Flag arrangements and tilings of simplices.

Federico Ardila
Sara Billey

FPSAC / SFCA / SFCA
San Diego, CA, USA
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The plan to follow: (or not to follow)

1. Arrangements of d flags in $\mathbb{C}^n$.
2. Tilings in two and more dimensions.
3. Applications to the flag Schubert calculus.

1. Arrangements of d flags in $\mathbb{C}^n$.

A complete flag $F_\bullet$ in $\mathbb{C}^n$ is

$$F_\bullet = \{\{0\} \subset \text{line} \subset \text{plane} \subset \cdots \subset \text{hyperplane} \subset \mathbb{C}^n\}.$$

Let $E^1_\bullet, \ldots, E^d_\bullet$ be d generically chosen complete flags in $\mathbb{C}^n$. Write

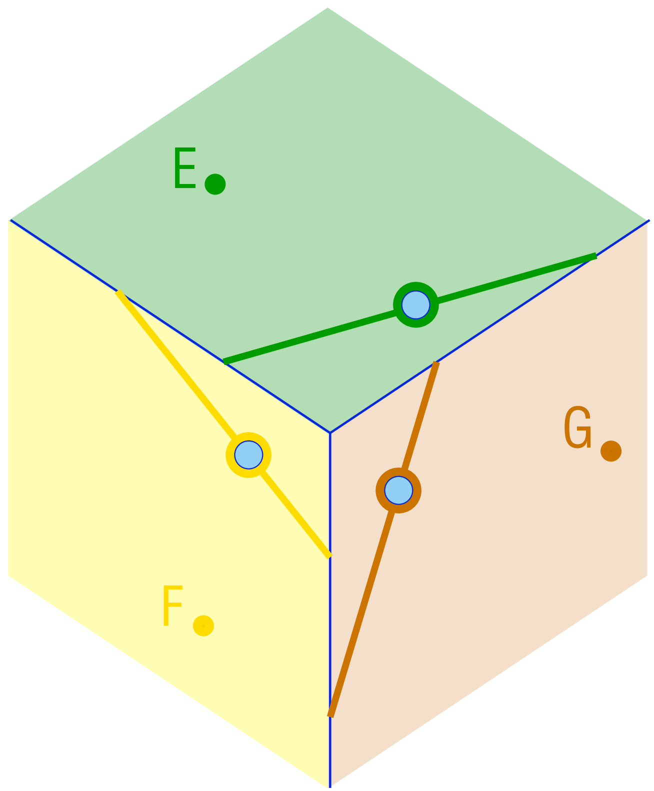
$$E^k_\bullet = \{\{0\} = E^k_0 \subset E^k_1 \subset \cdots \subset E^k_n = \mathbb{C}^n\},$$

where E^k_i is a vector space of dimension i.

Let $E^1_\bullet, \ldots, E^d_\bullet$ be d generically chosen complete flags in $\mathbb{C}^n$.

Example. $d = 3, n = 4$: flags $E_\bullet, F_\bullet, G_\bullet$ in $\mathbb{C}^4$ (projective picture)

Each flag is point $\subset$ line $\subset$ plane $\subset$ 3-space.



Goal. Study the set $\mathbf{E}_{n,d}$ of one-dimensional intersections determined by the flags; that is, all lines of the form

$$E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d},$$

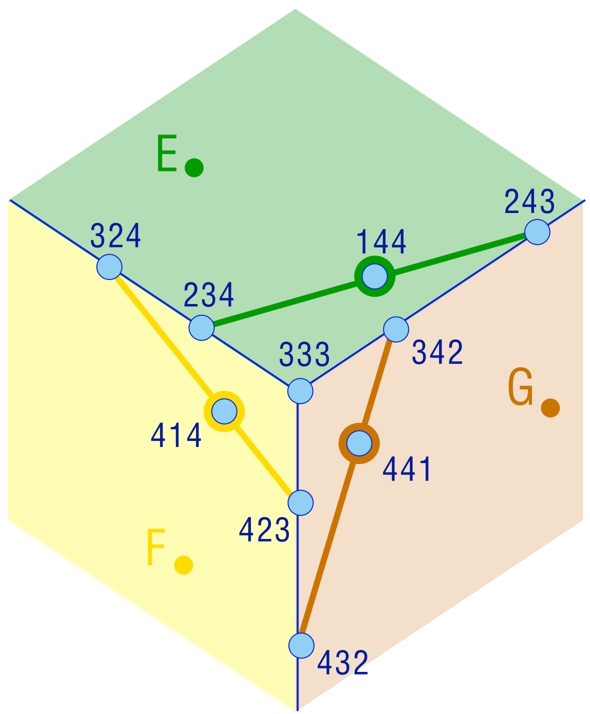
with $\sum(n - a_i) = n - 1$; that is, $\sum a_i = n(d-1) + 1$.

Example.

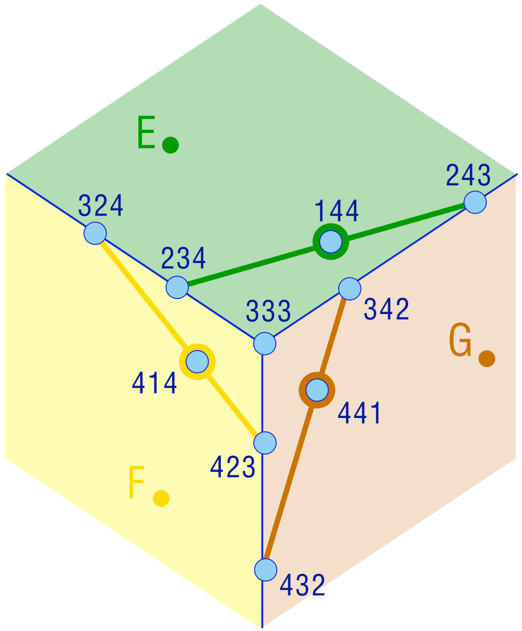
$\mathbf{E}_{4,3}$ consists of the ten lines:

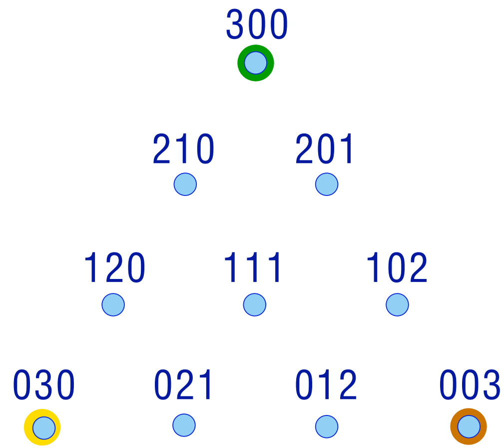
$$abc = E_a \cap F_b \cap G_c$$

for $a + b + c = 9$



Question. In $\mathbf{E}_{n,d}$, which sets are dependent/independent? What is the matroid?





First an encoding:

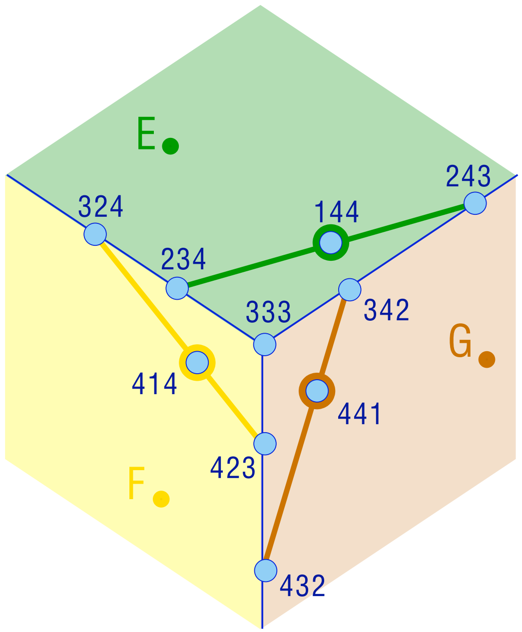
$$\text{lines in } \mathbf{E}_{n,d} \quad \leftrightarrow \quad \text{dots in "simplicial" array } T_{n,d}$$

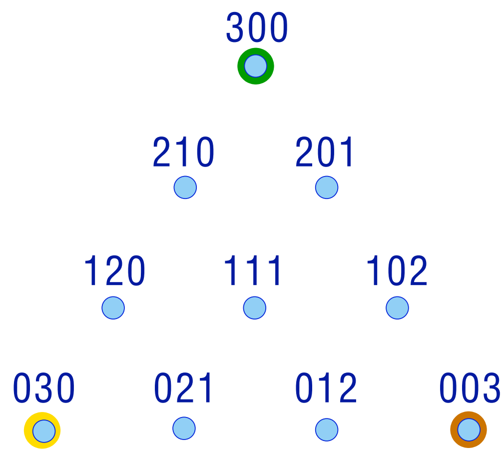
$$E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d} \quad \leftrightarrow \quad (n-a_1, \ldots, n-a_d)$$

Some easy dependence relations:

A k-dim $E^1_{b_1} \cap E^2_{b_2} \cap \cdots \cap E^d_{b_d}$ contains line $E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d}$ when $a_i \leq b_i$. Therefore, those lines have rank at most k.

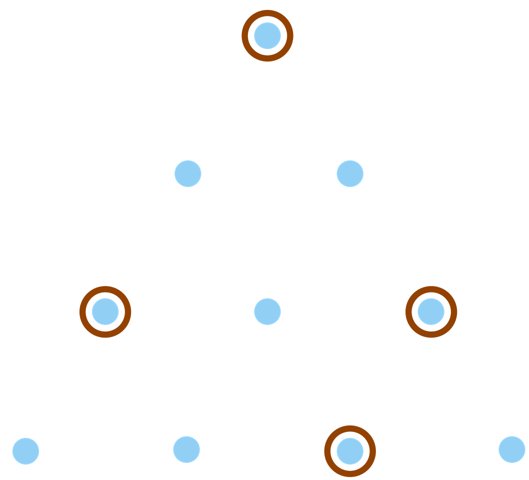
Combinatorial dependence relation.
Any $k+1$ dots in a simplex of size k are dependent.





Question. Are these the only dependence relations?

Answer. These **are** the only dependence relations.



Theorem. (Ardila, Billey, 2005.)

A set of dots in the array $T_{n,d}$ is independent if and only if no subarray $T_{k,d}$ of size k contains more than k dots.

The method of proof is constructive.

Goal:

How do we construct d "generic enough" flags in $\mathbb{C}^n$?

Reduce to:

How do we construct $(n-1)d$ "generic enough" hyperplanes in $\mathbb{C}^n$?

(Get a flag from $n-1$ hyps.: $A \supset (A \cap B) \supset (A \cap B \cap C) \supset \cdots$.)

Reduce to:

How do we construct a "generic enough" n-plane P in $\mathbb{C}^{(n-1)d}$?

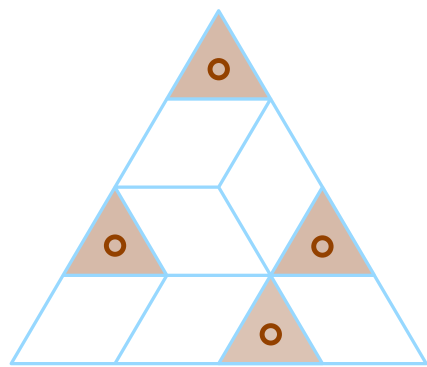
(Then intersect P with the nd coordinate hyperplanes in $\mathbb{C}^{(n-1)d}$.)

We do this using the theory of Dilworth truncations.

2. Tilings in two and more dimensions.

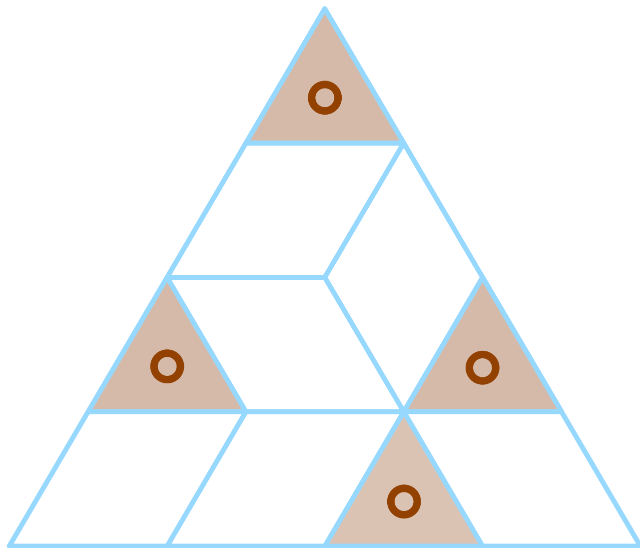
To tile the equilateral triangle $T(n)$ of size n with unit rhombi,

we first need to make $n = \binom{n+1}{2} - \binom{n}{2}$ holes.



Where can we put those holes?

Question. Given n holes in $T(n)$, is there a simple criterion to determine whether the resulting holey triangle can be tiled with unit rhombi?



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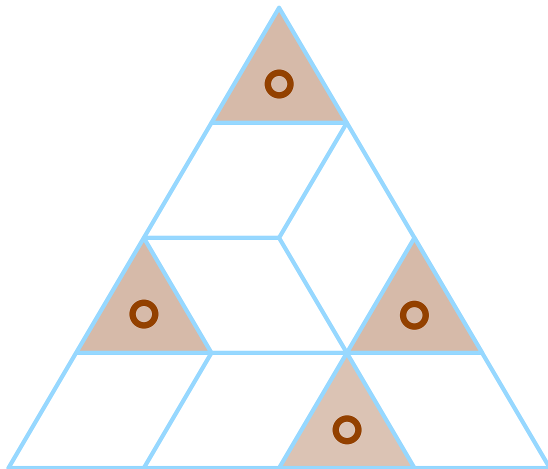


A necessary condition. If a holey triangle can be tiled with unit rhombi, then no $T(k)$ inside $T(n)$ can contain more than k holes.

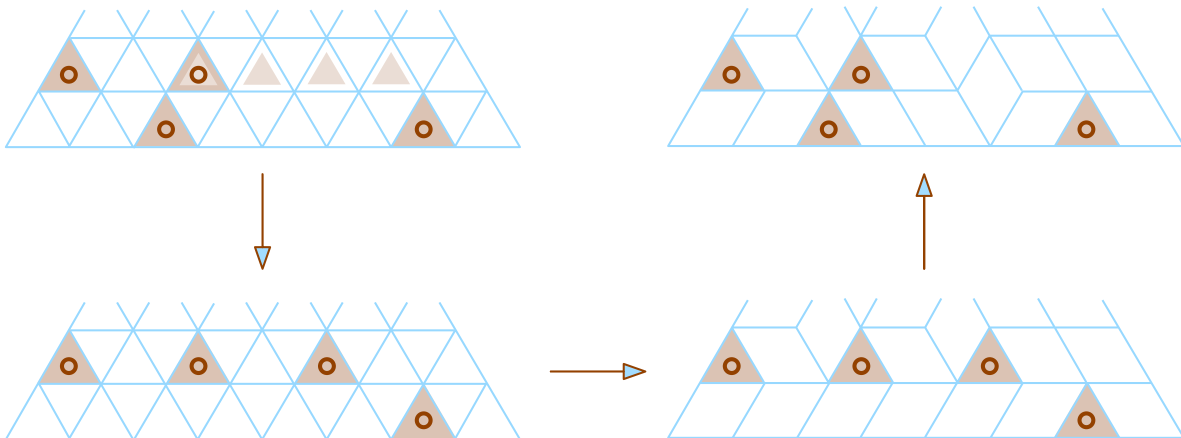
Proof. Count.

Theorem. (Ardila, Billey, 2005)

Consider a set of n holes in $T(n)$. The resulting holey triangle can be tiled with unit rhombi if and only if no $T(k)$ inside $T(n)$ contains more than k holes.

The possible locations of the holes are precisely the bases of $\mathcal{T}_{n,3}$!

The method of proof is constructive. Given a set of holes which is "not too crowded", we construct a tiling T with those holes. We start with a base tiling T_0, and arrive to T via local moves.

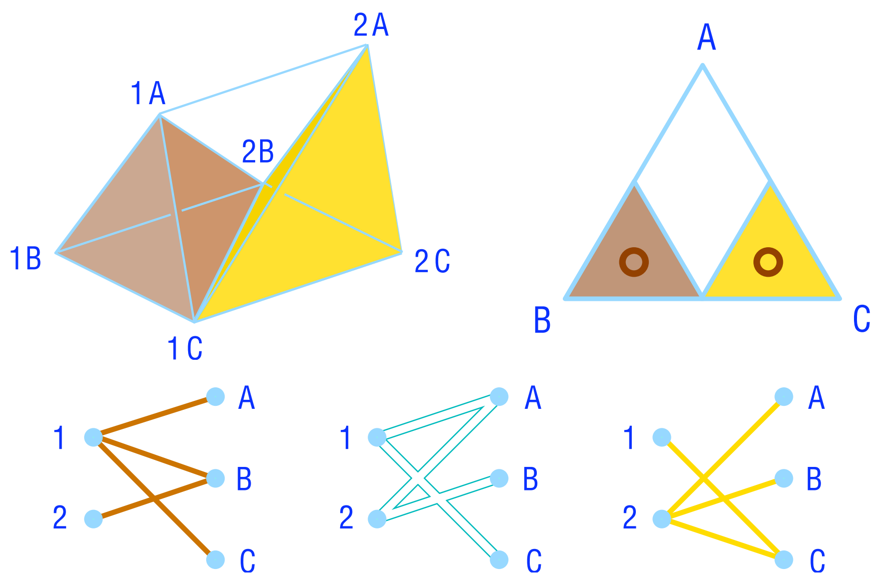


Generalization.

(geometry of 3 flags)	$\leftrightarrow$	(rhombus tilings of holey triangles)
(geometry of d flags)	$\leftrightarrow$	(fine mixed subdivisions of $n\Delta_{d-1}$)

A fine mixed subdivision of the simplex $n\Delta_{d-1}$ has tiles:

$$(d-1)\text{-dimensional products of faces of } \Delta_{d-1}$$



(See also: Babson-Billera, Diaconis-Sturmfels, Postnikov, Santos ...)

3. Applications to the flag Schubert calculus.

(Very) quick review of Schubert calculus of the flag manifold:

The relative position of two flags $E_\bullet$ and $F_\bullet$ in $\mathbb{C}^n$ is given by the $n \times n$ rank table whose (i,j) entry is $P[i,j] = \dim(E_i \cap F_j)$.

An example rank table:

$$P = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Each rank table comes from a permutation matrix:

$$P = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If $E_\bullet$ and $F_\bullet$ have rank table P, their relative position is $w = 53124$.

For fixed $E_\bullet$, divide all flags according to position with respect to $E_\bullet$:

The *Schubert cell* and *Schubert variety* be

$$\begin{aligned} X_w^\circ(E_\bullet) &= \{F_\bullet \mid E_\bullet \text{ and } F_\bullet \text{ have relative position } w\} \\ X_w(E_\bullet) &= \overline{X_w^\circ(E_\bullet)} \end{aligned}$$

Schubert problem. Given generic flags $E^1_\bullet, E^2_\bullet, E^3_\bullet$ in $\mathbb{C}^n$ and permutations u, v, w in S_n, how many flags $F_\bullet$ have relative positions u, v, w with respect to $E^1_\bullet, E^2_\bullet, E^3_\bullet$?

The answer, c_{uvw}, is independent of $E^1_\bullet, E^2_\bullet, E^3_\bullet$. The numbers c_{uvw} are very important. They are the multiplicative structure constants for the cohomology ring of the flag manifold.

Open problem. Given three permutations u, v, w, can we compute c_{uvw} combinatorially?

This question seems very difficult; the following may be easier:

Open problem. Can we describe the permutations u, v, w for which $c_{uvw} = 0$?

3.1. A vanishing criterion for c_{uvw}.

Proposition. (Billey-Vakil) If we know the relative positions u, v, w of $F_\bullet$ with respect to generic $E^1_\bullet, E^2_\bullet, E^3_\bullet$, then we know its relative position with respect to $E^1_\bullet \cap E^2_\bullet \cap E^3_\bullet$.

More concretely, if we know, for all a, b, c, j:

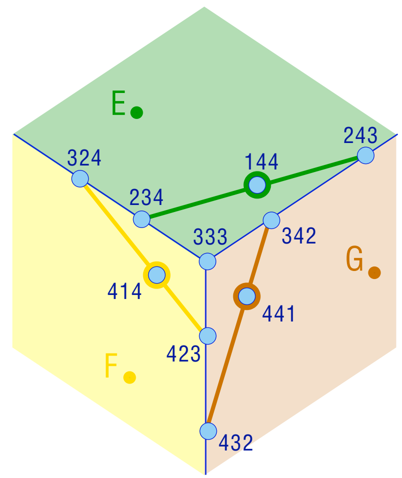
$$\dim(E^1_a \cap F_j), \qquad \dim(E^2_b \cap F_j), \qquad \dim(E^3_c \cap F_j),$$

then we can compute, for all a, b, c, j,

$$\dim(E^1_a \cap E^2_b \cap E^3_c \cap F_j)$$

and in particular, for $a + b + c = 2n + 1$,

$$\dim(abc \cap F_j).$$



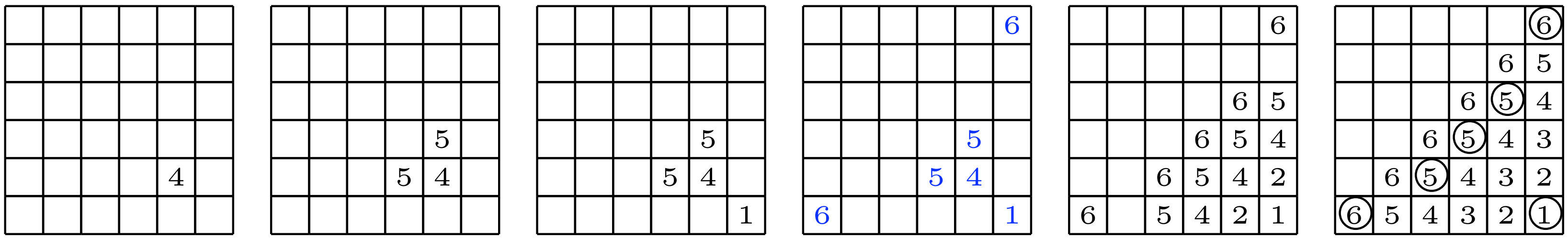
So we know the set $L(u, v, w)_j$ of lines abc which are in each F_j.

The relative positions u, v, w of $F_\bullet$ with respect to $E^1_\bullet, E^2_\bullet, E^3_\bullet$ determine the set $L(u,v,w)_j$ of lines abc which are in each F_j.

> **Observation.** The matroid $\mathcal{T}_{n,3}$ tells us $\text{rank}(L(u,v,w)_j)$. If that rank is greater than j, then $c_{uvw} = 0$.

This method already characterizes vanishing for $n \leq 5$; but this is just the easiest possible observation along these lines.

Example. $u = 231645, v = 231645, w = 326154$



Since F_4 would have to contain 5 independent vectors, $c_{uvw} = 0$.

(Doesn't follow from Knutson's descent cycling method. Compare with other methods: Lascoux-Schutzenberger, Purbhoo.)

3.2. Computing c_{uvw}.

Proposition. (Billey-Vakil) Given the permutations u, v, w, we can compute $\dim(E_a^1 \cap E_b^2 \cap E_c^3 \cap F_j)$. We can use these numbers to write down a simple and explicit set of equations cutting out

$$X = X_u(E_\bullet^1) \cap X_v(E_\bullet^2) \cap X_w(E_\bullet^3)$$

and just count the number c_{uvw} of flags in X.

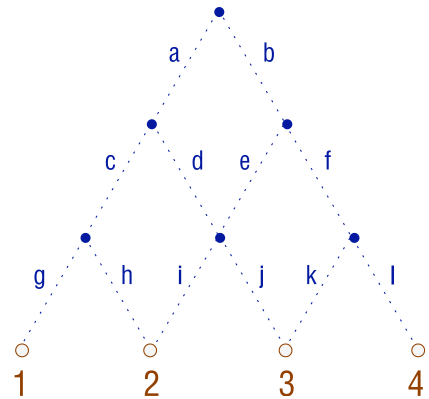
The equations are written in terms of the vectors

$$abc = E_a^1 \cap E_b^2 \cap E_c^3,$$

so it would be very useful to have a nice choice of abc.

Ultimately, we want a nice representation of the matroid $\mathcal{T}_{n,3}$.

We get this from $\mathcal{T}_{n,3}$ being a cotransversal matroid (via tilings!).



Assign weights to the edges. For each dot D, let $v_{D,i}$ be the sum of the weights of all paths from dot D to dot i on the bottom row.

For example, $v_{top} = (acg, ach + adi + bei, adj + bej + bfk, bfl)$.

Theorem. (Ardila-Billey, 2005) Vectors $v_D = (v_{D,1}, \ldots, v_{D,n})$ are a geometric representation of the matroid $\mathcal{T}_{n,3}$.

Result. (Billey-Vakil, 2004, Ardila-Billey, 2005) We get a method for computing c_{uvw} without reference to a fixed set of flags.

Thank you for your attention.

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