjecture (?): It is $O(\sqrt{n})$ for sequences on any finite alphabet.