Bijections of trees arising from Voiculescu's free probability theory

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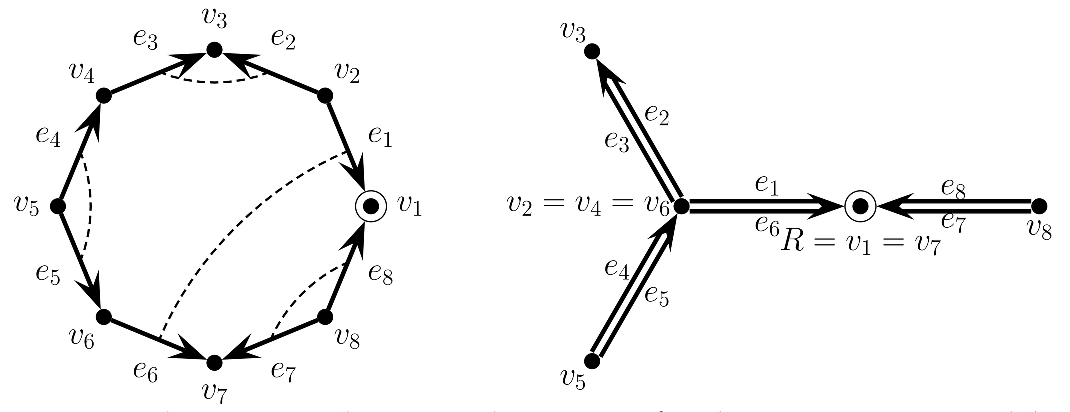
Quotient trees

Let a polygonal graph G with arbitrary orientations of the edges be fixed. Let σ be a pairing between the edges of G. We will always assume that σ is *non-crossing* and *compatible with the orientations of edges* (in each pair of connected edges one is oriented clockwise and the other counterclockwise).



Let us glue together each pair of edges connected by σ. The resulting graph T_σ is a tree (called *quotient tree*) and each edge inherits the orientation from the orientations of the edges in the original graph G.

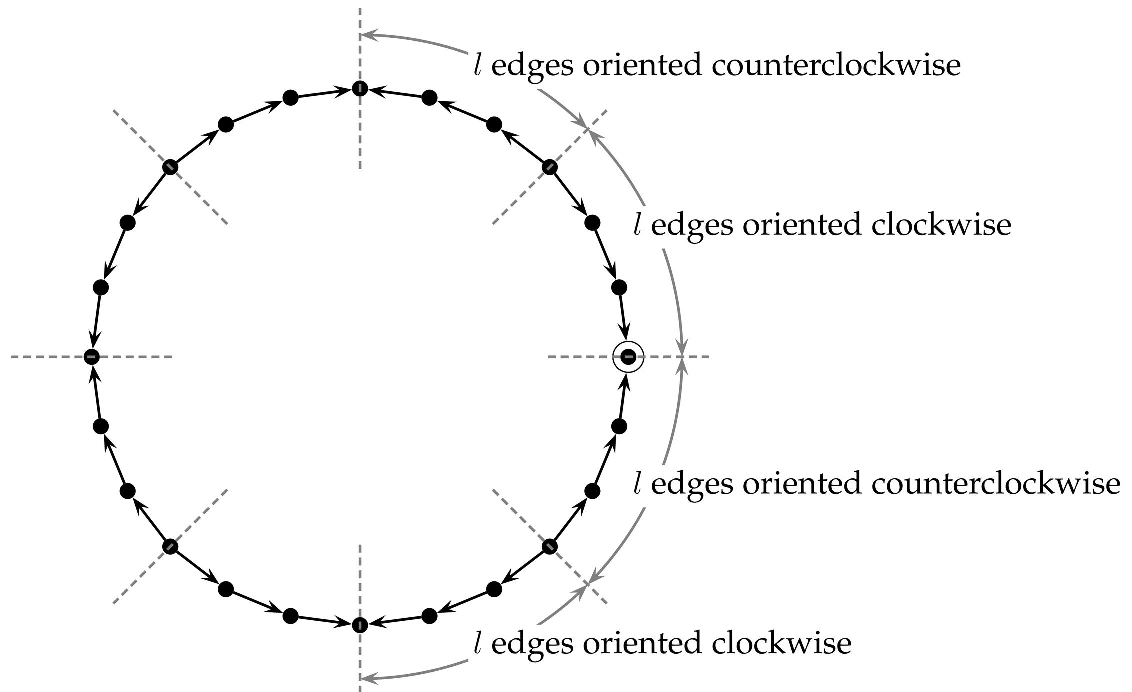
Orders on quotient trees

The orientations of edges define a partial order $\prec$ on the vertices of tree T_σ (convention: if $A \Longleftarrow B$ we write $A \prec B$). In the following we shall consider some total (linear) orders on the vertices of T_σ; we say that such an order $<$ is *compatible with the orientations of edges* if $<$ is an extension of the partial order $\prec$.

Quotient trees considered above naturally have a structure of planar rooted trees with a root R. By $\lhd$ we denote the order on the vertices given by pre-order. For example, in the above case we have $v_1 \lhd v_2 \lhd v_3 \lhd v_5 \lhd v_8$.

Regular polygonal graphs

For integers $l, m \geq 1$ we consider (l, m)*-regular graph*. It is the polygonal graph with $2lm$ edges of the form below. It consists of $2m$ groups of edges, each group consists of l edges with the same orientation, consecutive groups have opposite orientations.



Generalized parking functions

Let integers $l, m \geq 1$ be fixed. We say that $(a_1, \dots, a_{lm+1})$ is an (l, m)*-parking function* if

- $a_1, \dots, a_{lm+1} \in \{1, \dots, m\}$;
- for each $1 \leq n < m$ in the sequence $(a_1, \dots, a_{lm+1})$ there are at most ln elements which belong to $\{1, \dots, n\}$.

Raney lemma implies that the number of (l, m)-parking functions is equal to m^{ml}.

Main result: Bijection between ordered trees and parking functions

Theorem. *Let $l, m \geq 1$ be fixed. The algorithm* `MainBijection` *provides a bijection between*

- *the set of pairs $(T_\sigma, <)$, where T_σ is a quotient tree corresponding to the (l, m)-regular graph and $<$ is a total order on vertices of T_σ compatible with the orientations of edges;*
- *the set of (l, m)-parking functions.*

Corollary: generalized Cauchy identities

$$2^{2l} = \sum_{p+q=l} \binom{2p}{p} \binom{2q}{q}.$$

$$3^{3l} = \sum_{p+q=l} \binom{3p}{p,p,p} \binom{3q}{q,q,q} + 3 \sum_{\substack{p+q+r=l-1 \\ r'+q'=r+q+1 \\ p''+r''=p+r+1}} \binom{2p+p''}{p,p,p''} \binom{2q+q'}{q,q,q'} \binom{r+r'+r''}{r,r',r''}.$$

Auxiliary bijection between ordered trees

Theorem. *Let integers $i, l \geq 1$ be given. The algorithm* `SmallBijection` *provides a bijection between*

- *the set of quotient trees $(T_\sigma, <)$ corresponding to a (l, i)-regular graph equipped with a total order $<$ compatible with the orientation of the edges;*
- *the set of quotient trees $(T_\sigma, <)$ corresponding to a (l, i)-regular graph equipped with a total order $<$ on the vertices with the following two properties:*
  - *on the set $\{x \in T_\sigma : x \succeq R\}$ the orders $<$ and $\lhd$ coincide, where R denotes the root;*
  - *for all pairs of vertices $v, w \in T_\sigma$ such that $R \not\preceq v$ and $R \not\preceq w$ we have $v \prec w \implies v < w$.*

Applications

In the limit $l \to \infty$ orders on trees can be interpreted as stochastic processes in $\mathbb{R}^{m-1}$ (Brownian motions, Brownian bridges). Above bijections give rise to measure-preserving maps related to Pitman transform.

Where is Voiculescu's free probability theory?

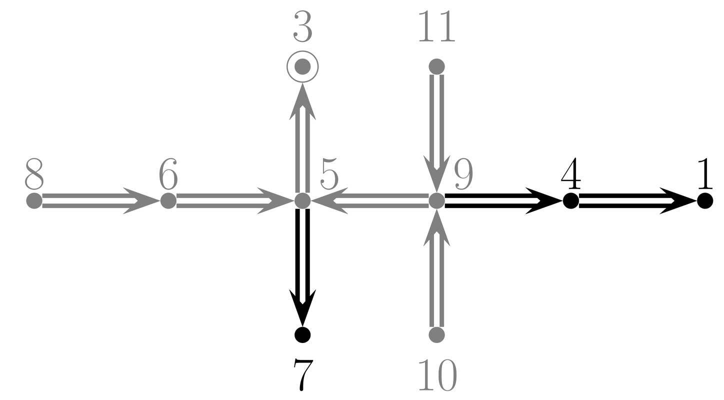
Please, ask me about it!

Main bijection

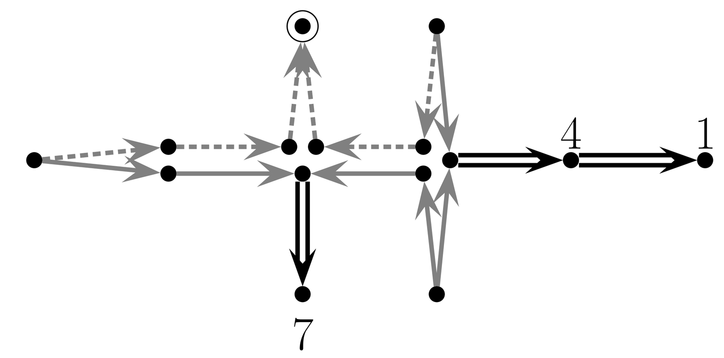
Parking function can be equivalently described as a tuple $(B_1, \dots, B_m)$ of disjoint sets such that $B_1 \cup \dots \cup B_m = \{1, \dots, ml+1\}$ and $|B_1| + \dots + |B_n| \leq ln$ holds true for each $1 \leq n \leq m-1$.

Function `MainBijection(T)`

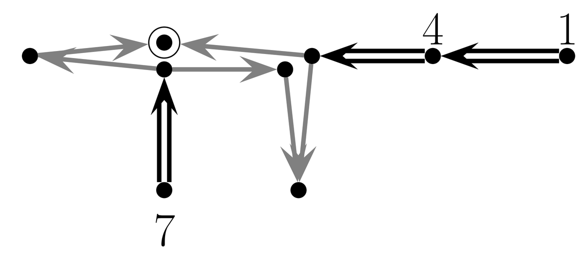
label all vertices of T with numbers $1, \dots, ml+1$ in such a way that each label appears exactly once and the order $<$ of vertices coincides with the order of the labels;
for i=m **downto** 1 **do**
  $T \leftarrow$ `SmallBijection`(T);
  $U \leftarrow$ tree $\{x \in T : x \succeq R\}$ (tree U is marked gray on example below);
  $B_i \leftarrow$ (labels of the vertices of U) $\cap \{1, \dots, ml+1\}$;



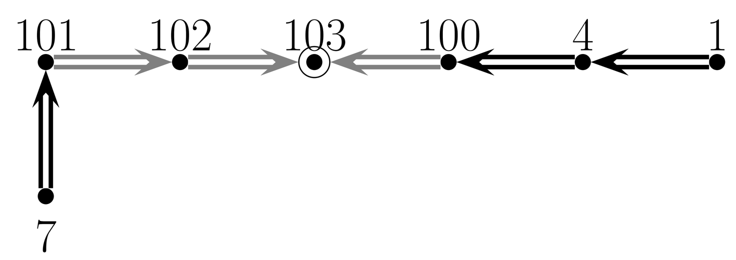
  remove the labels of the vertices of U;
  unglue all edges of tree U;



  remove l edges at each side of the vertex R;
  change the orientation of all edges and reverse the order $<$;



  create sufficiently many artifical labels (integer numbers all different from $1, \dots, ml+1$) which are smaller than any label on tree T;
  glue the remaining edges of tree U in such a way that $R \preceq X$ for every $X \in U$;
  label the unlabeled vertices with artificial labels in such a way that on tree U the orders $<$ and $\lhd$ coincide;

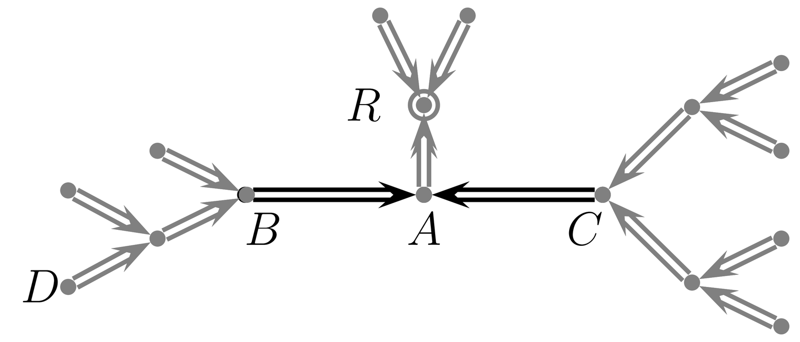


end
return $B_1, \dots, B_m$;

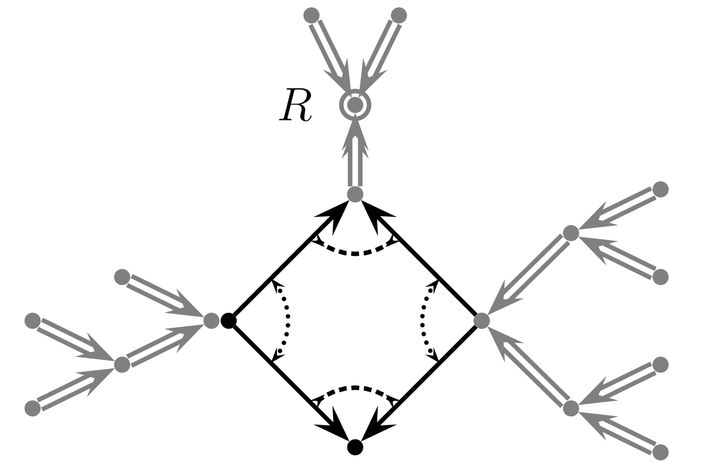
Auxiliary bijection

Function `SmallBijection(T)`

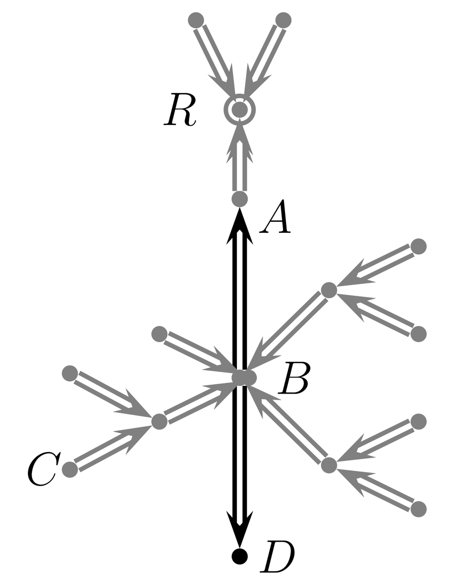
while *orders $<$ and $\lhd$ do not coincide on* $\{x \in T : x \succeq R\}$ **do**
  $D \leftarrow$the minimal element (with respect to $<$) such that $R \prec D$ and orders $<$ and $\lhd$ do not coincide on $\{x \in T : R \preceq x$ and $x \leq D\}$;
  $U \leftarrow$ tree $\{x \in T : R \preceq x$ and $x \leq D\}$;
  $C \leftarrow$the successor of D in U with respect to $\lhd$;
  $A \leftarrow$father of C;
  $B \leftarrow$son of A in U which is to the left of C;
  labels $\leftarrow$set of labels carried by the vertices A, B, C, D;



  remove the labels from the vertices A, B, C, D;
  unglue the edges BA and CA;



  reglue these edges in the other possible way;
  to unlabeled vertices give labels from labels in such a way that for each pair of newly labeled vertices $x < y$ iff $x \lhd y$;



end
return T;

References

[Śni04] Piotr Śniady. Generalized Cauchy identities, trees and multidimensional Brownian motions. Part I: bijective proof of generalized Cauchy identities. Preprint arXiv:math.CO/0412043, 2004.