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The Incidence Algebra of a Composition Poset

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A *composition* is just a sequence $w = k_1 k_2 \dots k_r$ of positive integers. A number of partial orders on the set of all compositions have been studied recently. For example, Björner and Stanley have defined a poset on compositions which has many of the properties of Young's lattice for partitions. We consider a ordering that was first defined by Bergeron, Bousquet-Mélou, and Dulucq: Given $u = k_1 \dots k_r$ and $w = l_1 \dots l_s$ then we have $u \leq w$ if there is a subsequence $l_{i_1} \dots l_{i_r}$ of w which is componentwise bigger than u, i.e.,

$$k_j \leq l_{i_j} \quad \text{for } 1 \leq j \leq r. \tag{1}$$

Call this poset C. It is interesting, in part, because it is related to the poset of all permutations ordered by pattern containment.

In the first half of the talk, we will study the zeta function, ζ, of the incidence algebra $I(C)$. This is joint work with Anders Björner and full details can be found in the paper at

http://www.math.msu.edu/~sagan/Papers/rmf.pdf

If $w = k_1 \dots k_r$ satisfies $\sum_i k_i = N$, then w is said to be a *composition of* N and we write $|w| = N$. Let c_N be the number of compositions of N. It is well known (and easy to prove) that

$$\sum_{N \geq 0} c_N x^N = \frac{1-x}{1-2x} \tag{2}$$

which is a rational function of x. Now given $u \in C$, consider the generating function

$$Z(u) = \sum_{w \geq u} x^{|w|} = \sum_{w \in C} \zeta(u,w) x^{|w|}.$$

So equation 2 is just $Z(\epsilon)$, where ϵ is the empty composition. We show that $Z(u)$ is always a rational function by using techniques from the theory of formal languages. We also investigate similar generating function for powers of ζ. Surprisingly, to evaluate the sums, hypergeometric series identities are needed.

In the second half of the talk, we will study the Möbius function, $\mu = \zeta^{-1}$, in $I(C)$. This is joint work with Vincent Vatter and full details can be found in the paper at

http://www.math.msu.edu/~sagan/Papers/mfc.pdf

A set of indices $I = \{i_1, \dots, i_r\}$ such that 1 holds is called an *embedding* of u into w. We show that $\mu(u,w)$ gives a signed counting of certain embeddings of u into w. In fact, there are three proofs of this result: one combinatorial via an involution, one topological using discrete Morse theory, and one using the machinery discussed in the previous paragraph (this last being work with Björner). We will present the topological proof, giving an introduction to discrete Morse theory in the process.

The results above have analogues in the work Björner, some of it with Reutenauer, on subword order. We will show that both their results and ours are part of a more general framework. In particular, let P be any poset and consider the *Kleene closure* of all words over P:

$$P^* = \{w = k_1 k_2 \dots k_r \; : \; k_i \in P \text{ for all } i \text{ and } r \geq 0\}.$$

Then 1 defines a partial order $u \leq w$ on elements of P^*, where the inequalities $k_j \leq l_{i_j}$ are taken in P. With little extra effort, one can prove theorems about P^* which specialize to those about composition order or about subword order by taking P to be a chain or an antichain, respectively.

We will end with a list of intriguing conjectures and open questions concerning these ideas. As another surprise, the Tchebyshev polynomials of the first kind enter into a conjectured formula for the Möbius function of certain intervals of P^* for a particular 3-element poset, P.

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