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Octahedrons with equally many lattice points and generalizations

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Abstract. While counting lattice points in octahedra of different dimensions n and m, it is an interesting question to ask, how many octahedra exist containing equally many such points. This gives rise to the Diophantine equation $P_n(x) = P_m(y)$ in rational integers x, y, where $\{P_k(x)\}$ denote special Meixner polynomials $\{M_k^{(\beta,c)}(x)\}$ with $\beta = 1$, $c = -1$. We join the purely algebraic criterion of Y. Bilu and R. F. Tichy (*The Diophantine equation* $f(x) = g(y)$, Acta Arith. **95** (2000), no. 3, 261–288) with a famous result of P. Erdös and J. L. Selfridge (*The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292–301) and prove that

$$M_n^{(\beta,c_1)}(x) = M_m^{(\beta,c_2)}(y)$$

with $m, n \geq 3$, $\beta \in \mathbb{Z} \setminus \{0, -1, -2, -\max(n, m) + 1\}$ and $c_1, c_2 \in \mathbb{Q} \setminus \{0, 1\}$ only admits a finite number of integral solutions x, y. Some more results on polynomial families in three-term recurrences are presented.

Résumé. Dans l'étude du dénombrement de sommets d'octaèdres de dimensions n et m se pose la question intéressante de connaître combien d'octaèdres existent possédant le même nombre de sommets. Ce problème se traduit par l'équation diophantienne $P_n(x) = P_m(y)$, avec x, y entiers relatifs et où $\{P_k(x)\}$ sont les polynômes spéciaux de Meixner avec $\beta = 1$, $c = -1$. Nous joignons au critère purement algébrique de Y. Bilu et R. F. Tichy (*The Diophantine equation* $f(x) = g(y)$, Acta Arith. **95** (2000), no. 3, 261–288) un fameux résultat dû à P. Erdös et J. L. Selfridge (*The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292–301) et prouvons que

$$M_n^{(\beta,c_1)}(x) = M_m^{(\beta,c_2)}(y)$$

avec $m, n \geq 3$, $\beta \in \mathbb{Z} \setminus \{0, -1, -2, -\max(n, m) + 1\}$ et $c_1, c_2 \in \mathbb{Q} \setminus \{0, 1\}$ n'admet qu'un nombre fini de solutions entières x, y. De plus, nous présentons quelques résultats portant sur des familles polynômiales avec triple récurrence.

1. Introduction

An n-dimensional octahedron of radius r is the convex body in $\mathbb{R}^n$ defined by $|x_1| + \cdots + |x_n| \leq r$. In this talk we investigate the following problem and some algebraic generalizations:

Problem: *Given distinct positive integers n, m, how often can two octahedrons of dimensions n and m, respectively, contain equally many integral points?*

Obviously, it is sufficient to consider octahedrons of integral radius r. Also, any positive odd number can occur as the number of integers in the "one-dimensional octahedron" $[-r, r]$. Hence, it is natural to assume that $n, m \geq 2$.

Denote by $P_n(r)$ the number of integral points $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ satisfying $|x_1| + \cdots + |x_n| \leq r$. In 1967, Erhardt [**5**] proved that $P_n(r)$ is a polynomial in r of degree n indeed for any general lattice polytope described by

$$\frac{|x_1|}{a_1} + \frac{|x_2|}{a_2} + \cdots + \frac{|x_n|}{a_n} \leq r,$$

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where $a_1, \dots, a_n$ are positive integers. In general, the Ehrhart polynomial is difficult to access and its coefficients involve Dedekind sums and their higher analogues [**1**]. However, in the special case of symmetric octahedra, which we are dealing with here, Kirschenhofer, Pethö and Tichy [**10**] could show that $P_n(r)$ can be made explicit, namely

$$P_n(r) = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{r}{i} = {}_2F_1\left[\begin{matrix} -n, -r \\ 1 \end{matrix} ; 2\right], \tag{1.1}$$

where

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; z\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\, z^k$$

is the Gauss hypergeometric function with $(a)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1)$ the Pochhammer symbol. Thus, the original combinatorial counting problem can be restated by means of a polynomial Diophantine equation:

Problem, restated: *How many solutions $x, y \in \mathbb{Z}$ can the equation $P_n(x) = P_m(y)$ have?*

According to the modern Askey-scheme [**14**] and (1.1), we note that

$$P_k(x) = M_k^{(1,-1)}(x), \tag{1.2}$$

where

$$M_k^{(\beta,c)}(x) = {}_2F_1\left[\begin{matrix} -k, -x \\ \beta \end{matrix} ; 1 - \frac{1}{c}\right]$$

denote the well-known *Meixner* polynomials.

2. Historical remarks

Hajdu [**7, 8**] studied the problem for small n and m. For the cases

$$(n, m) \in \{(3, 2), (4, 2), (6, 2), (4, 3), (6, 4)\}$$

he completely determined all integral solutions of $P_n(x) = P_m(y)$. He also conjectured that the equation has finitely many solutions when $n > m = 2$. This was confirmed by Kirschenhofer, Pethő and Tichy [**10**], who reduced it to the Siegel–Baker theorem about the hyperelliptic equation $y^2 = f(x)$ in order to give a computable bound for integral solutions x, y of the equation $P_n(x) = P_2(y)$. Moreover, finiteness is also shown in the following three cases: $m = 4$; $2 \le m < n \le 103$; $n \not\equiv m \bmod 2$. The two latter results are no longer effective (i.e., no upper bound for x, y can be retrieved from the proof), because they depend on the non-effective Davenport–Lewis–Schinzel [**4**] theorem about the Diophantine equation $f(x) = g(y)$. The general answer to the problem has been obtained in [**2**]:

Theorem 2.1 (Bilu-Stoll-Tichy, 2000). *Let n and m be distinct integers satisfying $m, n \ge 2$. Then the equation*

$$P_n(x) = P_m(y)$$

has only finitely many solutions in rational integers x, y.

In other words, sufficiently large octahedra of distinct dimensions n, m cannot have equally many lattice points. The proof of Theorem 2.1 is based on a non-effective result of Bilu and Tichy [**3**], thus, we cannot make “sufficiently large” more explicit.

3. Generalizations

Several new questions arise in this context. For instance, it is well-known that the general family $\{M_k^{(\beta,c)}(x)\}$ defines a discrete orthogonal polynomial family if and only if $\beta > 0$ and $0 < c < 1$. Since the original case $\beta = 1$, $c = -1$ (see (1.2)) does not fit in, we are interested in a more general statement, which handles both the original and the orthogonal case.

Question 1: *Is it possible to derive a similar result to Theorem 2.1 for more general β and c, including the orthogonal case?*

Furthermore, one may also ask, whether it is possible to replace the family of Meixner polyomials by some other polynomial family $\{p_k(x)\}$. Since orthogonal polynomials are closely related to polynomials in three-term recurrences by Favard's theorem, the following question seems of interest.

Question 2: *Let $\{p_k(x)\}$ be a sequence of polynomials defined by*

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x - c_0 \\ p_{k+1}(x) &= (x - c_k)p_n(x) - d_k p_{k-1}(x), \qquad k = 1, 2, \dots, \end{aligned} \tag{3.1}$$

where c_k and d_k are parameters depending only on k. For which c_k, d_k the equation $p_n(x) = p_m(y)$ only has finitely many integral solutions x, y?

Note again, that by the Askey scheme, the Meixner polynomials satisfy a normalized recurrence relation with $c_k = (k + (k+\beta)c)/(c-1)$ and $d_k = (k(k+\beta-1)c)/(c-1)^2$.

Diophantine equations of the form $p_m(x) = p_n(y)$ with polynomials in three-term recurrences have been studied recently by Kirschenhofer and Pfeiffer [**11, 12**]. They point out several striking connections to enumeration problems (for instance, to permutations with coloured cycles).

4. Main results

4.1. Concerning 'Question 1'. Question 1 is settled by the following result [**17**]:

Theorem 4.1. *Let n and m be distinct integers satisfying $m, n \geq 3$, further let $c_1, c_2 \in \mathbb{Q} \setminus \{0, 1\}$ and $\beta \in \mathbb{Z} \setminus \{0, -1, -2, -\max(n, m) + 1\}$. Then the equation*

$$M_n^{(\beta, c_1)}(x) = M_m^{(\beta, c_2)}(y)$$

has only finitely many solutions in integers x, y.

Denote by $K_n^{(p,N)}(x)$ the two-parametric *Krawtchouk* polynomials given in [**14**]:

$$K_n^{(p,N)}(x) = {}_2F_1\left[\begin{matrix} -n, -x \\ -N \end{matrix} ; \frac{1}{p}\right] \qquad n = 0, 1, 2, \dots, N.$$

Since

$$K_n^{(p,N)}(x) = M_n^{(-N, p/(p-1))}(x),$$

we also have

Theorem 4.2. *Let n and m be distinct integers satisfying $m, n \geq 3$, further let $N \geq \max(m, n)$ and $p_1, p_2 \in \mathbb{Q} \setminus \{0, 1\}$. Then the equation*

$$K_n^{(p_1,N)}(x) = K_m^{(p_2,N)}(y) \tag{4.1}$$

has only finitely many solutions in integers x, y.

4.2. Concerning 'Question 2'. We obtain sufficient conditions on c_k and d_k in order to state an again more general finiteness theorem [**18**]:

Theorem 4.3. *Let $\{p_k(x)\}$ be a polynomial sequence satisfying (3.1). Assume one of the following conditions ($A, B, C \in \mathbb{Q}$)*

(1) $c_0 = A, \quad c_k = A, \quad d_k = B$ *with* $A \neq 0$ *and* $B > 0$,
(2) $c_0 = A + B, \quad c_k = A, \quad d_k = B^2$ *with* $B \neq 0$,
(3) $c_0 = A, \quad c_k = Bk + A, \quad d_k = \frac{1}{4}B^2k^2 + Ck$ *with* $C > -\frac{1}{4}B^2$.

Then the Diophantine equation

$$\mathcal{A}p_m(x) + \mathcal{B}p_n(y) = \mathcal{C}$$

with $m > n \geq 4$, $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{Q}$, $\mathcal{A}\mathcal{B} \neq 0$ has at most finitely many solutions in rational integers x, y.

Note that, for instance, in case (3) there are the six rational parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}, A, B, C$ involved, thus, the generality of Theorem 4.3 should well fit specific combinatorial applications. Furthermore, well-known orthogonal families are covered by the statement. So, for example, in the first case of Theorem 4.3 we deal with (shifted) *Jacobi* polynomials, while the third case corresponds to modified *Hermite* and *Laguerre* polynomials.

5. Methods and tools

5.1. The Bilu-Tichy method. The proofs of Theorem 4.1, Theorem 4.2 and Theorem 4.3 are basically algebraic, as they are based on an explicit algorithmic criterion of Bilu and Tichy [**3**], which only involves knowledge of the coefficients of the polynomials under consideration. In order to state that result, we have to introduce some more notation.

Let $\gamma, \delta \in \mathbb{Q}\backslash\{0\}$, $q, s, t \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{\geq 0}$ and $v(x) \in \mathbb{Q}[x]$ a non-zero polynomial (which may be constant). Further let $D_s(x,\gamma)$ denote the *Dickson* polynomials which can be defined via

$$D_s(x,\gamma) = \sum_{i=0}^{\lfloor s/2 \rfloor} d_{s,i} x^{s-2i} \qquad \text{with} \qquad d_{s,i} = \frac{s}{s-i}\binom{s-i}{i}(-\gamma)^i.$$

The pair $(f(x), g(x))$ or viceversa $(g(x), f(x))$ is called a *standard pair over* $\mathbb{Q}$ if it can be represented by an explicit form listed below. In such a case we call (f, g) a standard pair of the *first*, *second*, *third*, *fourth*, *fifth kind*, respectively.

kind	explicit form of (f,g) resp. (g,f)	parameter restrictions
first	$(x^q, \gamma x^r v(x)^q)$	with $0 \leq r < q,\ (r,q) = 1,\ r + \deg v > 0$
second	$(x^2, (\gamma x^2 + \delta) v(x)^2)$	–
third	$(D_s(x,\gamma^t), D_t(x,\gamma^s))$	with $(s,t) = 1$
fourth	$\left(\gamma^{-s/2} D_s(x,\gamma), -\delta^{-t/2} D_t(x,\delta)\right)$	with $(s,t) = 2$
fifth	$\left((\gamma x^2 - 1)^3, 3x^4 - 4x^3\right)$	–

These standard pairs are important in view of the following characterization result [**3**].

Theorem 5.1 (Bilu-Tichy, 2000). *Let $p(x), q(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent:*

(a) *The equation $p(x) = q(y)$ has infinitely many rational solutions with a bounded denominator.*
(b) *We can express $p \circ \kappa_1 = \phi \circ f$ and $q \circ \kappa_2 = \phi \circ g$ where $\kappa_1, \kappa_2 \in \mathbb{Q}[x]$ are linear, $\phi(x) \in \mathbb{Q}[x]$, and (f, g) is a standard pair over $\mathbb{Q}$.*

If we are able to get contradictions for decompositions of p and q as demanded in (b) of Theorem 5.1 then finiteness of number of integral solutions x, y of the original Diophantine equation $p(x) = q(y)$ is guaranteed. Note that this approach is basically an algebraic one and does involve an accurate comparison of coefficients.

5.2. Erdös-Selfrdige tool. As an additional tool, we restate a well-known result obtained by Erdös and Selfridge [**6**]:

Theorem 5.2 (Erdös-Selfridge, 1975). *The equation*

$$x(x+1)\cdots(x+k-1) = y^l$$

has no solution in rational integers $x > 0$, $k > 1$, $l > 1$, $y > 1$.

Interestingly, simple comparison of the leading coefficients of the *Meixner* polynomials gives an equation very similar to that of Theorem 5.2. Therefore, there are no parameters that satisfy such a coefficient equation. In other words, we can easily derive a contradiction if we suppose a higher degree polynomial representation in Theorem 5.1.

5.3. Lesky tool. There is a close connection beween three-term recurrences and Sturm-Liouville differential equations [**13**]:

Theorem 5.3 (Koepf-Schmersau, 2002). *The following conditions are equivalent:*

(1) *The second-order Sturm-Liouville differential equation ($k \geq 0$)*

$$\sigma(x) p_k''(x) + \tau(x) p_k'(x) - k((k-1)a + d) p_k(x) = 0, \tag{5.1}$$

with $\sigma(x) = ax^2 + bx + c \not\equiv 0$, $\tau = dx + e$, $a, b, c, d, e \in \mathbb{R}$, $d \neq -ta$ for all $t \in \mathbb{Z}_{\geq 0}$ has a (up to a factor depending on k) unique infinite polynomial family solution $\{p_k(x)\}$ of exact degree k.

(2) *The family $\{p_k(x)\}$ satifies a three-term recurrence of type (3.1) with*

$$c_0 = -\frac{e}{d},$$
$$c_k = -\frac{2kb((k-1)a+d) - e(2a-d)}{(2ka+d)((2k-2)a+d)},$$
$$d_k = \frac{k((k-2)a+d)}{((2k-1)a+d)((2k-3)a+d)}\left(-c + \frac{((k-1)b+e)(((k-1)a+d)b-ae)}{((2k-2)a+d)^2}\right).$$

The properties of Theorem 5.3 are shared by all classical orthogonal polynomials (Jacobi, Laguerre, Hermite). On the other hand, one has by Favard's Theorem (see for instance [**19**]), that all polynomial families defined by a three-term recurrence of shape (3.1) are orthogonal with respect to some moment functional. If one demands orthogonality with respect to a positive definite moment functional (in order to use all known facts about zeros of orthogonal polynomials etc.), then one exactly gets only Jacobi, Laguerre and Hermite up to a linear transformation $x \mapsto \nu_1 x + \nu_2$ with $\nu_1, \nu_2 \in \mathbb{R}$ (see the results of Lesky in [**15**]). Hence, one can completely characterize all positive definite orthogonal solutions of (5.1) just by looking at the coefficients a, b, c, d, e (see [**9**]). This can be translated into conditions on c_k and d_k for the general equation

$$\mathcal{A}p_m(x) + \mathcal{B}p_n(y) = \mathcal{C}.$$

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