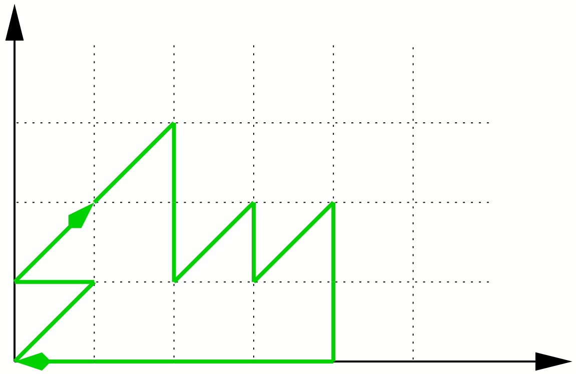
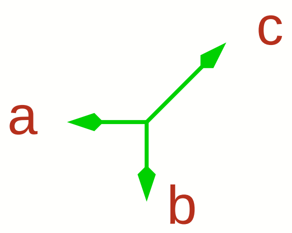
Kreweras Walks and Loopless Triangulations

Olivier Bernardi - LaBRI, Bordeaux

FPSAC 2006, San Diego

Kreweras walks





Walks made of $West$, $South$ and $North - East$ steps, starting and ending at the origin and confined in the first quadrant.

Preliminary remarks

Kreweras walks are words w on $\{a, b, c\}$ such that

- $|w|_a = |w|_b = |w|_c$,
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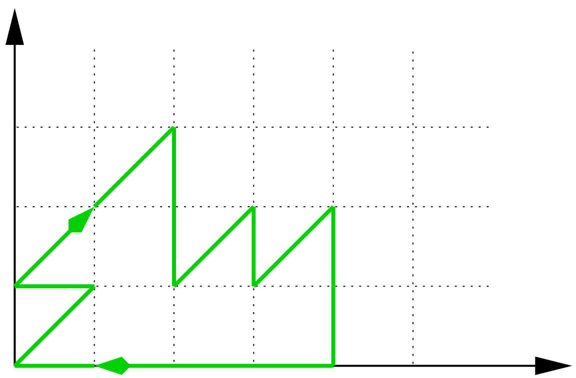

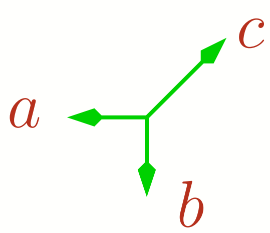

$$w = caccbbcbcbbaaaa$$

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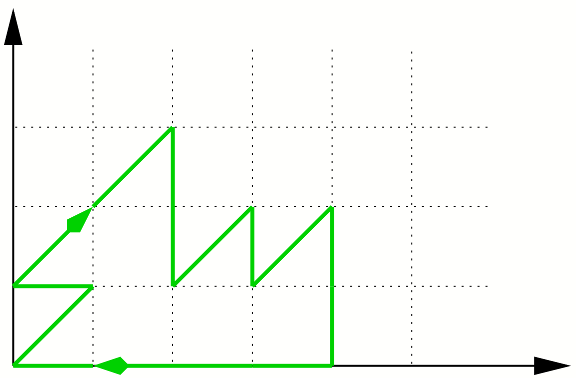

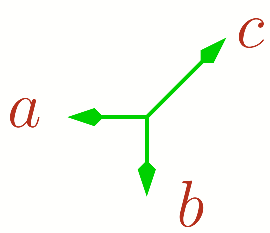

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Kreweras walks

Theorem (Kreweras 65): The number of Kreweras walks of size n ($3n$ steps) is

$$k_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

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[Kreweras 65, Niederhausen 82, 83, Gessel 86, Bousquet-Mélou 05]

Kreweras walks and cubic maps

- Cubic maps and depth-trees.
- Bijection:

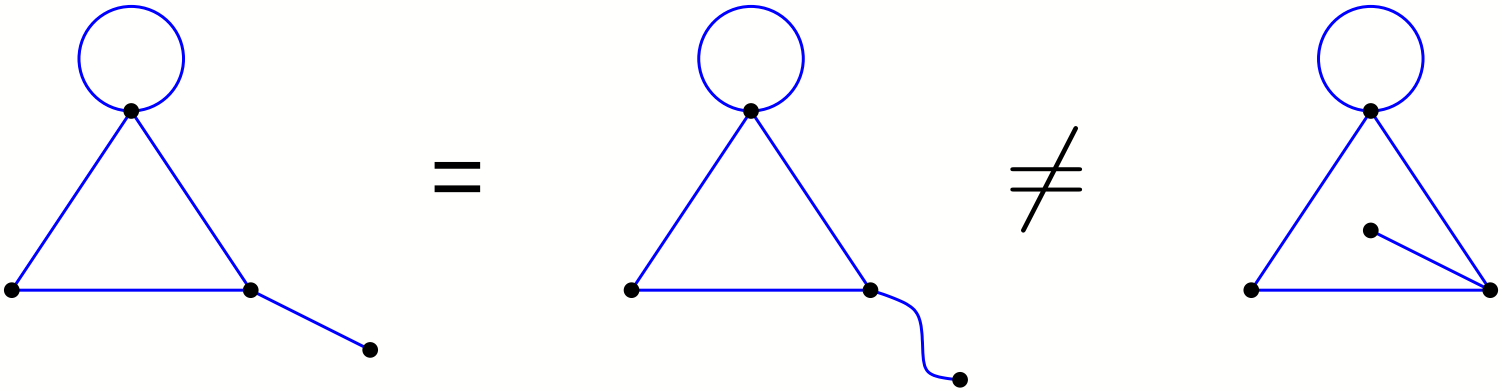
  Kreweras walk $\Longleftrightarrow$ Cubic map + Depth-tree.

- Counting Kreweras walks and cubic maps.
- Open problems.

Cubic maps and depth-trees

Maps

A map is a connected planar graph properly embedded in the sphere.
The map is considered up to deformation.



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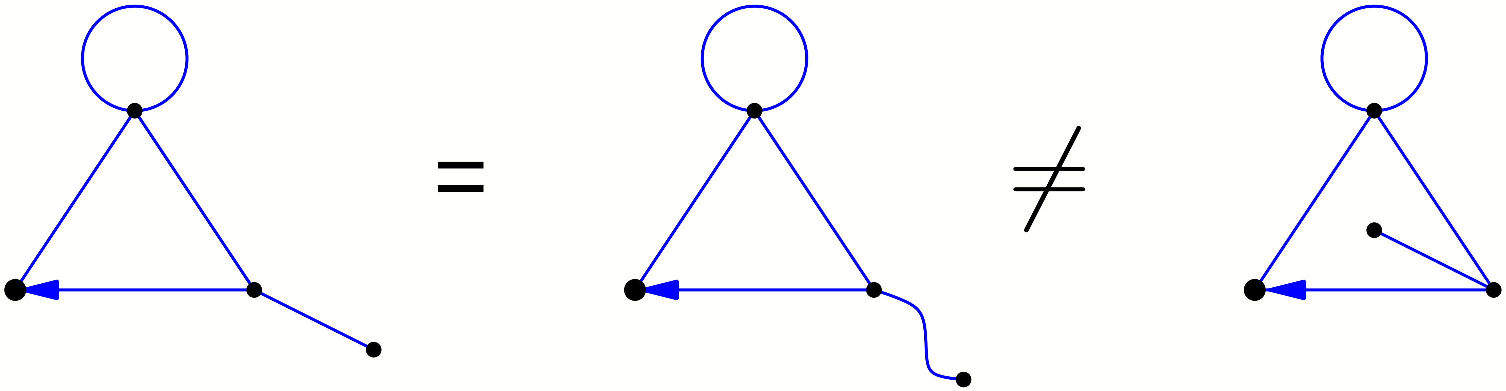
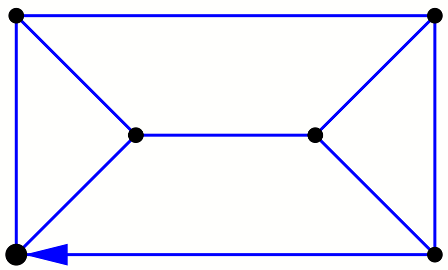


A map is rooted if a half-edge is distinguished as the root.

Cubic maps

A map is cubic if every vertex has degree 3.



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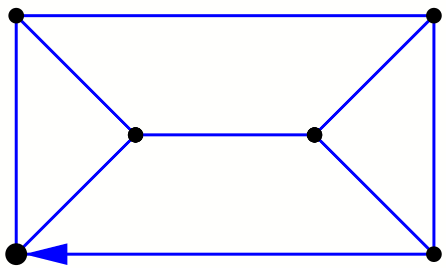
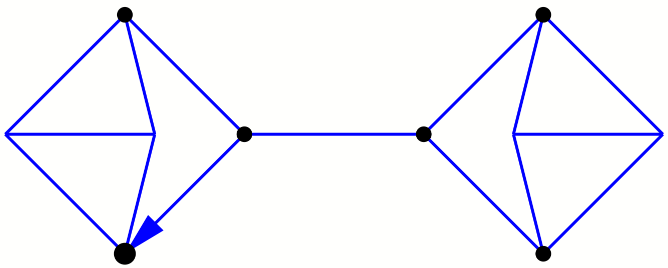


We focus on cubic maps *without isthmus*.



Cubic maps and triangulations

Cubic maps without isthmus are the dual of loopless *triangulations*.

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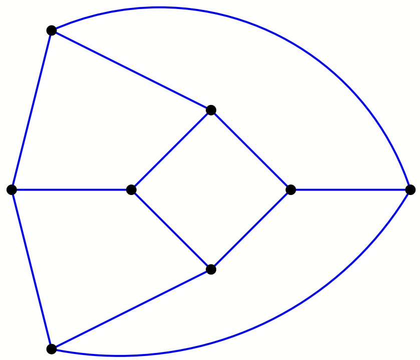

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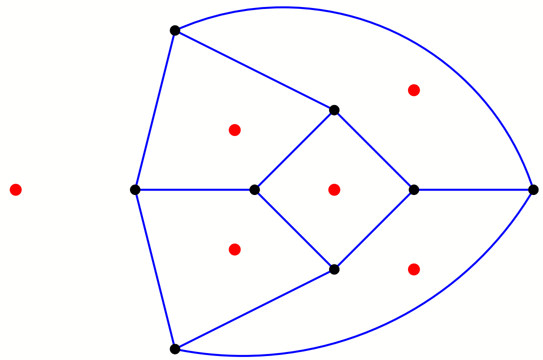

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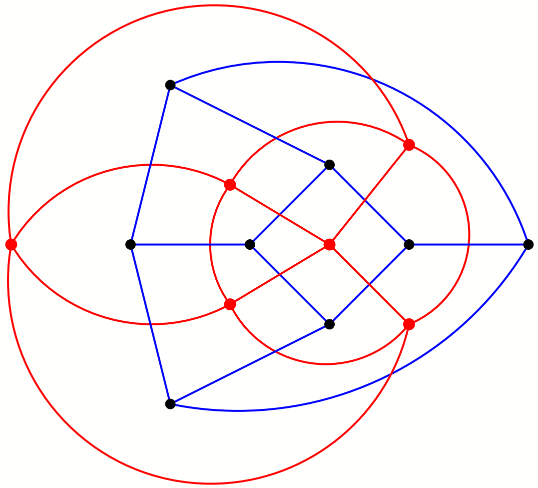

Cubic maps - counting result

Remark: The number of edges of a cubic map is always a multiple of 3.

A cubic map of size n has $3n$ edges, $2n$ vertices and $n+2$ faces.

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Theorem [Mullin 65, Poulalhon & Schaeffer 03]:
The number of cubic maps without isthmus of size n is

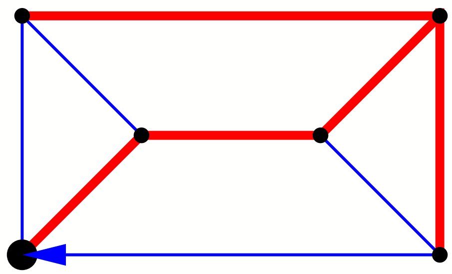
$$c_n = \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n} = \frac{k_n}{2^n}.$$

Depth-trees

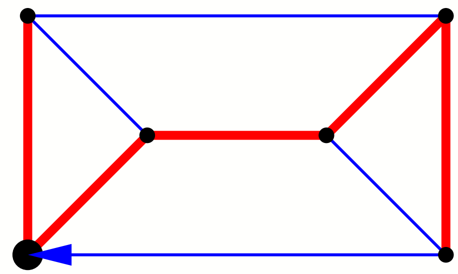
We consider *spanning trees* of rooted maps.

Depth-trees

A spanning tree of a rooted map is a depth-tree if every external edge links a vertex to one of its ancestors.



YES



NO

Counting depth-trees

Theorem: For any cubic map of size n ($3n$ edges), there are 2^n depth-trees not containing the root.

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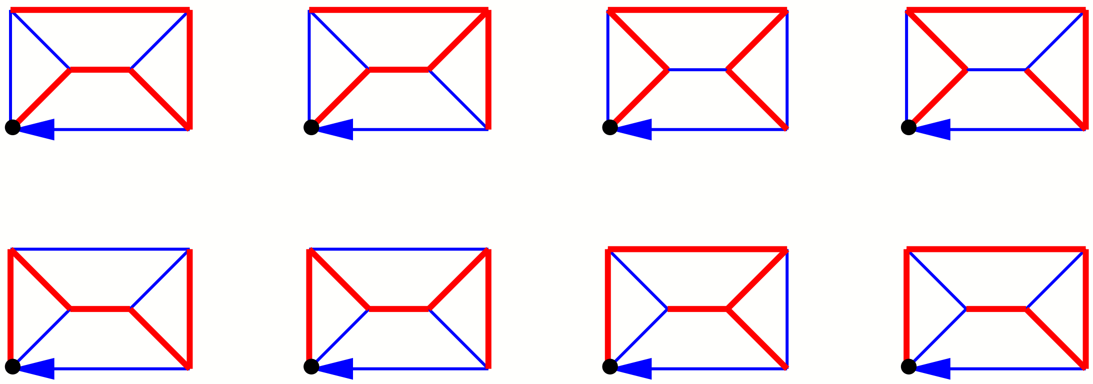
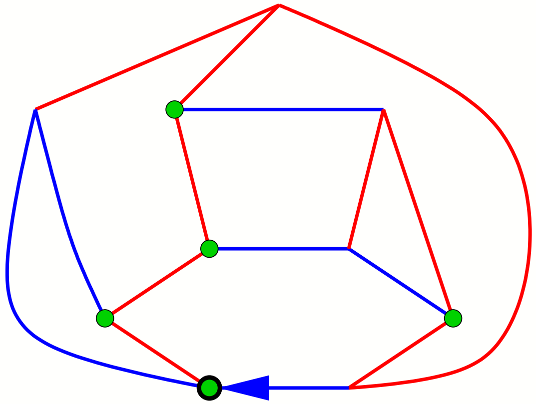
Example: n=3



Counting depth-trees

(Idea of the) proof:

- The depth-trees are the trees that can be obtained by a *depth-first search algorithm* (DFS).
- During a DFS, there are n *real* binary choices. (One for each external edge.)



$rrllr$

Kreweras walk

$\Longleftrightarrow$

Cubic map + Depth-tree

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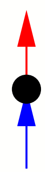
$$k_n = c_n \times 2^n$$

Bijection

Example:

$w = caccbbcbcbbaaaa$



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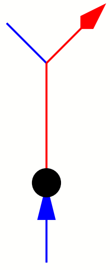

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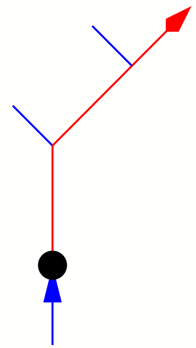

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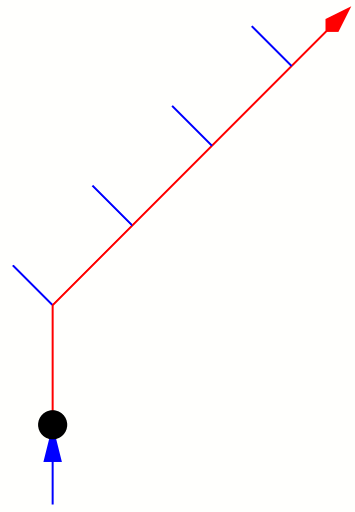

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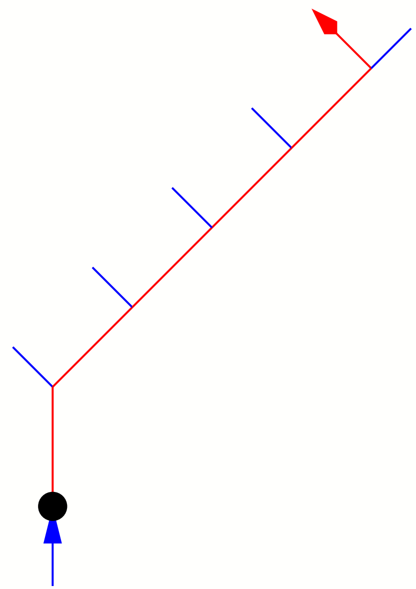

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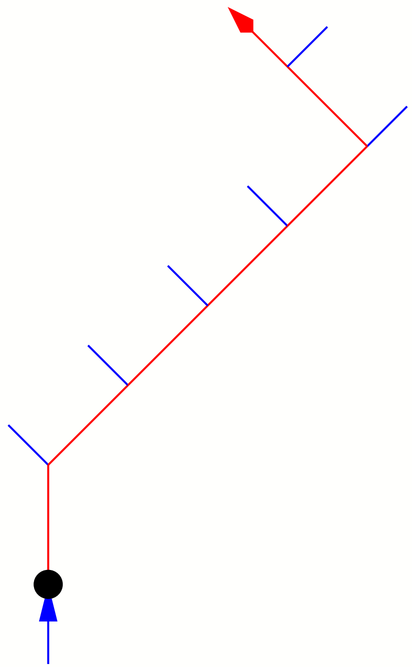

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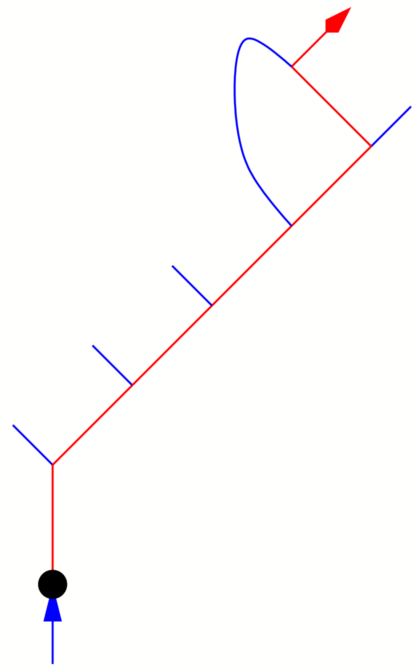

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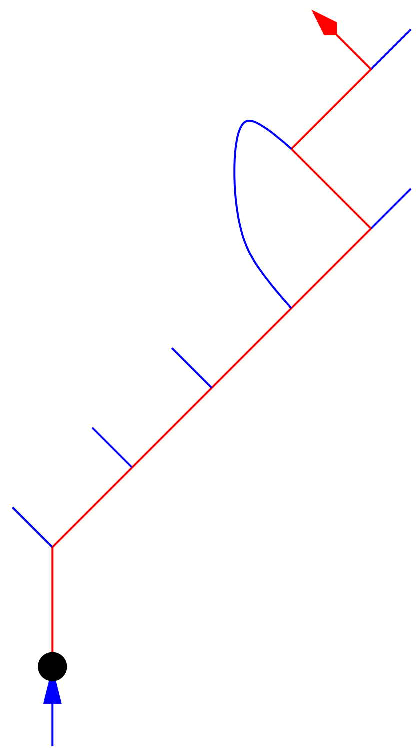

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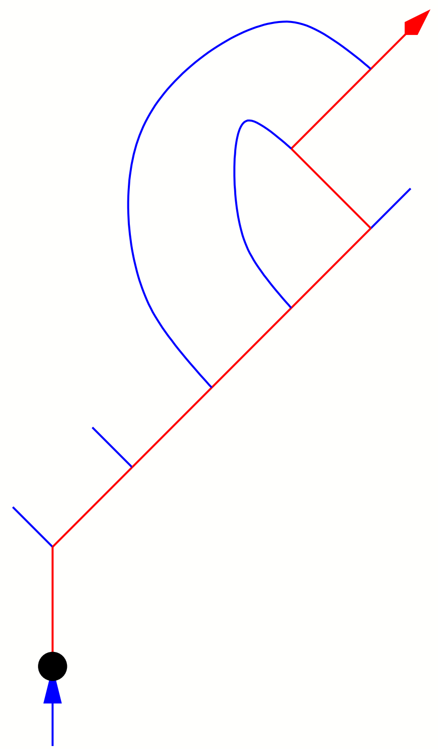

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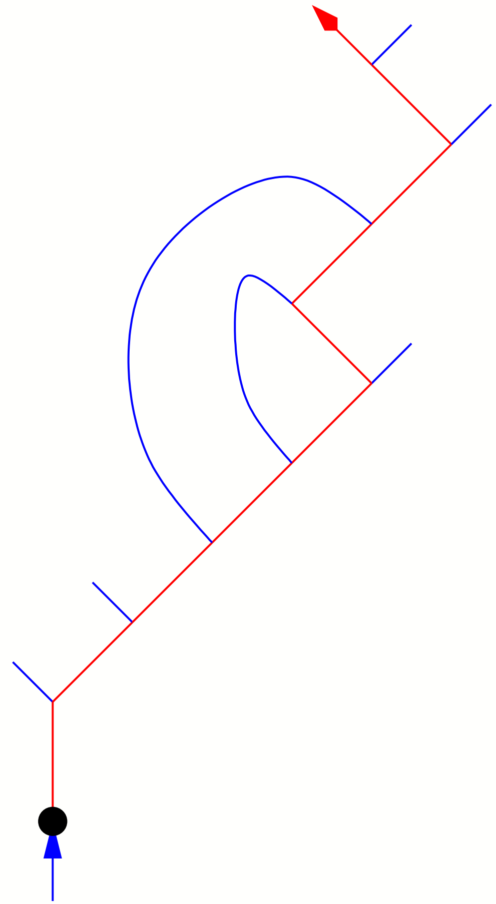

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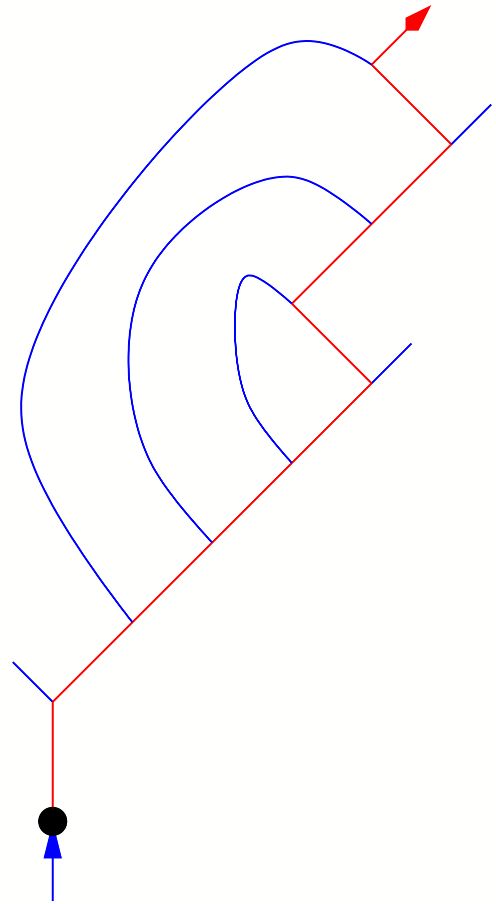

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Example:

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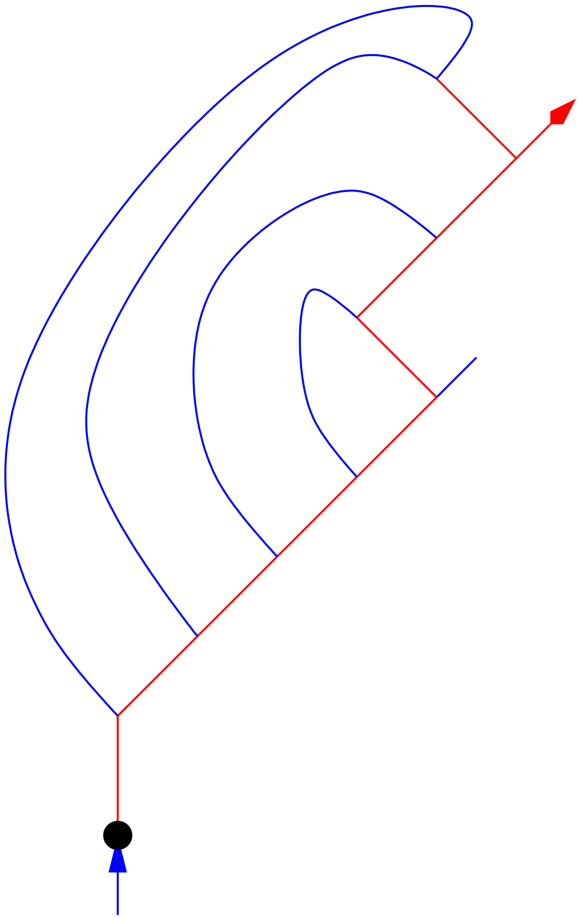

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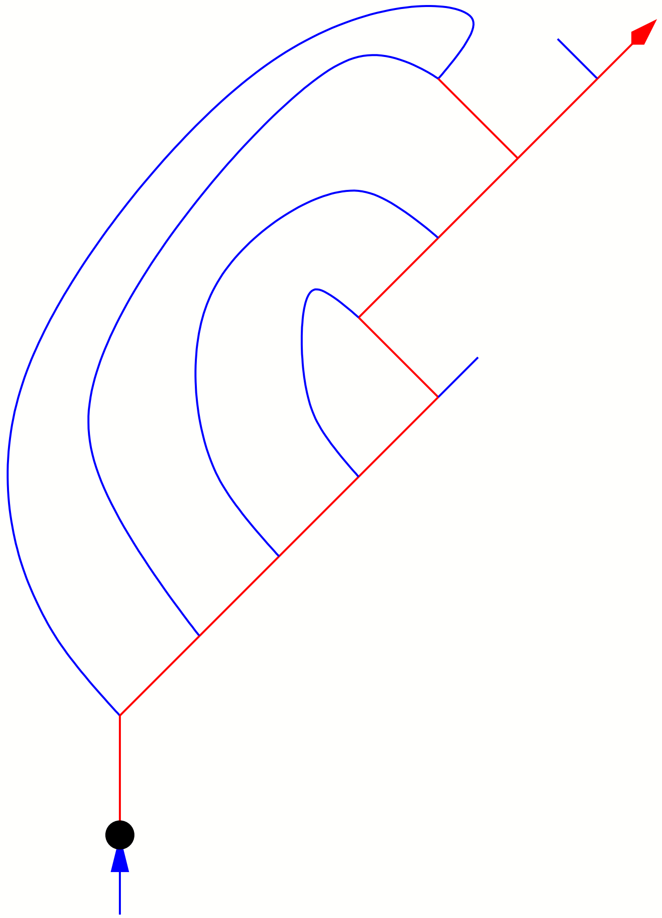

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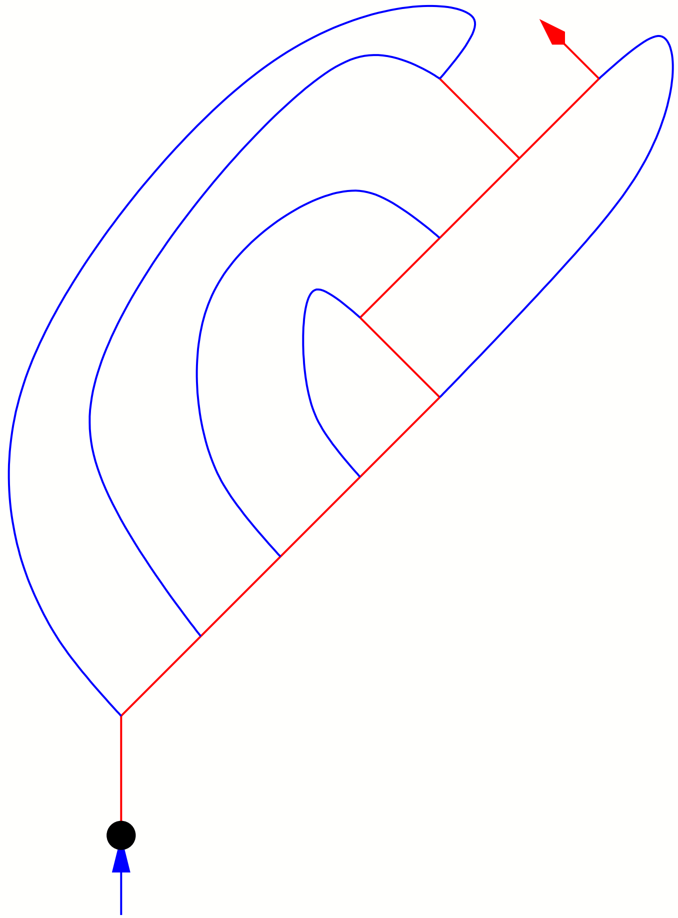

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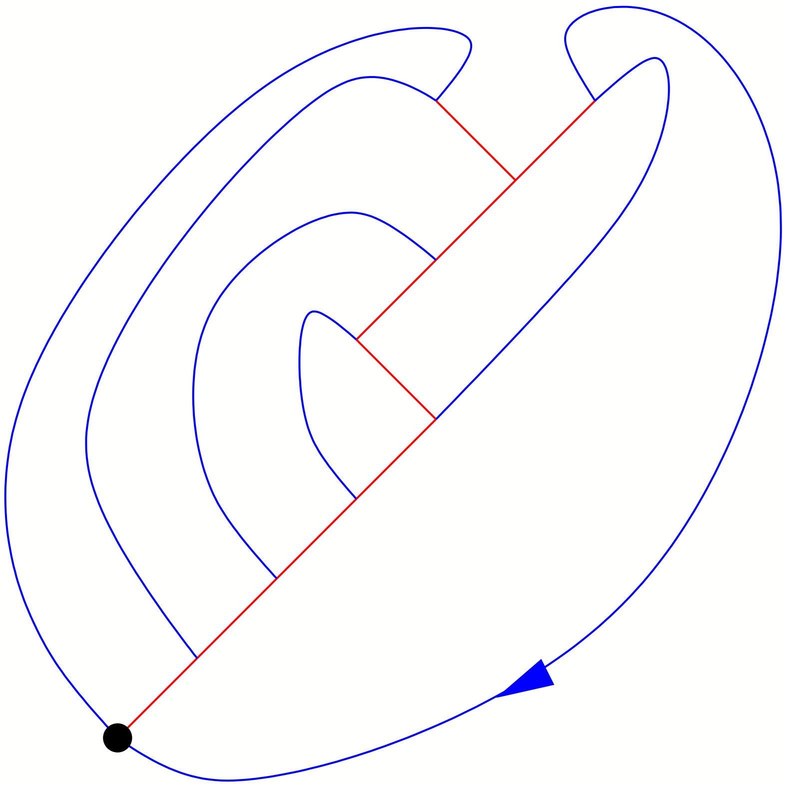
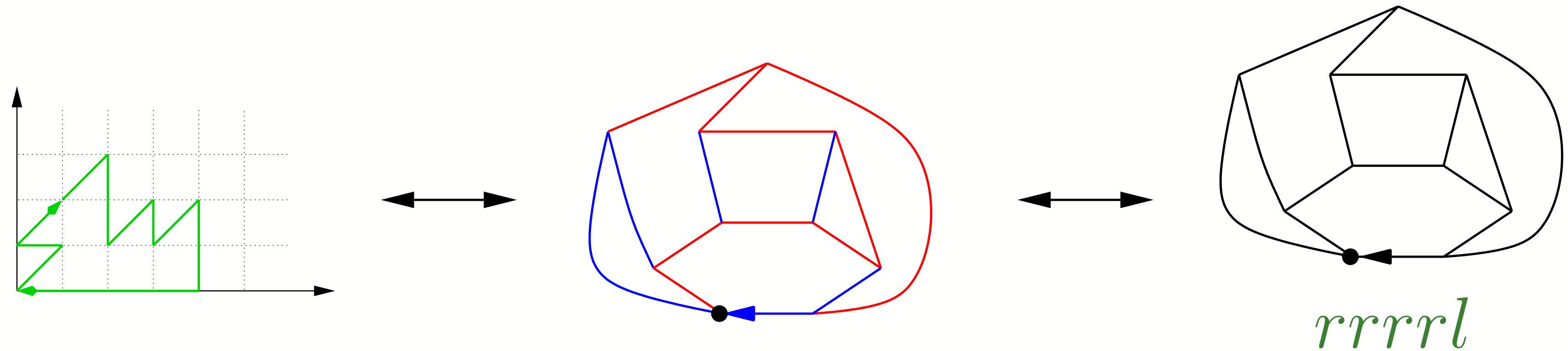


Theorem: This construction is a bijection between Kreweras walks of size n and cubic maps of size n **+** depth-tree.

Corollary: $k_n = c_n \times 2^n$.



Proof: The reverse bijection

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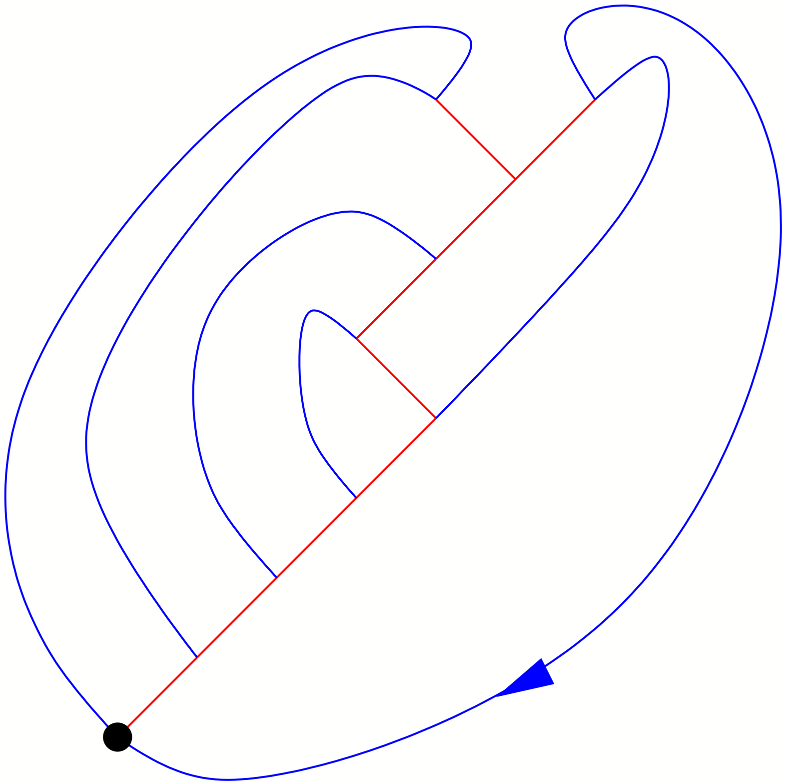

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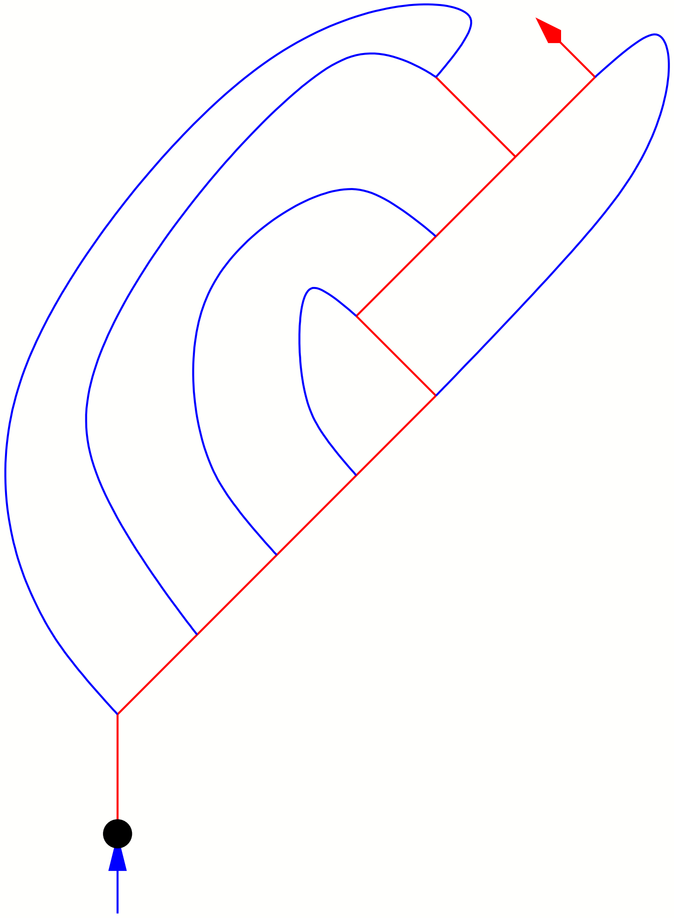

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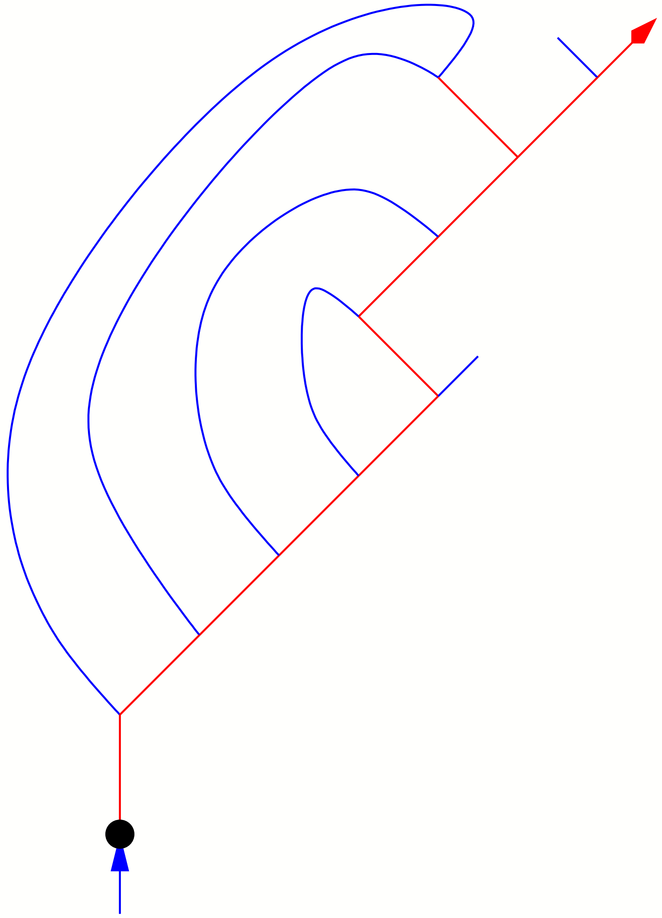

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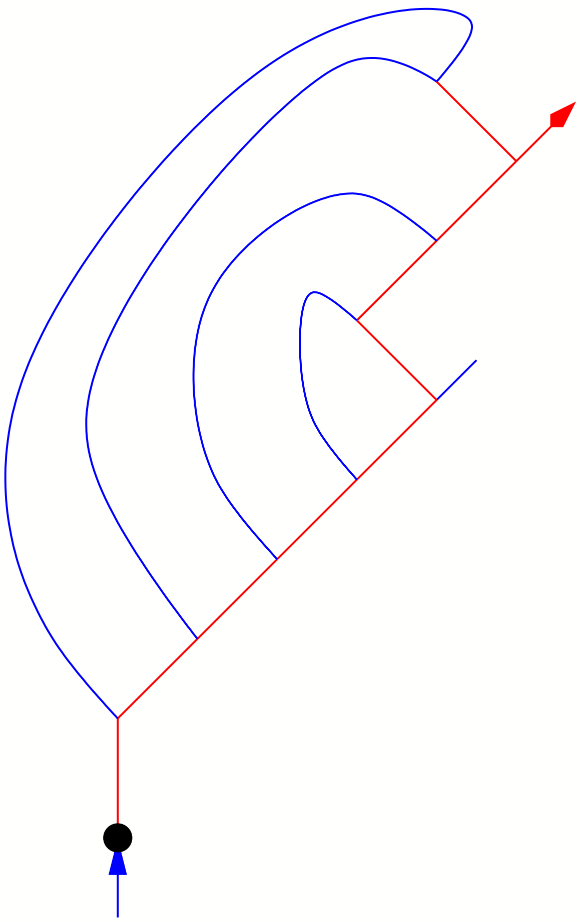
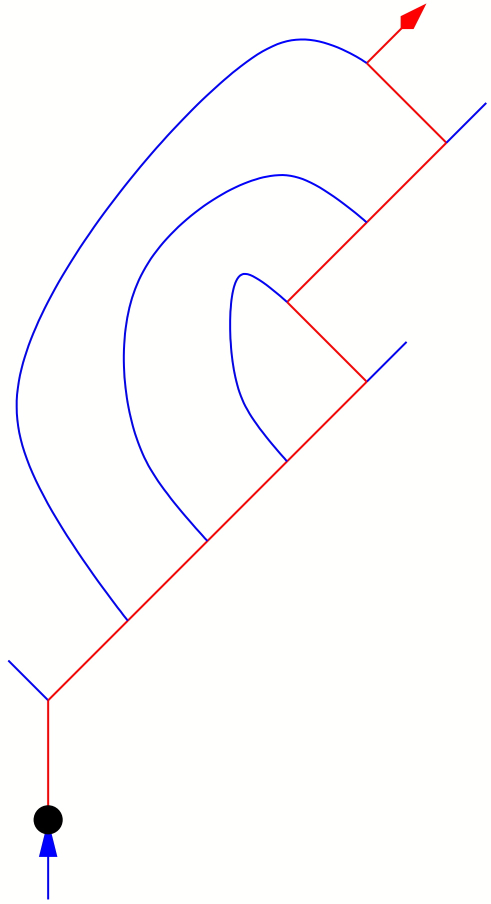


Proof: The reverse bijection

$w = cac\color{red}{cbbcbcbbaaaa}$



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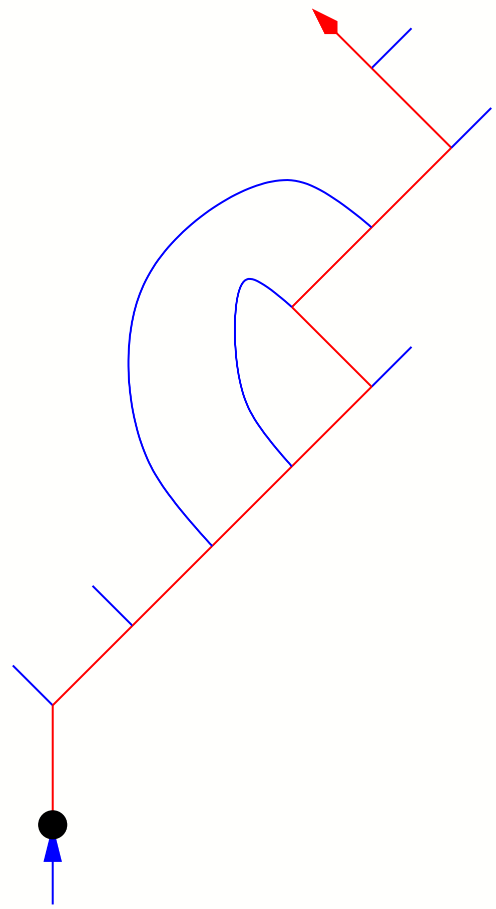

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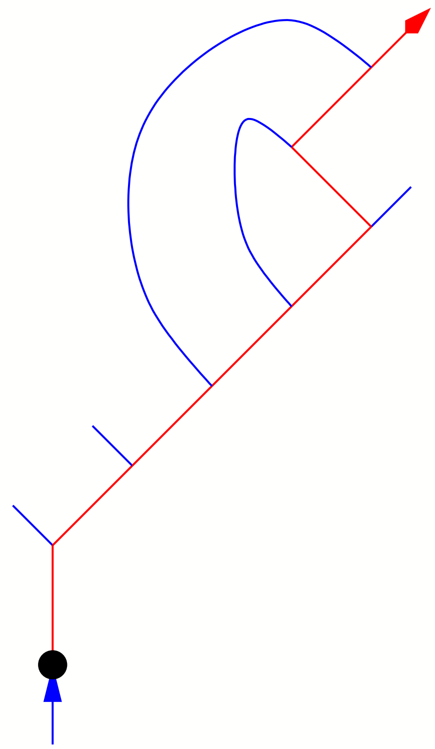

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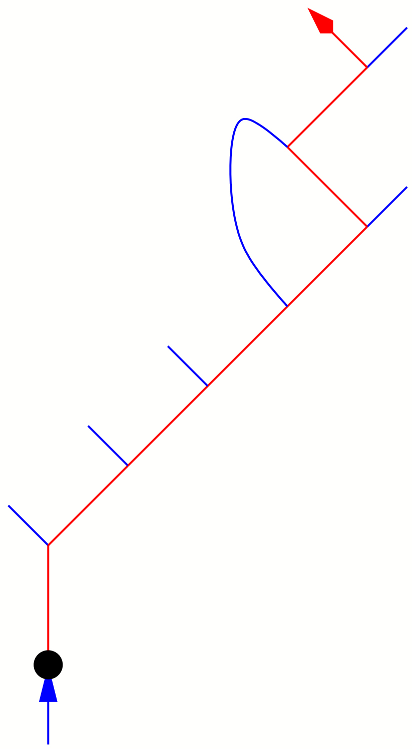

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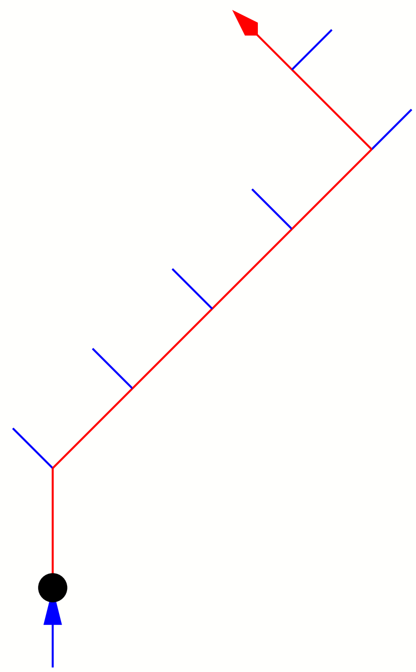

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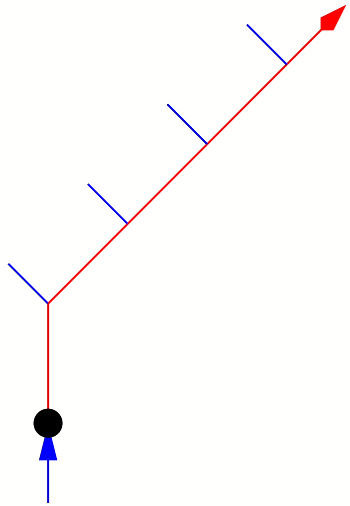

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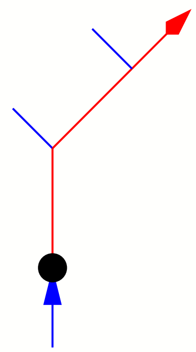
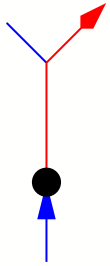


Proof: The reverse bijection

$w = caccbbcbcbbaaa{\color{red}a}$



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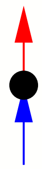

Counting Kreweras walks and cubic maps

Relaxing some constraints

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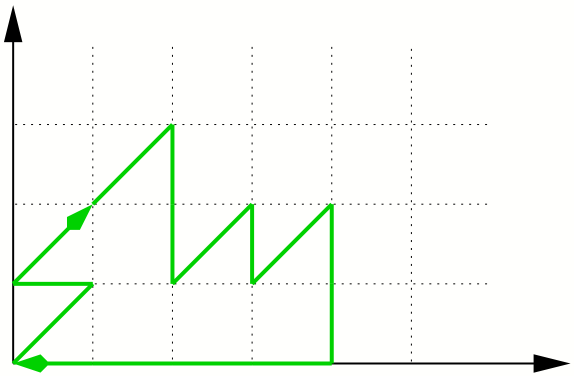
What about words w on $\{a, b, c\}$ such that

- $|w|_a + |w|_b = 2|w|_c$,
- for any prefix w', $|w'|_a + |w'|_b \leq 2|w'|_c$?

We call them excursions.

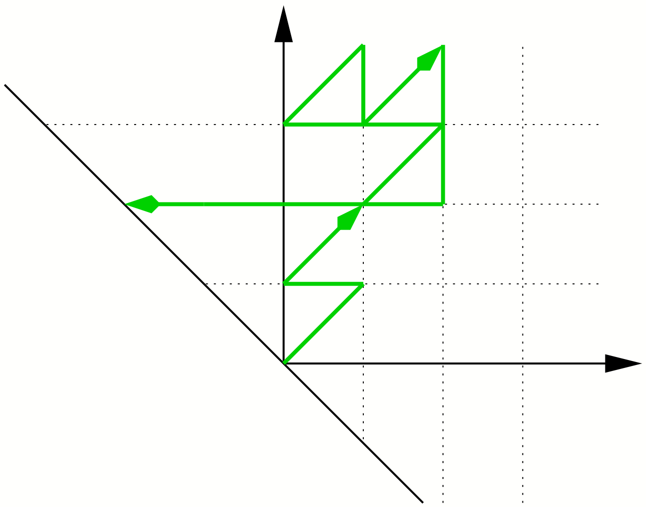
Kreweras



$$w = caccaacbcbbaaaa$$

$|w'|_a \leq |w'|_c$ and $|w'|_b \leq |w'|_c$

Excursion



$$w = caccbbcbcbbaaaa$$

$$|w'|_a + |w'|_b \leq 2|w'|_c$$

Proposition: There are $e_n = \frac{4^n}{2n+1}\binom{3n}{n}$ excursions of size n.

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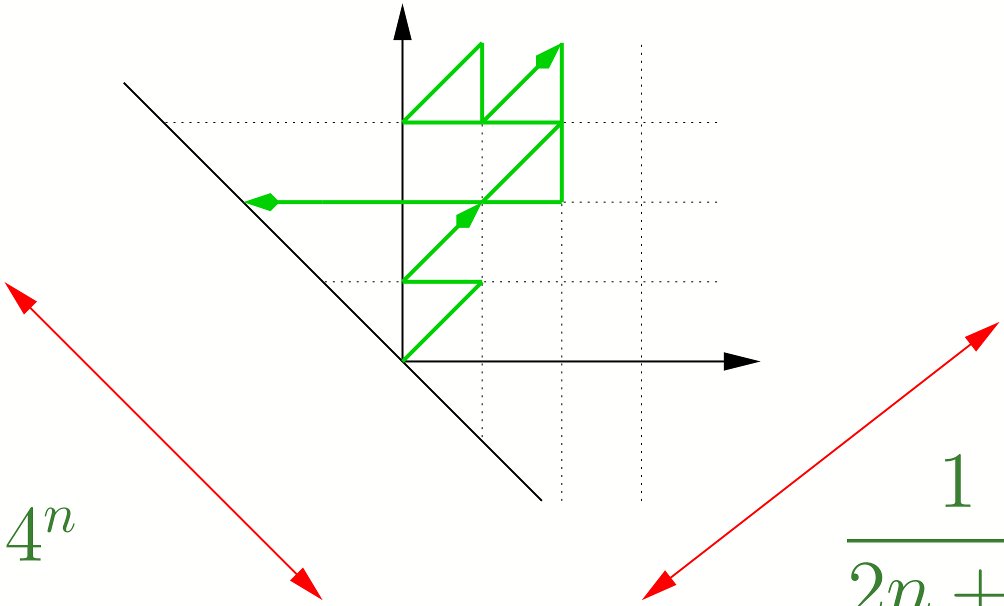


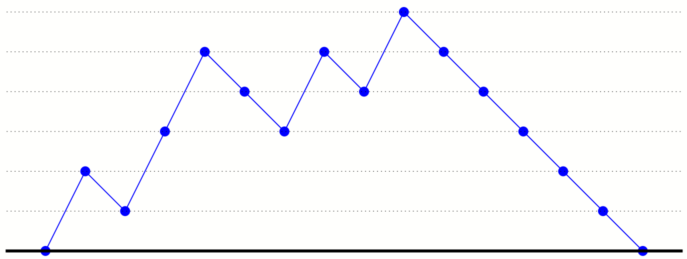
4^n

$\frac{1}{2n+1}\binom{3n}{n}$

Proof: The excursions w are such that:
$|w|_a + |w|_b = 2|w|_c$,
for all prefix w', $|w'|_a + |w'|_b \leq 2|w'|_c$.

- **Position of the c's:** $\frac{1}{2n+1}\binom{3n}{n}$.

  Cycle lemma: There are $\frac{1}{2n+1}\binom{3n}{n}$ (one-dimensional) walks with $3n$ steps +2 and -1.

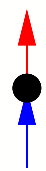


- **Position of the a's and b's:** 2^{2n}.

Extending the bijection

Example:

$w = caccaacbcbbaaaa$



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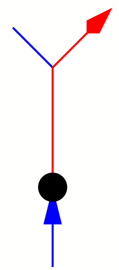

Extending the bijection

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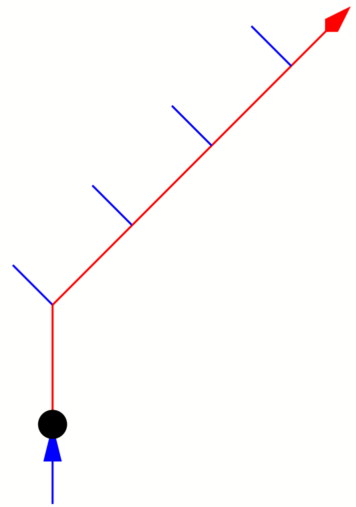
$w = caccaacbcbbaaaa$



Extending the bijection

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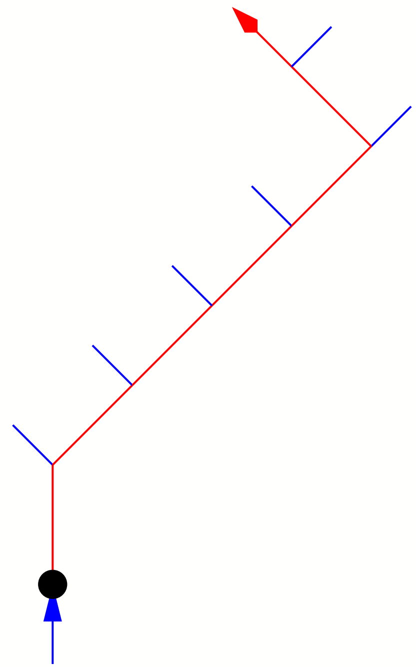

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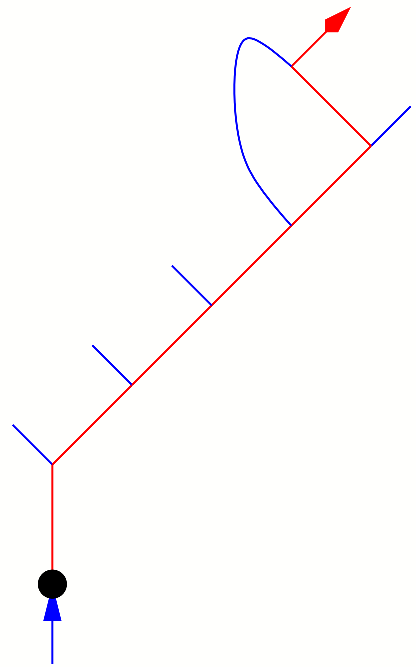

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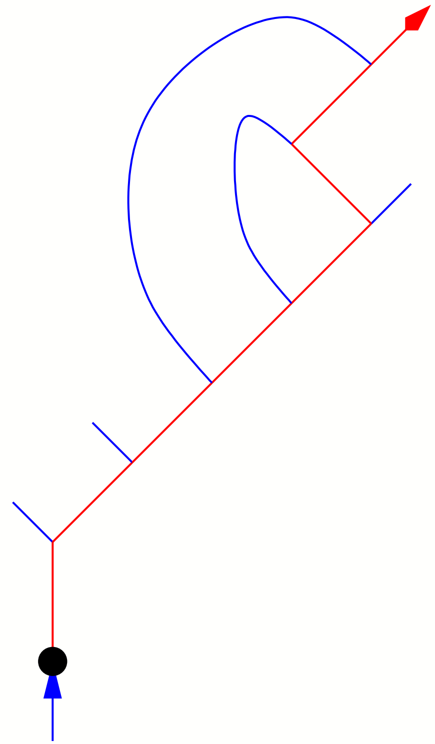
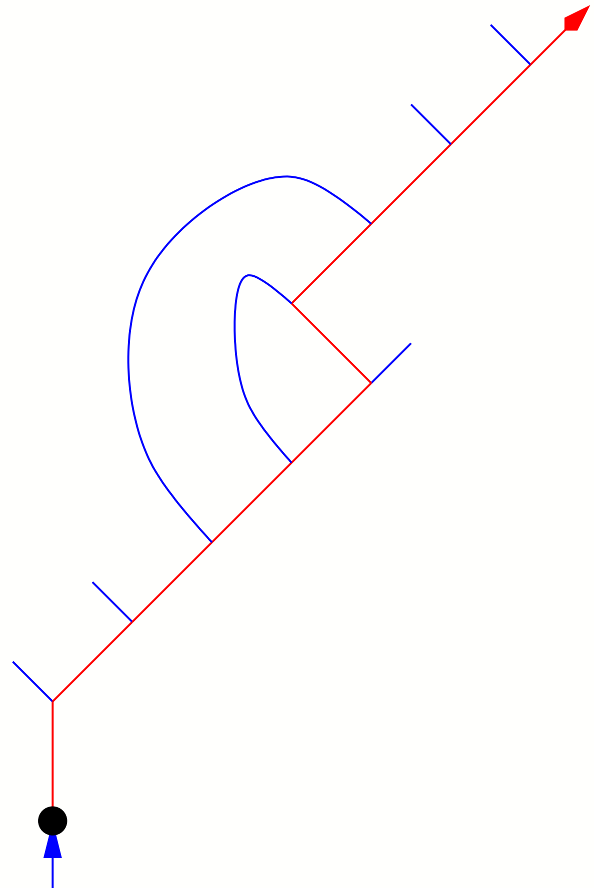


Extending the bijection

Example:

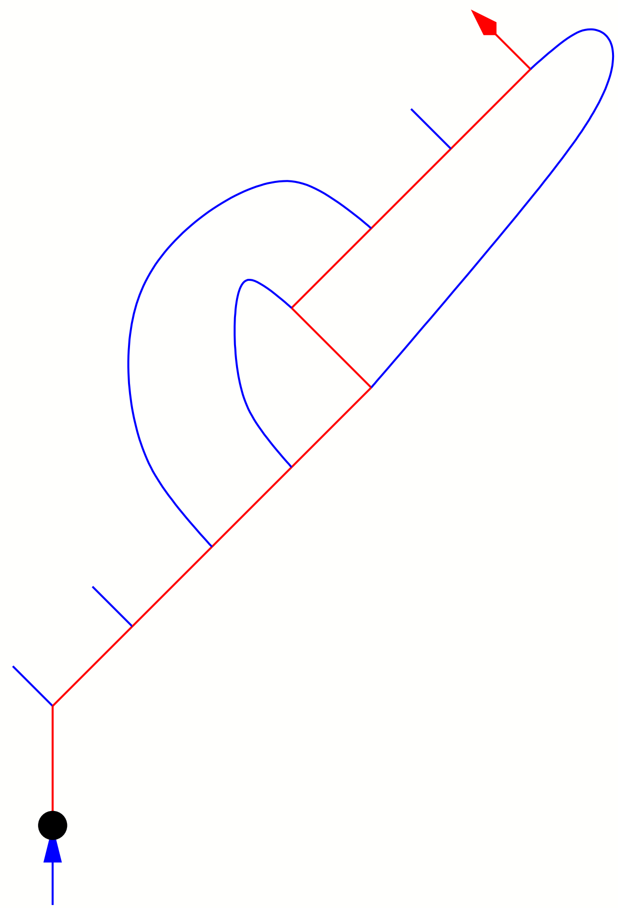
$w = cacc\textcolor{red}{aacbcbbaaaa}$



Extending the bijection

Example:

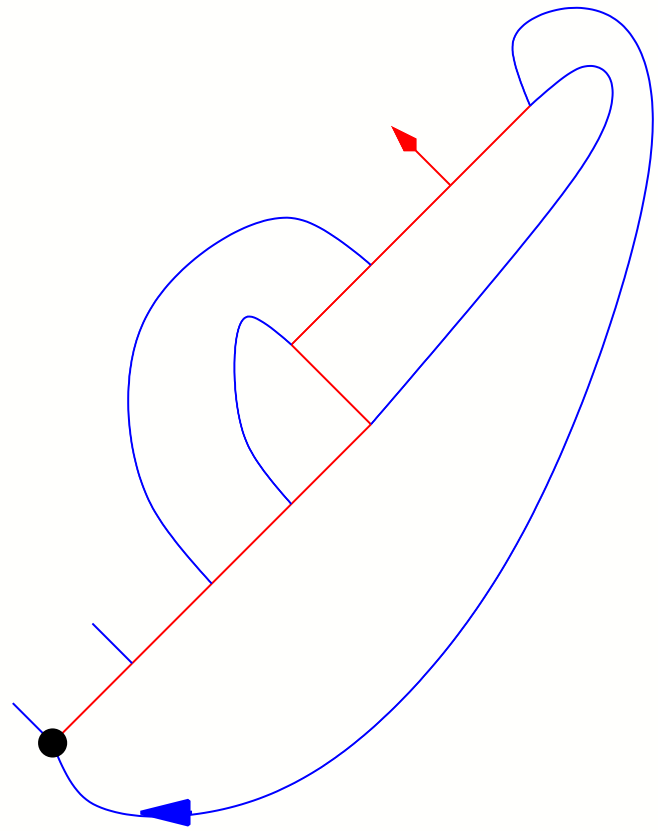
$w = cac\textcolor{red}{caacbcbbaaaa}$



Extending the bijection

Example:

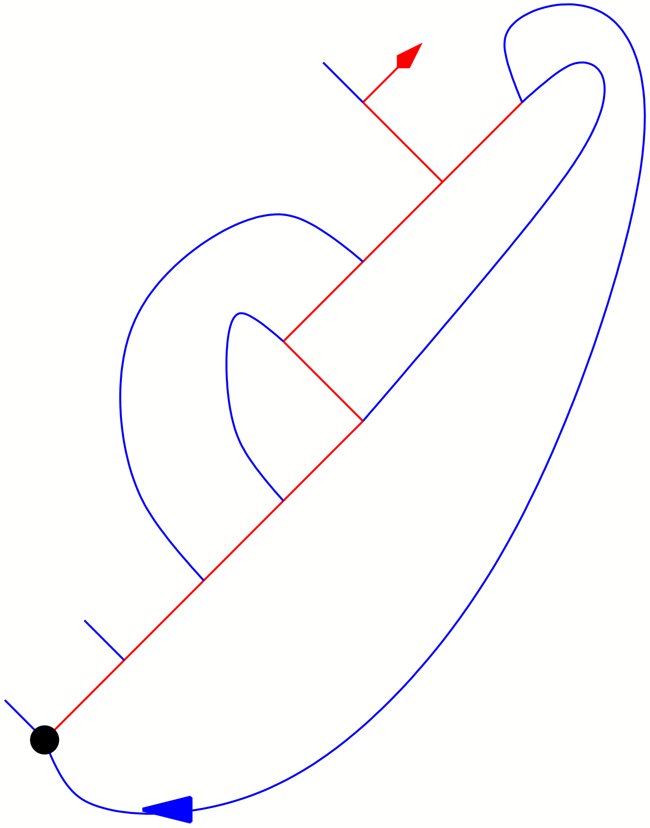
$w = ca\color{red}{ccaacbcbbaaaa}$



Extending the bijection

Example:

$w = c\color{red}{accaacbcbbaaaa}$



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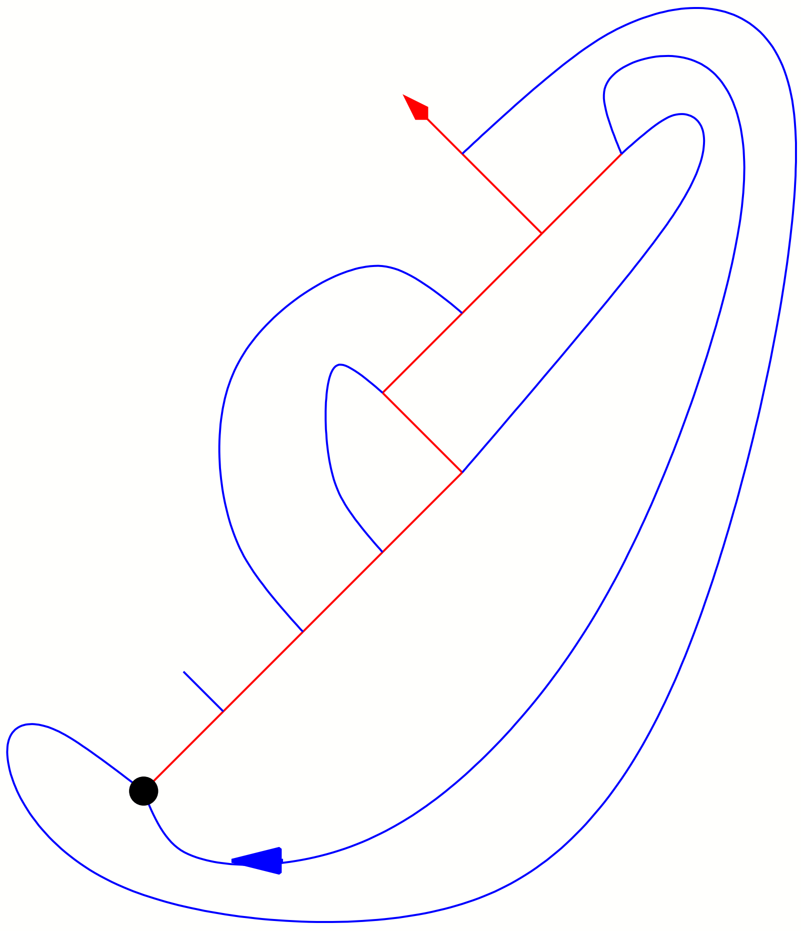

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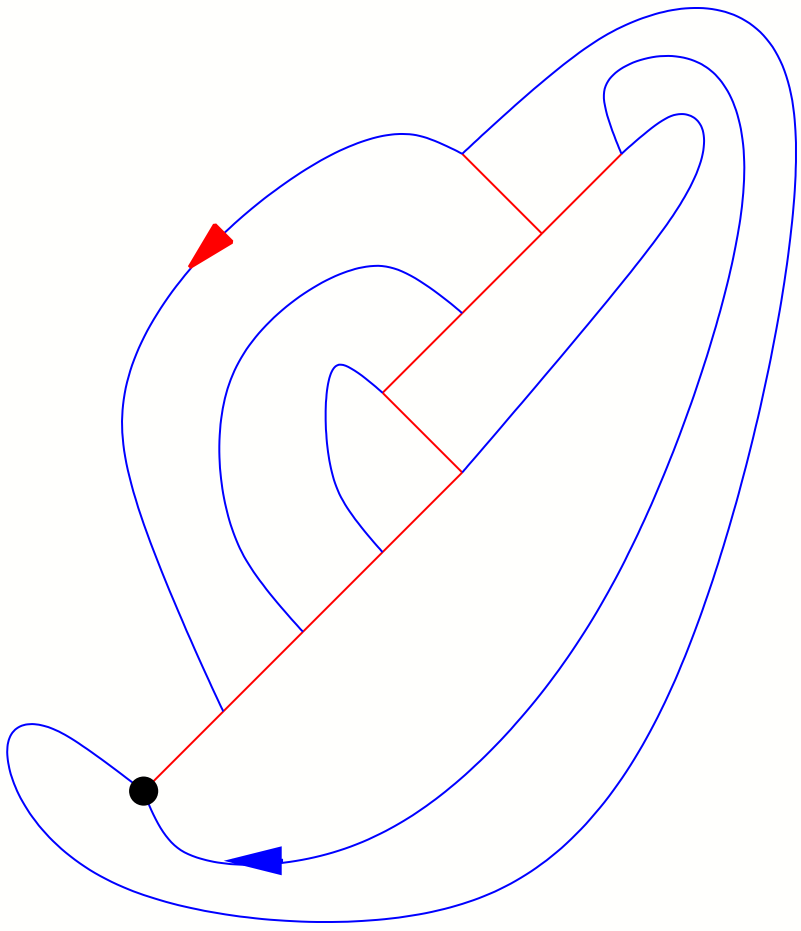

Theorem: This construction is a bijection between ecursions of size n and cubic maps of size n **+** depth-tree **+** marked external edge.

Corollary: $e_n = c_n \times 2^n \times (n+1)$.

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Thus,

$c_n = \dfrac{2^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ and $k_n = \dfrac{4^n}{(n+1)(2n+1)}\dbinom{3n}{n}$.

Concluding remarks

Results

- We established a bijection between Kreweras walks and cubic maps with a depth-tree.

  $\Rightarrow$ Coding of triangulations with $\log_2(27)$ bits per vertex. (Optimal coding: $\log_2(27) - 1$ bits per vertex.)

Results

- We established a bijection between Kreweras walks and cubic maps with a depth-tree.

  $\Rightarrow$ Coding of triangulations with $\log_2(27)$ bits per vertex. (Optimal coding: $\log_2(27) - 1$ bits per vertex.)

- We extended the bijection to a more general class of walks.

  $\Rightarrow$ Counting results.

  $\Rightarrow$ Random sampling of triangulations in linear time.

$$k_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

Open problems

- Can we count Kreweras walks ending at $(i, 0)$? at (i, j) ?

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Kreweras walks ending at $(i,0)$.

Remark: Kreweras walks ending at $(i,0)$ and $(i+2)$-near-cubic maps are related :

$$k_{n,i} \; = \; 2^n \times c_{n,i}.$$

Open problems

- There are similar counting results:

  - Non-separable maps **[Tutte]**.

  - Two-stack sortable permutations **[West, Zeilberger]**.

$$\mathcal{NS}_n = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

[Dulucq, Gire & Guibert 96, Goulden & West 96]

Thanks.