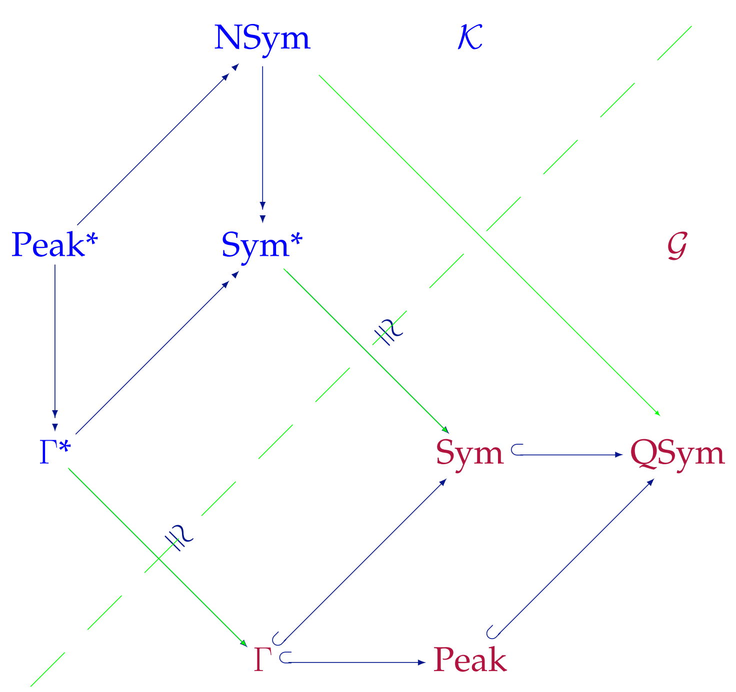
Algebraic Structures on Grothendieck Groups of a Tower of Algebras

HUILAN LI
Department of Mathematics and Statistics, York University, Canada, M3J 1P3, lihuilan@mathstat.yorku.ca



Hopf Algebra	Sym	QSym	Γ	Peak
Duality	Sym	NSym	Γ	Peak*
Tower	$\oplus\mathbb{C}\mathfrak{S}_n$	$\oplus H_n(0)$	$\oplus \mathrm{Se}_n$	$\oplus HCl_n(0)$

1

Let B be a finite-dimensional algebra.

$$G_0(B) = \frac{\text{Span}\{\text{isomophic classes of f.g. } B\text{-modules}\}}{<(M)-(L)-(N)>_{0\to L\to M\to N\to 0 \text{ exact}}} \ni [M]$$

$$K_0(B) = \text{Span}\{\text{isomophic classes of f.g. proj. } B\text{-modules}\} \ni [P]$$

$\{V_1, V_2, \ldots, V_s\}$: complete list of noniso. simple B-module

$\{P_1, P_2, \ldots, P_s\}$: their proj. covers, complete list of noniso. proj. B-module

$G_0(B) = \bigoplus \mathbb{Z}[V_i]$ $\qquad$ $K_0(B) = \bigoplus \mathbb{Z}[P_i]$

2

Graded bialgebra $H = \bigoplus_{n\geq 0} H_n$ $\left\{ \begin{array}{l} \text{Algebra } H_m \otimes H_n \xrightarrow{\pi} H_{m+n} \\ \text{Coalgebra } H_n \xrightarrow{\Delta} \bigoplus_{k+l=n} H_k \otimes H_l \end{array} \right.$

Graded dual $H^{*gr} = \bigoplus_{n\geq 0} H_n^*$ is also a graded bialgebra.

Hopf algebra: biagebra + antipode

Graded connected bialgebra $\Rightarrow$ Hopf algebra

3

$A = \bigoplus_{n\geq 0} A_n$, a tower of algebras:

(1) A_n is a finite-dimensional algebra with unit 1_n, for each n. $A_0 \cong \mathbb{C}$.

(2) $\exists$ an external multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$, for all $m, n \geq 0$,

(a) $\rho_{m,n}$ is injective with $\rho_{m,n}(1_m \otimes 1_n) = 1_{m+n}$;

(b) ρ is associative.

(3) A_{m+n} is a two-sided projective $A_m \otimes A_n$-module for all $m, n \geq 0$.

(4) For every primitive idempotent g in A_{m+n},

$$A_{m+n}g \cong \bigoplus (A_m \otimes A_n)(e \otimes f) \Leftrightarrow gA_{m+n} \cong \bigoplus (e \otimes f)(A_m \otimes A_n).$$

(5) An analogue of Mackey's formula holds for $G_0(A)$ or $K_0(A)$

$$[\mathrm{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}} \mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}(M \otimes N)]$$
$$= \sum_{t+s=k} [\widetilde{\mathrm{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(\mathrm{Res}^{A_m}_{A_t\otimes A_{m-t}} M \otimes \mathrm{Res}^{A_n}_{A_s\otimes A_{n-s}} N)]$$

Implications

(1) $\Rightarrow$ Graded connected

(2), (3) $\Rightarrow$ Inductions and Restrictions well-defined. Furthermore, Multiplications and Comultiplications well-defined.

(4) $\Rightarrow$ Duality

(5) $\Rightarrow$ Compatibility of algebra and coalgebra structures

4

	$\mathcal{G}$	$\mathcal{K}$
Induction	$i_{m,n} : G_0(A_m) \bigotimes_{\mathbb{Z}} G_0(A_n) \to G_0(A_{m+n})$ $[M] \otimes [N] \mapsto [A_{m+n} \bigotimes_{A_m\otimes A_n} (M \otimes N)]$	$i'_{m,n} : K_0(A_m) \bigotimes_{\mathbb{Z}} K_0(A_n) \to K_0(A_{m+n})$ $[P] \otimes [Q] \mapsto [A_{m+n} \bigotimes_{A_m\otimes A_n} (P \otimes Q)]$
Restriction	$r_{k,l} : G_0(A_n) \to G_0(A_k) \bigotimes_{\mathbb{Z}} G_0(A_l)$ with $k + l = n$ $[N] \mapsto [\mathrm{Hom}_{A_n}(A_n, N)]$	$r'_{k,l} : K_0(A_n) \to K_0(A_k) \bigotimes_{\mathbb{Z}} K_0(A_l)$ with $k + l = n$ $[P] \mapsto [\mathrm{Hom}_{A_n}(A_n, P)]$
Grothendieck Group	$\mathcal{G} = G_0(A) = \bigoplus_{n\geq 0} G_0(A_n)$	$\mathcal{K} = K_0(A) = \bigoplus_{n\geq 0} K_0(A_n)$
Multiplication	$\pi : G_0(A) \bigotimes_{\mathbb{Z}} G_0(A) \to G_0(A)$ $\pi\vert_{G_0(A_k)\bigotimes G_0(A_l)} = i_{k,l}$	$\pi' : K_0(A) \bigotimes_{\mathbb{Z}} K_0(A) \to K_0(A)$ $\pi'\vert_{K_0(A_k)\bigotimes K_0(A_l)} = i'_{k,l}$
Comultiplication	$\Delta : G_0(A) \to G_0(A) \bigotimes_{\mathbb{Z}} G_0(A)$ $\Delta\vert_{G_0(A_n)} = \sum_{k+l=n} r_{k,l}$	$\Delta' : K_0(A) \to K_0(A) \bigotimes_{\mathbb{Z}} K_0(A)$ $\Delta'\vert_{K_0(A_n)} = \sum_{k+l=n} r'_{k,l}$
Unit	$\mu : \mathbb{Z} \to G_0(A)$ $\mu(a) = a[K] \in G_0(A_0)$, for $a \in \mathbb{Z}$	$\mu' : \mathbb{Z} \to K_0(A)$ $\mu'(a) = a[K] \in K_0(A_0)$, for $a \in \mathbb{Z}$
Counit	$\epsilon : G_0(A) \to \mathbb{Z}$ $\epsilon([M]) = \begin{cases} a & \text{if } [M] = a[K], a \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$	$\epsilon' : K_0(A) \to \mathbb{Z}$ $\epsilon'([M]) = \begin{cases} a & \text{if } [M] = a[K], a \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

Pairing $\quad <,>: K_0(A) \times G_0(A) \to \mathbb{Z}$, where $<[P],[M]> = \begin{cases} \dim_K(\mathrm{Hom}_{A_n}(P, M)) & \text{if } [P] \in K_0(A_n) \text{ and } [M] \in G_0(A_n) \\ 0 & \text{otherwise} \end{cases}$

RESULT 1 $\qquad$ $G_0(A)$ Hopf $\xleftrightarrow[\text{Dual}]{*}$ $K_0(A)$ Hopf

5

Modify	(2)(a)... $\rho_{m,n}(1_m \otimes 1_n) \neq 1_{m+n}$	
Induction	same	$A_{m+n} \bigotimes_{A_m\otimes A_n} (A_m e \otimes A_n f) = A_{m+n}\rho_{m,n}(e \otimes f)$
Restriction	$\mathrm{Res}^{A_n}_{A_k\otimes A_l} N = \{u \in N : \rho_{k,l}(1_k \otimes 1_l)u = u\} \subseteq N$	$\mathrm{Res}^{A_n}_{A_k\otimes A_l} P = \{p \in P : \rho_{k,l}(1_k \otimes 1_l)p = p\} \subseteq P$

RESULT 2 $\qquad$ $G_0(A)$ Hopf $\xleftrightarrow[\text{Dual}]{*}$ $K_0(A)$ Hopf

6