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# Flag arrangements and triangulations of products of simplices.

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ABSTRACT. We investigate the line arrangement that results from intersecting $d$ complete flags in $\mathbb{C}^n$. We give a combinatorial description of the matroid $\mathcal{T}_{n,d}$ that keeps track of the linear dependence relations among these lines.

We prove that the bases of the matroid $\mathcal{T}_{n,3}$ characterize the triangles with holes which can be tiled with unit rhombi. More generally, we provide evidence for a conjectural connection between the matroid $\mathcal{T}_{n,d}$, the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$, and the arrangements of $d$ tropical hyperplanes in tropical $(n-1)$-space.

Our work provides a simple and effective criterion to ensure the vanishing of many Schubert structure constants in the flag manifold, and a new perspective on Billey and Vakil's method for computing the non-vanishing ones.

RÉSUMÉ. Nous étudions l'arrangement de droites qui résulte de l'intersection de $d$ drapeaux complets dans $\mathbb{C}^n$. Nous donnons une description combinatoire du matroide $\mathcal{T}_{n,d}$ défini par les dépendances linéaires entre ces droites.

Nous démontrons que les bases du matroide $\mathcal{T}_{n,3}$ caractérisent les triangles sans trou qui peuvent être pavés par des losanges unitaires. Plus généralement, nous étayons une relation conjecturale entre le matroide $\mathcal{T}_{n,d}$, les triangulations du produit de simplexes $\Delta_{n-1} \times \Delta_{d-1}$ et les arrangements de $d$ hyperplans tropicaux dans l'espace tropical de dimension $n-1$.

Nos travaux produisent un critère simple et efficace pour déterminer quand de nombreuses constantes de structure de Schubert sont nulles, et une nouvelle façon de voir la méthode de Billey et Vakil pour calculer celles qui sont non-nulles..

## 1. Introduction.

Let $E_\bullet^1, \ldots, E_\bullet^d$ be $d$ generically chosen complete flags in $\mathbb{C}^n$. Write

$$E_\bullet^k = \{\{0\} = E_0^k \subset E_1^k \subset \cdots \subset E_n^k = \mathbb{C}^n\},$$

where $E_i^k$ is a vector space of dimension $i$. Consider the set $\mathbf{E}_{n,d}$ of one-dimensional intersections determined by the flags; that is, all lines of the form $E_{a_1}^1 \cap E_{a_2}^2 \cap \cdots \cap E_{a_d}^d$.

The initial goal of this paper is to characterize the line arrangements $\mathbb{C}^n$ which arise in this way from $d$ generically chosen complete flags. We will then show an unexpected connection between these line arrangements and an important and ubiquitous family of subdivisions of polytopes: the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$. These triangulations appear naturally in studying the geometry of the product of all minors of a matrix [**1**], tropical geometry [**4**], and transportation problems [**17**]. To finish, we will illustrate some of the consequences that the combinatorics of these line arrangements have on the Schubert calculus of the flag variety.

The results of the paper are roughly divided into four parts as follows. First of all, Section 2 is devoted to studying the line arrangement determined by the intersections of a generic arrangement of hyperplanes. This will serve as a warmup before we investigate generic arrangements of complete flags, and the results we obtain will be useful in that investigation.

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The second part consists of Sections 3 and 4, where we will characterize the line arrangements that arise as intersections of a "matroid-generic" arrangement of $d$ flags in $\mathbb{C}^n$. Section 3 is a short discussion of the combinatorial setup that we will use to encode these geometric objects. In Section 4, we propose a combinatorial definition of a matroid $\mathcal{T}_{n,d}$, and show that it is the matroid of the line arrangement of any $d$ flags in $\mathbb{C}^n$ which are generic enough. Finally, we show that these line arrangements are completely characterized combinatorially: any line arrangement in $\mathbb{C}^n$ whose matroid is $\mathcal{T}_{n,d}$ arises as an intersection of $d$ flags.

The third part establishes a surprising connection between these line arrangements and an important class of subdivisions of polytopes. The bases of $\mathcal{T}_{n,3}$ exactly describe the ways of punching $n$ triangular holes into the equilateral triangle of size $n$, so that the resulting holey triangle can be tiled with unit rhombi. A consequence of this is a very explicit geometric representation of $\mathcal{T}_{n,3}$. We show these results in Section 5. We then pursue a higher-dimensional generalization of this result. In Section 6, we suggest that the fine mixed subdivisions of the Minkowski sum $n\Delta_{d-1}$ are an adequate $(d-1)$-dimensional generalization of the rhombus tilings of holey triangles. We give a completely combinatorial description of these subdivisions. Finally, in Section 7, we prove that each pure mixed subdivision of the Minkowski sum $n\Delta_{d-1}$ (or equivalently, each triangulation of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$) gives rise to a basis of $\mathcal{T}_{n,d}$. We conjecture that every basis of $\mathcal{T}_{n,d}$ arises in this way. In fact, we conjecture that every basis of $\mathcal{T}_{n,d}$ arises from a *coherent* subdivision or, equivalently, from an arrangement of $d$ tropical hyperplanes in tropical $(n-1)$-space.

The fourth and last part of the paper, Section 8, presents some of the consequences of our work in the Schubert calculus of the flag variety. We start by recalling Eriksson and Linusson's permutation arrays, and Billey and Vakil's related method for explicitly intersecting Schubert varieties. In Section 8.1 we show how the geometric representation of the matroid $\mathcal{T}_{n,3}$ of Section 5 gives us a new perspective on Billey and Vakil's method for computing the structure constants $c_{uvw}$ of the cohomology ring of the flag variety. Finally, Section 8.2 presents a simple and effective criterion for guaranteeing that many Schubert structure constants are equal to zero.

## 2. The lines in a generic hyperplane arrangement.

Before thinking about flags, let us start by studying the slightly easier problem of understanding the matroid of lines of a generic arrangement of $m$ hyperplanes in $\mathbb{C}^n$. We will start by presenting, in Proposition 2.1, a combinatorial definition of this matroid $\mathcal{H}_{n,m}$. Theorem 2.2 then shows that this is, indeed, the right matroid. As it turns out, this warmup exercise will play an important role in Section 4.

Throughout this section, we will consider an arrangement of $m$ generically chosen hyperplanes $H_1, \ldots, H_m$ in $\mathbb{C}^n$ passing through the origin. For each subset $A$ of $[m]$, let

$$H_A = \bigcap_{a \in A} H_a.$$

By genericity,

$$\dim H_A = \begin{cases} n - |A| & \text{if } |A| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the set $L_{n,m}$ of one-dimensional intersections of the $H_i$s consists of the $\binom{m}{n-1}$ lines $H_A$ for $|A| = n-1$.

There are several "combinatorial" dependence relations among the lines in $L_{n,m}$, as follows. Each $t$-dimensional intersection $H_B$ (where $B$ is an $(n-t)$-subset of $[m]$) contains the lines $H_A$ with $B \subseteq A$. Therefore, in an independent set $H_{A_1}, \ldots, H_{A_k}$ of $L_{n,m}$, we cannot have $t+1$ $A_i$s which contain a fixed $(n-t)$-set $B$.

At first sight, it seems intuitively clear that, in a generic hyperplane arrangement, these will be the only dependence relations among the lines in $L_{n,m}$. This is not as obvious as it may seem: let us illustrate a situation in $L_{4,5}$ which is surprisingly close to a counterexample to this statement. For simplicity, we will draw the projective picture, and denote hyperplanes $H_1, \ldots, H_5$ simply by $1, \ldots, 5$, and an intersection like $H_{124}$ simply by 124.

In Figure 1, we have started by drawing the triangles $T$ and $T'$ with vertices $124, 234, 134$ and $125, 235, 135$, respectively. The three lines connecting the pairs $(124, 125)$, $(234, 235)$ and $(134, 135)$, are the lines 12, 23, and 13, respectively. They intersect at the point 123, so that the triangles $T$ and $T'$ are perspective with respect to this point.

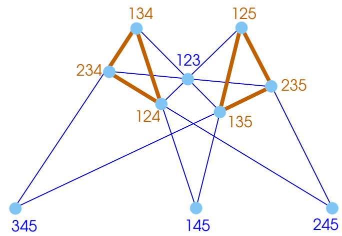


Figure 1. The Desargues configuration in $L_{4,5}$.

Now, Desargues' theorem applies, and it predicts an unexpected dependence relation. It tells us that the three points of intersection of the corresponding sides of $T$ and $T'$ are collinear. The lines 14 (which connects 124 and 134) and 15 (which connects 125 and 135) intersect at the point 145. Similarly, 24 and 25 intersect at 245, and 34 and 35 intersect at 345. Desargues' theorem says that the points 145, 245, and 345 are collinear. In principle, this new dependence relation does not seem to be one of our predicted "combinatorial relations". Somewhat surprisingly, it is: it simply states that these three points are on the line 45.

The previous discussion illustrates two points. First, it shows that Desargues' theorem is really a combinatorial statement about incidence structures, rather than a geometric statement about points on the Euclidean plane. Second, and more important to us, it shows that even five generic hyperplanes in $\mathbb{C}^4$ give rise to interesting geometric configurations. It is not unreasonable to think that larger arrangements $L_{n,m}$ will contain other configurations, such as the Pappus configuration, which have nontrivial and honestly geometric dependence relations that we may not have predicted.

Having told our readers what they might need to worry about, we now intend to convince them not to worry about it.

First we show that the combinatorial dependence relations in $L_{n,m}$ are consistent, in the sense that they define a matroid.

Proposition 2.1. *Let $\mathcal{I}$ consist of the collections $I$ of subsets of $[m]$, each containing $n-1$ elements, such that no $t+1$ of the sets in $I$ contain an $(n-t)$-set. In symbols,*

$$\mathcal{I} := \left\{ I \subseteq \binom{[m]}{n-1} \text{ such that for all } S \subseteq I,\ \Big|\bigcap_{A \in S} A\Big| \leq n - |S| \right\}.$$

*Then $\mathcal{I}$ is the collection of independent sets of a matroid $\mathcal{H}_{n,m}$.*

Proof. Omitted. $\square$

Then we show that this matroid $\mathcal{H}_{n,m}$ is the one determined by the lines in a generic hyperplane arrangement.

Theorem 2.2. *If a central*[1] *hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{C}^n$ is generic enough, then the matroid of the $\binom{m}{n-1}$ lines $H_A$ is isomorphic to $\mathcal{H}_{n,m}$.*

Proof. We already observed that the one-dimensional intersections of $\mathcal{A}$ satisfy all the dependence relations of $\mathcal{H}_{n,m}$. Now we wish to show that, if $\mathcal{A}$ is "generic enough", these are the only relations.

It is enough to construct one "generic enough" hyperplane arrangement, and we do it as follows. Consider the $m$ coordinate hyperplanes in $\mathbb{C}^m$, numbered $J_1, \ldots, J_m$. Pick a sufficiently generic $n$-dimensional subspace $V$ of $\mathbb{C}^m$, and consider the ($(n-1)$-dimensional) hyperplanes $H_1 = J_1 \cap V, \ldots, H_m = J_m \cap V$ in $V$. The theory of Dilworth truncations of matroids precisely guarantees that $V$ can be chosen in such a way that the lines determined by the $H_i$s satisfy no new relations. We omit the details. $\square$

## 3. From lines in a flag arrangement to lattice points in a simplex.

Having understood the matroid of lines in a generic hyperplane arrangement, we proceed to study the case of complete flags. In the following two sections, we will describe the matroid of lines of a generic arrangement of $d$ complete flags in $\mathbb{C}^n$. We start, in this section, with a short discussion of the combinatorial

[1] A hyperplane arrangement is *central* if all its hyperplanes go through the origin.

setup that we will use to encode these geometric objects. We then propose, in Section 4, a combinatorial definition of the matroid $\mathcal{T}_{n,d}$, and show that this is, indeed, the matroid we are looking for.

Let $E^1_\bullet, \ldots, E^d_\bullet$ be $d$ generically chosen complete flags in $\mathbb{C}^n$. Write

$$E^k_\bullet = \{\{0\} = E^k_0 \subset E^k_1 \subset \cdots \subset E^k_n = \mathbb{C}^n\},$$

where $E^k_i$ is a vector space of dimension $i$.

These $d$ flags determine a line arrangement $\mathbf{E}_{n,d}$ in $\mathbb{C}^n$ as follows. Look at all the possible intersections of the subspaces under consideration; they are of the form $E_{a_1,\ldots,a_d} = E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d}$. We are interested in the one-dimensional intersections. Since the $E^k_\bullet$s were chosen generically, $E_{a_1,\ldots,a_d}$ has codimension $(n-a_1)+\ldots+(n-a_d)$ (or $n$ if this sum exceeds $n$). Therefore, the one-dimensional intersections are the lines $E_{a_1,\ldots,a_d}$ for $a_1+\cdots+a_d = (d-1)n+1$. There are $\binom{n+d-2}{d-1}$ such lines, corresponding to the ways of writing $n-1$ as a sum of $d$ nonnegative integers $n-a_1, \ldots, n-a_d$.

Let $T_{n,d}$ be the set of lattice points in the following $(d-1)$-dimensional simplex in $\mathbb{R}^d$:

$$\{\,(x_1,\ldots,x_d) \in \mathbb{R}^d \mid x_1+\cdots+x_d = n-1 \text{ and } x_i \geq 0 \text{ for all } i\}.$$

The $d$ vertices of this simplex are $(n-1,0,0,\ldots,0), (0,n-1,0,\ldots,0), \ldots, (0,0,\ldots,n-1)$.

For example, $T_{n,3}$ is simply a triangular array of dots *of size* $n$; that is, with $n$ dots on each side. We will call $T_{n,d}$ the $(d-1)$-*simplex of size* $n$.

It will be convenient to identify the line $E_{a_1,\ldots,a_d}$ (where $a_1+\cdots+a_d = (d-1)n+1$ and $1 \leq a_i \leq n$) with the vector of codimensions $(n-a_1,\ldots,n-a_d)$. This clearly gives us a one-to-one correspondence between the set $T_{n,d}$ and the lines in our line arrangement $\mathbf{E}_{n,d}$.

We illustrate this correspondence for $d=3$ and $n=4$ in Figure 2. This picture is easier to visualize in real projective 3-space. Now each one of the flags $E_\bullet$, $F_\bullet$, and $G_\bullet$ is represented by a point in a line in a plane. The lines in our line arrangement are now the 10 intersection points we see in the picture.

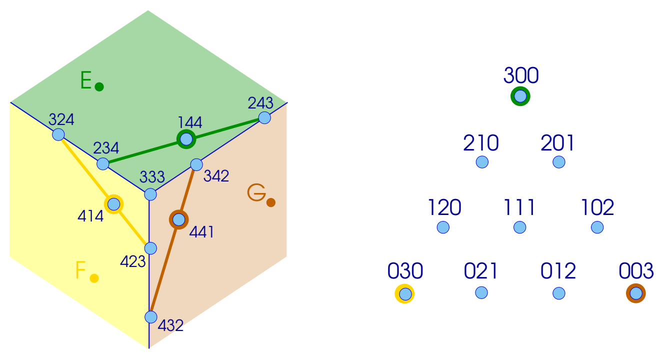


Figure 2. The lines determined by three flags in $\mathbb{C}^4$, and the array $T_{4,3}$.

We are interested in the dependence relations among the lines in the line arrangement $\mathbf{E}_{n,d}$. As in the case of hyperplane arrangements, there are several *combinatorial relations* which arise as follows. Consider a $k$-dimensional subspace $E_{b_1,\ldots,b_d}$ with $b_1+\cdots+b_d = (d-1)n+k$. Every line of the form $E_{a_1,\ldots,a_d}$ with $a_i \leq b_i$ is in this subspace, so no $k+1$ of them can be independent. The corresponding points $(n-a_1,\ldots,n-a_d)$ are the lattice points inside a parallel translate of $T_{k,d}$, the simplex of size $k$, in $T_{n,d}$. In other words, in a set of independent lines of our arrangement, we cannot have more than $k$ lines whose corresponding dots are in a simplex of size $k$ in $T_{n,d}$.

For example, no four of the lines $E_{144}, E_{234}, E_{243}, E_{324}, E_{333}$, and $E_{342}$ are independent, because they are in the 3-dimensional hyperplane $E_{344}$. The dots corresponding to these six lines form the upper $T_{3,3}$ found in our $T_{4,3}$.

In principle, there could be other hidden dependence relations among the lines in $\mathbf{E}_{n,d}$. The goal of the next section is to show that this is not the case. In fact, these combinatorial relations are the only dependence relations of the line arrangement associated to $d$ generically chosen flags in $\mathbb{C}^n$.

We will proceed as in the case of hyperplane arrangements. We will start by showing that the combinatorial relations do give rise to a matroid $\mathcal{T}_{n,d}$. We will then show that this is, indeed, the matroid we are looking for.

## 4. The lines in a generic flag arrangement.

We first show that the combinatorial dependence relations defined in Section 3 do determine a matroid.

THEOREM 4.1. *Let $\mathcal{I}_{n,d}$ be the collection of subsets $I$ of $T_{n,d}$ such that every parallel translate of $T_{k,d}$ contains at most $k$ points of $I$, for every $k \leq n$.*

*Then $\mathcal{I}_{n,d}$ is the collection of independent sets of a matroid $\mathcal{T}_{n,d}$ on the ground set $T_{n,d}$.*

PROOF. Omitted. □

We now show that the matroid $\mathcal{T}_{n,d}$ of Section 4 is, indeed, the matroid that arises from intersecting $d$ flags in $\mathbb{C}^n$ which are generic enough.

THEOREM 4.2. *If $d$ complete flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$ are generic enough, then the matroid of the $\binom{n+d-2}{d-1}$ lines $E_{a_1,\ldots,a_d}$ is isomorphic to $\mathcal{T}_{n,d}$.*

PROOF. As mentioned in Section 3, the one-dimensional intersections of the $E^i_\bullet$s satisfy the following combinatorial relations: each $k$ dimensional subspace $E_{b_1\ldots b_d}$ with $b_1 + \cdots + b_d = (d-1)n + k$, contains the lines $E_{a_1\ldots a_d}$ with $a_i \leq b_i$; therefore, it is impossible for $k+1$ of these lines to be independent. The subspace $E_{b_1\ldots b_d}$ corresponds to the simplex of dots which is labelled $T_{n-b_1,\ldots,n-b_d}$, and has size $n - \sum(n-b_i) = k$. The lines $E_{a_1\ldots a_d}$ with $a_i \leq b_i$ correspond precisely the dots in this copy of $T_{k,d}$. So these "combinatorial relations" are precisely the dependence relations of $\mathcal{T}_{n,d}$.

Now we need to show that, if the flags are "generic enough", these are the only linear relations among these lines. It is enough to construct one set of flags which satisfies no other relations.

Consider a set $\mathcal{H}$ of $d(n-1)$ hyperplanes $H^i_j$ in $\mathbb{C}^n$ (for $1 \leq i \leq d$ and $1 \leq j \leq n-1$) which are generic in the sense of Theorem 2.2, so the only dependence relations among their one-dimensional intersections are the combinatorial ones. Now, for $i = 1, \ldots, d$, define the flag $E^i_\bullet$ by:

$$\begin{aligned}
E^i_{n-1} &= H^i_{n-1} \\
E^i_{n-2} &= H^i_{n-1} \cap H^i_{n-2} \\
&\vdots \\
E^i_1 &= H^i_{n-1} \cap H^i_{n-2} \cap \cdots \cap H^i_1,
\end{aligned}$$

We show that these $d$ flags are generic enough; in other words, the matroid of their one-dimensional intersections is $\mathcal{T}_{n,d}$. We omit the details. □

With Theorem 4.2 in mind, we will say that the complete flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$ are *matroid-generic* if the matroid of the $\binom{n+d-2}{d-1}$ lines $E_{a_1,\ldots,a_d}$ is isomorphic to $\mathcal{T}_{n,d}$.

We conclude this section by showing that the one-dimensional intersections of matroid-generic flag arrangements are completely characterized by their combinatorial properties.

PROPOSITION 4.3. *If a line arrangement $\mathcal{L}$ in $\mathbb{C}^n$ has matroid $\mathcal{T}_{n,d}$, then it can be realized as the arrangement of one-dimensional intersections of $d$ complete flags in $\mathbb{C}^n$.*

PROOF. Omitted. □

## 5. Rhombus tilings of holey triangles and the matroid $\mathcal{T}_{n,3}$.

Let us change the subject for a moment.

Let $T(n)$ be an equilateral triangle with side length $n$. Suppose we wanted to tile $T(n)$ using unit rhombi with angles equal to $60°$ and $120°$. It is easy to see that this task is impossible, for the following reason. Cut $T(n)$ into $n^2$ unit equilateral triangles; $n(n+1)/2$ of these triangles point upward, and $n(n-1)/2$ of them point downward. Since a rhombus always covers one upward and one downward triangle, we cannot use them to tile $T(n)$.

Suppose, then, that we make $n$ holes in the triangle $T(n)$, by cutting out $n$ of the upward triangles. Now we have an equal number of upward and downward triangles, and it may or may not be possible to tile the remaining shape with rhombi.

The main question we address in this section is the following:

QUESTION 5.1. *Given $n$ holes in $T(n)$, is there a simple criterion to determine whether there exists a rhombus tiling of the holey triangle that remains?*

A rhombus tiling is equivalent to a perfect matching between the upward triangles and the downward triangles. Hall's theorem then gives us an answer to Question 5.1: It is necessary and sufficient that any $k$ downward triangles have a total of at least $k$ upward triangles to match to.

However, the geometry of $T(n)$ allows us to give a simpler criterion. Furthermore, in view of Theorem 4.1, this criterion reveals an unexpected connection between these rhombus tilings and the line arrangement determined by 3 generically chosen flags in $\mathbb{C}^n$.

THEOREM 5.2. *Let $S$ be a set of $n$ holes in $T(n)$. The triangle $T(n)$ with holes at $S$ can be tiled with rhombi if and only if every $T(k)$ in $T(n)$ contains at most $k$ holes, for all $k \leq n$.*

PROOF. Omitted. □

COROLLARY 5.3. *The possible locations of $n$ holes for which a rhombus tiling of the holey triangle $T(n)$ exists correspond to the bases of the matroid $\mathcal{T}_{n,3}$.*

PROOF. This is just a restatement of Theorem 5.2. □

Corollary 5.3 allows us to say more about the structure of the matroid $\mathcal{T}_{n,3}$. We first remind the reader of the definition of an important family of matroids, called *cotransversal* matroids. For more information, we refer the reader to [**13**].

Let $G$ be a directed graph with vertex set $V$, and let $A = \{v_1, \dots, v_r\}$ be a subset of $V$. We say that an $r$-subset $B$ of $V$ *can be linked to* $A$ if there exist $r$ vertex-disjoint directed paths whose initial vertex is in $B$ and whose final vertex is in $A$. We will call these $r$ paths a *routing* from $B$ to $A$. The collection of $r$-subsets which can be linked to $A$ are the bases of a matroid denoted $L(G, A)$. Such a matroid is called a *strict gammoid* or a *cotransversal* matroid.

THEOREM 5.4. *The matroid $\mathcal{T}_{n,3}$ is cotransversal.*

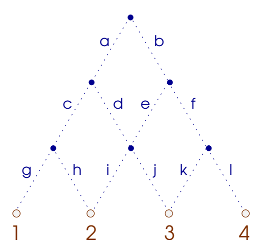


FIGURE 3. The graph $G_4$.

PROOF. Let $G_n$ be the directed graph whose set of vertices is the triangular array $T_{n,3}$, where each dot not on the bottom row is connected to the two dots directly below it. Label the dots on the bottom row $1, 2, \dots, n$. Figure 3 shows $G_4$; all the edges of the graph point down.

There is a bijection between the rhombus tilings of the holey triangles of size $n$, and the routings (sets of $n$ non-intersecting paths) in the graph $G_n$ which end at vertices $1, 2, \dots, n$. This correspondence is best understood in an example; see Figure 4. We leave it to the reader to check the details.

In this correspondence, the holes of the holey triangle correspond to the starting points of the $n$ paths in the graph. From Corollary 5.3, it follows that $\mathcal{T}_{n,3}$ is the cotransversal matroid $L(G_n, [n])$. □

THEOREM 5.5. *Assign algebraically independent weights to the edges of $G_n$.*[2] *For each dot $D$ in the triangular array $T_{n,3}$ and each $1 \leq i \leq n$, let $v_{D,i}$ be the sum of the weights of all paths*[3] *from dot $D$ to dot $i$ on the bottom row.*

*Then the* path vectors $v_D = (v_{D,1}, \dots, v_{D,n})$ *are a geometric representation of the matroid $T_{n,3}$.*

[2]Integer weights which increase extremely quickly will also work.

[3]The weight of a path is defined to be the product of the weights of its edges.

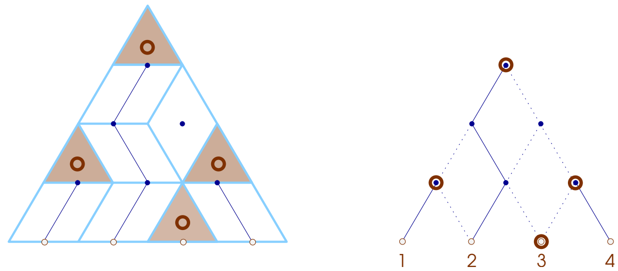


FIGURE 4. A tiling of a holey $T(4)$ and the corresponding routing of $G_4$.

For example, the top dot of $T_{4,3}$ in Figure 3 would be assigned the *path vector* $(acg, ach+adi+bei, adj+bej+bfk, bfl)$ Similarly, focusing our attention on the top three rows, the representation we obtain for the matroid $\mathcal{T}_{3,3}$ is given by the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & c & 0 & ac \\ 0 & 1 & 0 & d & e & ad+be \\ 0 & 0 & 1 & 0 & f & bf \end{pmatrix}$$

PROOF OF THEOREM 5.5. This is a consequence of the Lindström-Gessel-Viennot lemma [**7, 8, 10, 12**]; we omit the details. □

The very simple and explicit representation of $\mathcal{T}_{n,3}$ of Theorem 5.5 will be shown in Section 8 to have an unexpected consequence in the Schubert calculus: it provides us with a reasonably efficient method for computing Schubert structure constants in the flag variety.

## 6. Fine mixed subdivisions of $n\Delta_{d-1}$ and triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

The surprising relationship between the geometry of three flags in $\mathbb{C}^n$ and the rhombus tilings of holey triangles is useful to us in two ways: it explains the structure of the matroid $\mathcal{T}_{n,3}$, and it clarifies the conditions for a rhombus tiling of such a region to exist. We now investigate a similar connection between the geometry of $d$ flags in $\mathbb{C}^n$, and certain well-studied $(d-1)$-dimensional analogs of these tilings.

Instead of thinking of rhombus tilings of a holey triangle, it will be slightly more convenient to think of them as *lozenge tilings* of the triangle: these are the tilings of the triangle using unit rhombi and upward unit triangles. A good high-dimensional analogue of the lozenge tilings of the triangle $n\Delta_2$ are the *fine mixed subdivisions* of the simplex $n\Delta_{d-1}$; we briefly recall their definition. Define a *fine mixed cell* of the simplex $\Delta_{d-1}$ to be a Minkowski sum $B_1 + \cdots + B_n$, where the $B_i$s are faces of $\Delta_{d-1}$ which lie in independent affine subspaces, and whose dimensions add up to $d-1$. A *fine mixed subdivision* of $n\Delta_{d-1}$ is a subdivision of $n\Delta_{d-1}$ into fine mixed cells[**15**, Theorem 2.6].

In the same way that we identified arrays of triangles with triangular arrays of dots in Section 5, we can identify the array of possible locations of the simplices in $n\Delta_{d-1}$ with the array of dots $T_{n,d}$ defined in Section 3. A conjectural generalization of Corollary 5.3, which we now state, would show that fine mixed subdivisions of $n\Delta_{d-1}$ are also closely connected to the matroid $\mathcal{T}_{n,d}$.

CONJECTURE 6.1. *The possible locations of the simplices in a fine mixed subdivision of $n\Delta_{d-1}$ are precisely the bases of the matroid $\mathcal{T}_{n,d}$.*

In the remainder of this section, we will give a completely combinatorial description of the fine mixed subdivisions of $n\Delta_{d-1}$. We will use this description to prove one direction of this conjecture in Section 7.

We start by recalling the one-to-one correspondence between the fine mixed subdivisions of $n\Delta_{d-1}$ and the triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$. This equivalent point of view has the drawback of bringing us to a higher-dimensional picture. Its advantage is that it simplifies greatly the combinatorics of the tiles, which are now just simplices.

Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_d$ be the vertices of $\Delta_{n-1}$ and $\Delta_{d-1}$, so that the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ are of the form $v_i \times w_j$. A triangulation $T$ of $\Delta_{n-1} \times \Delta_{d-1}$ is given by a collection of simplices. For each simplex $t$ in $T$, consider the fine mixed cell whose $i$-th summand is $w_a w_b \ldots w_c$, where $a, b, \ldots, c$ are the indexes $j$ such that $v_i \times w_j$ is a vertex of $t$. These fine mixed cells constitute the fine mixed subdivision of

$n\Delta_{d-1}$ corresponding to $T$. (This bijection is only a special case of the more general Cayley trick, which is discussed in detail in [**15**].)

For instance, Figure 5 shows a triangulation of the triangular prism $\Delta_1 \times \Delta_2 = 12 \times ABC$, and the corresponding fine mixed subdivision of $2\Delta_2$, whose three tiles are $ABC + B$, $AC + AB$, and $C + ABC$.

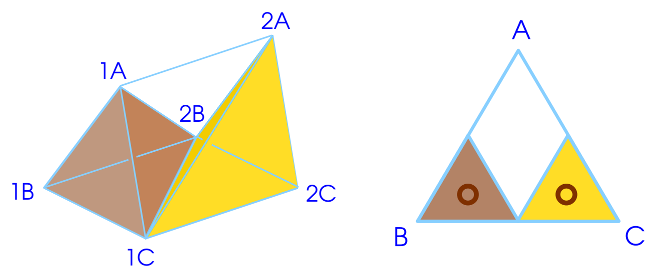


FIGURE 5. The Cayley trick.

Consider the complete bipartite graph $K_{n,d}$ whose vertices are $v_1, \ldots, v_n$ and $w_1, \ldots, w_d$. Each vertex of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to an edge of $K_{n,d}$. The vertices of each simplex in $\Delta_{n-1} \times \Delta_{d-1}$ determine a subgraph of $K_{n,d}$. Each triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is then encoded by a collection of subgraphs of $K_{n,d}$. Figure 6 shows the three trees that encode the triangulation of Figure 5.

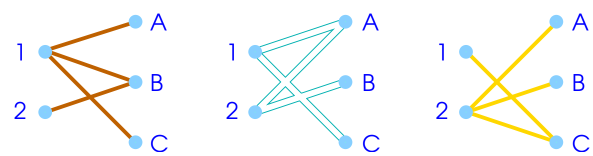


FIGURE 6. The trees corresponding to the triangulation of Figure 5.

Our next result is a combinatorial characterization of the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

PROPOSITION 6.2. *A collection of subgraphs $t_1, \ldots, t_k$ of $K_{n,d}$ encodes a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ if and only if:*

(1) *Each $t_i$ is a spanning tree.*
(2) *For each $t_i$ and each internal[4] edge $e$ of $t_i$, there exists an edge $f$ and a tree $t_j$ with $t_j = t_i - e \cup f$.*
(3) *There do not exist two trees $t_i$ and $t_j$, and a circuit $C$ of $K_{n,d}$ which alternates between edges of $t_i$ and edges of $t_j$.*

PROOF. Omitted. □

In light of Proposition 6.2, we will call a collection of spanning trees satisfying the above properties a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

## 7. Subdivisions of $n\Delta_{d-1}$ and the matroid $\mathcal{T}_{n,d}$.

Having given a combinatorial characterization of the triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ in Proposition 6.2, we are now in a position to prove the forward direction of Conjecture 6.1, which relates these triangulations to the matroid $\mathcal{T}_{n,d}$. The following combinatorial lemma will play an important role in our proof.

PROPOSITION 7.1. *Let $n, d$, and $a_1, \ldots, a_d$ be non-negative integers such that $a_1 + \cdots + a_d \le n - 1$. Suppose we have a coloring of the $n(n-1)$ edges of the directed complete graph $K_n$ with $d$ colors, such that each color defines a poset on $[n]$; in other words,*

(a) *the edges $u \to v$ and $v \to u$ have different colors, and*
(b) *if $u \to v$ and $v \to w$ have the same color, then $u \to w$ has that same color.*

[4] An edge of a tree is *internal* if it is not a leaf.

*Call a vertex $v$* outgoing *if, for every $i$, there exist at least $a_i$ vertices $w$ such that $v \to w$ has color $i$. Then the number of outgoing vertices is at most $n - a_1 - \cdots - a_d$.*

PROOF. Omitted. The intuition is the following. We have $d$ poset structures on the set $[n]$, and this statement essentially says that we cannot have "too many" elements which are "very large" in all the posets. $\square$

We have now laid down the necessary groundwork to prove one direction of Conjecture 6.1.

PROPOSITION 7.2. *In any fine mixed subdivision of $n\Delta_{d-1}$,*
- (a) *there are exactly $n$ tiles which are simplices, and*
- (b) *the locations of the $n$ simplices give a basis of the matroid $\mathcal{T}_{n,d}$.*

PROOF OF PROPOSITION 7.2. Let us look back at the way we defined the correspondence between a triangulation $T$ of $\Delta_{n-1} \times \Delta_{d-1}$ and a fine mixed subdivision $f(T)$ of $n\Delta_{d-1}$. It is clear that the simplices $f(t)$ of $f(T)$ arise from those simplices $t$ of $T$ whose vertices are $v_i \times w_1, \ldots, v_i \times w_d$ (for some $i$), and one $v_j \times w_{g(j)}$ for each $j \neq i$. Furthermore, the location of $f(t)$ in $n\Delta_{d-1}$ is given by the sum of the $w_{g(j)}$s.

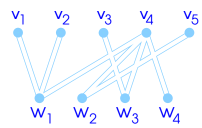


FIGURE 7. A spanning tree of $K_{5,4}$.

For instance the spanning tree of $K_{5,4}$ shown in Figure 7 gives rise to a simplex in a fine mixed subdivision of $5\Delta_3 = 5w_1w_2w_3w_4$ given by the Minkowski sum $w_1 + w_1 + w_3 + w_1w_2w_3w_4 + w_2$. The location of this simplex in $5\Delta_3$ corresponds to the point $(2,1,1,0)$ of $T_{5,4}$, because the Minkowski sum above contains two $w_1$ summands, one $w_2$, and one $w_3$.

The simplices of the fine mixed subdivision of $n\Delta_{d-1}$ come from spanning trees $t$ of $K_{n,d}$ for which one vertex $v_i$ has degree $d$ and the other $v_j$s have degree 1. The coordinates of the location of $f(t)$ in $n\Delta_{d-1}$ are simply $(\deg_t w_1 - 1, \ldots, \deg_t w_d - 1)$. Call such a simplex, and the corresponding tree, *$i$-pure*. For instance, in Figure 5, there is a 1-pure tree and a 2-pure tree, which give simplices in locations $(0,1,0)$ and $(0,0,1)$ of $2\Delta_2$, respectively.

*Proof of (a).* We prove that in a triangulation $T$ of $\Delta_{n-1} \times \Delta_{d-1}$ there is exactly one $i$-pure simplex for each $i$ with $1 \leq i \leq n$. The details are omitted.

*Proof of (b).* The idea is to construct a coloring of the directed complete graph $K_n$ which economically stores a description of the $n$ pure trees, and invoke Proposition 7.1. Again, we omit the details. $\square$

For the converse of Conjecture 6.1, we would need to show that every basis of $\mathcal{T}_{n,d}$ arises from a fine mixed subdivision of $n\Delta_{d-1}$. We conjecture a stronger result.

CONJECTURE 7.3. *For any basis $B$ of $\mathcal{T}_{n,d}$, there is a* **coherent** *fine mixed subdivision of $n\Delta_{d-1}$ whose $n$ simplices are located at $B$.*

Given the correspondence between coherent fine mixed subdivisions of $n\Delta_{d-1}$ and the combinatorial types of arrangements of $d$ generic tropical hyperplanes in tropical $(n-1)$-space [**4, 15**], Conjecture 7.3 is an invitation to study more closely those combinatorial types. This can naturally be thought of as the study of *tropical oriented matroids.*

## 8. Applications to Schubert calculus.

In this section, we show some of the implications of our work in the Schubert calculus of the flag variety. Throughout this section, we will assume some familiarity with the Schubert calculus, though we will recall some of the definitions and conventions that we will use; for more information, see for example [**6, 11**]. We will also need some of the results of Eriksson and Linusson [**5**] and Billey and Vakil [**2**] on Schubert varieties and permutation arrays.

Eriksson and Linusson [**5**] introduced certain higher-dimensional analogs of permutation matrices, called *permutation arrays.* A permutation array is an array of dots in the cells of a $d$-dimensional $n \times n \times \cdots \times n$

box, satisfying some quite restrictive properties. From a permutation array $P$, via a simple combinatorial rule, one can construct a *rank array* of integers, also of shape $[n]^d$. We denote it $\operatorname{rk} P$.

This definition is motivated by the observation that the relative position of $d$ flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathcal{F}\ell_n$ is described by a unique *permutation array* $P$, via the equations

$$\dim\bigl(E^1_{x_1} \cap \cdots \cap E^d_{x_d}\bigr) = \operatorname{rk} P[x_1, \ldots, x_d] \qquad \text{for all } \ 1 \le x_1, \ldots, x_d \le n.$$

This result initiated the study of *permutation array schemes*, which generalize Schubert varieties in the flag variety $\mathcal{F}\ell_n$.

The relative position of $d$ generic flags is described by the *transversal permutation array*

$$\Bigl\{(x_1, \ldots, x_d) \in [n]^d \mid \sum_{i=1}^{d} x_i = (d-1)n + 1\Bigr\}.$$

The dot at position $(x_1, \ldots, x_d)$ represents a one-dimensional intersection $E^1_{x_1} \cap \cdots \cap E^d_{x_d}$. Naturally, we identify the dots in the transversal permutation array with the corresponding element of the matroid $\mathcal{T}_{n,d}$.

Given a fixed flag $E_\bullet$ in $\mathbb{C}^n$ and a permutation $w$ in $S_n$, denote the *Schubert cell* and *Schubert variety* by

$$\begin{aligned}
X^\circ_w(E_\bullet) &= \{F_\bullet \mid E_\bullet \text{ and } F_\bullet \text{ have relative position } w\} \\
&= \{F_\bullet \mid \dim(E_i \cap F_j) = \operatorname{rk} w[i,j] \ \text{ for all } \ 1 \le i, j \le n.\}, \ \text{ and} \\
X_w(E_\bullet) &= \{F_\bullet \mid \dim(E_i \cap F_j) \ge \operatorname{rk} w[i,j] \ \text{ for all } \ 1 \le i, j \le n.\},
\end{aligned}$$

respectively.

A *Schubert problem* asks for the number of flags $F_\bullet$ whose relative positions with respect to $d$ given fixed flags $E^1_\bullet, \ldots, E^d_\bullet$ are given by the permutations $w^1, \ldots, w^d$. This question only makes sense when

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is 0-dimensional; that is, when $l(w^1) + \cdots + l(w^d) = \binom{n}{2}$. If $E^1_\bullet, \ldots, E^d_\bullet$ are sufficiently generic, the intersection $X$ has a fixed number of points $c_{w^1 \ldots w^d}$ which only depends on the permutations $w^1, \ldots, w^d$.

This question is a fundamental one for several reasons. The numbers $c_{w^1 \ldots w^d}$ which answer this question appear in several different contexts. For instance, the cycles $[X_w]$ corresponding to the Schubert varieties form a $\mathbb{Z}$-basis for the cohomology ring of the flag variety $\mathcal{F}\ell_n$, and the numbers $c_{uvw}$ are the multiplicative structure constants. (For this reason, if we know the answer to all Schubert problems with $d = 3$, we can easily obtain them for higher $d$.) The analogous structure constants in the Grassmannian are the Littlewood-Richardson coefficients, which are much better understood. For instance, even though the $c_{uvw}$s are known to be positive integers, it is a long standing open problem to find a combinatorial interpretation of them.

Billey and Vakil [**2**] showed that the permutation arrays of Eriksson and Linusson can be used to explicitly intersect Schubert varieties, and compute the numbers $c_{w^1 \ldots w^d}$.

Theorem 8.1. *(Billey-Vakil, [**2**]) Suppose that*

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

*is a 0-dimensional and nonempty intersection, with $E^1_\bullet, \ldots, E^d_\bullet$ generic.*

(1) *There exists a unique permutation array $P \subset [n]^{d+1}$, easily constructed from $w^1, \ldots, w^d$, such that*

$$\dim\bigl(E^1_{x_1} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}\bigr) = rk\, P[x_1, \ldots, x_d, x_{d+1}],$$

*for all $F_\bullet \in X$ and all $1 \le x_1, \ldots, x_{d+1} \le n$.*

(2) *Given the permutation array $P$, and a vector $v_{a_1,\ldots,a_d}$ in each one-dimensional intersection $E_{a_1,\ldots,a_d} = E^1_{a_1} \cap \cdots \cap E^d_{a_d}$, we can write down an explicit set of polynomial equations defining $X$.*

Theorem 8.1 highlights the importance of studying the line arrangements $\mathbf{E}_{n,d}$ determined by intersecting $d$ generic complete flags in $\mathbb{C}^n$. In principle, if we are able to construct such a line arrangement, we can compute the structure constants $c_{uvw}$ for any $u, v, w \in S_n$. (In practice, we still have to solve the system of polynomial equations, which is not easy for large $n$.) Let us make two observations in this direction.

**8.1. Matroid genericity versus Schubert genericity.** We have been talking about the line arrangement $\mathbf{E}_{n,d}$ determined by a generic flag arrangement $E^1_\bullet, \dots, E^d_\bullet$ in $\mathbb{C}^n$. We need to be careful, because we have given two different meanings to the word *generic*.

In Sections 3 and 4, we have shown that, if $E^1_\bullet, \dots, E^d_\bullet$ are sufficiently generic, then the linear dependence relations in the line arrangement $\mathbf{E}_{n,d}$ are described by a fixed matroid $\mathcal{T}_{n,d}$. Let us say that the flags are *matroid-generic* if this is the case.

Recall that in the Schubert problem described by permutations $w^1, \dots, w^d$ with $\sum l(w^i) = \binom{n}{2}$, the 0-dimensional intersection

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

contains a fixed number of points $c_{w^1 \dots w^d}$, provided that $E^1_\bullet, \dots, E^d_\bullet$ are sufficiently generic. Let us say that the flags are *Schubert-generic* if they are sufficiently generic for any Schubert problem.

These notions depend only on the line arrangement $\mathbf{E}_{n,d}$. The line arrangement $\mathbf{E}_{n,d}$ is matroid-generic if its matroid is $\mathcal{T}_{n,d}$, and it is Schubert-generic if the equations of Theorem 8.1 give the correct number of solutions to every Schubert problem.

Our characterization of matroid-generic line arrangements (*i.e.*, our description of the matroid $\mathcal{T}_{n,d}$) does not tell us how to construct a Schubert-generic line arrangement. However, when $d = 3$ (which is the interesting case in the Schubert calculus),the cotransversality of the matroid $\mathcal{T}_{n,3}$ allows us to present such a line arrangement explicitly.

Proposition 8.2. *The $\binom{n}{2}$ path vectors of Theorem 5.5 are Schubert-generic.*

Proof. Omitted. □

Proposition 8.2 shows that when we plug the path vectors into the polynomial equations of Theorem 8.1, and compute the intersection $X$, we will have $|X| = c_{uvw}$. The advantage of this point of view is that the equations are now written in terms of combinatorial objects, without any reference to an initial choice of flags.

Problem 8.3. *Interpret combinatorially the $c_{uvw}$ solutions of the above system of equations, thereby obtaining a combinatorial interpretation for the structure constants $c_{uvw}$.*

**8.2. A criterion for vanishing Schubert structure constants.** Consider the Schubert problem

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet).$$

Let $P \in [n]^{d+1}$ be the permutation array which describes the dimensions $\dim(E^1_{x_1} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}})$ for any flag $F_\bullet \in X$. Let $P_1, \dots, P_n$ be the $n$ "floors" of $P$, corresponding to $F_1, \dots, F_n$, respectively. Each one of them is itself a permutation array of shape $[n]^d$.

Billey and Vakil proposed a simple criterion which is very efficient in detecting that many Schubert structure constants are equal to zero.

Proposition 8.4. *(Billey-Vakil, [**2**]) If $P_n$ is not the transversal permutation array, then $X = \emptyset$ and $c_{w^1 \dots w^d} = 0$.*

Knowing the structure of the matroid $\mathcal{T}_{n,d}$, we can strengthen this criterion as follows.

Proposition 8.5. *Suppose $P_n$ is the transversal permutation array, and identify it with the set $T_{n,d}$. If, for some $k$, the rank of $P_k \cap P_n$ in $\mathcal{T}_{n,d}$ is greater than $k$, then $X = \emptyset$ and $c_{w^1 \dots w^d} = 0$.*

Proof. Each dot in $P_n$ corresponds to a one-dimensional intersection of the form $E^1_{x_1} \cap \cdots \cap E^d_{x_d}$. Therefore, each dot in $P_k \cap P_n$ corresponds to a line that $F_k$ is supposed to contain, if $F_\bullet$ is a solution to the Schubert problem. The rank of $P_k \cap P_n$ is the dimension of the subspace spanned by those lines; if $F_\bullet$ exists, that dimension must be at most $k$. □

Let us see how to apply Proposition 8.5 in an example. Following the algorithm of [**2**], the permutations $u = v = w = 213$ in $S_3$ give rise to the four-dimensional permutation array consisting of the dots $(3,3,1,1)$, $(1,3,3,2)$, $(3,1,3,2)$, $(3,3,1,2)$, $(1,3,3,3)$, $(2,2,3,3)$, $(2,3,2,3)$, $(3,1,3,3)$, $(3,2,2,3)$, and $(3,3,1,3)$. We follow [**5, 18**] in representing it as follows:

| | | |
|---|---|---|
| | | |
| | | |
| | | 1 |

| | | |
|---|---|---|
| | | 3 |
| | | |
| 3 | | 1 |

| | | |
|---|---|---|
| | | 3 |
| | 3 | 2 |
| 3 | 2 | 1 |

The three boards shown represent the three-dimensional floors $P_1, P_2$, and $P_3$ of $P$, form left to right. In each one of them, a dot in cell $(i, j, k)$ is represented in two dimensions by a number $k$ in cell $(i, j)$.

It takes some practice to interpret these tables; but once one is used to them, it is very easy to proceed. Simply notice that $P_2 \cap P_3$ is a set of rank 3 in the matroid $\mathcal{T}_{3,3}$, and we are done! We conclude that $c_{213,213,213} = 0$. For $n = 3$, this is the only vanishing $c_{uvw}$ which is not explained by Proposition 8.4.

We remark that there are other methods for detecting the vanishing of Schubert structure constants, due to Knutson, Lascoux and Schutzenberger, and Purbhoo. In comparing these methods for small values of $n$, we have found Proposition 8.5 to be quicker and simpler, but less complete than some of these methods.

However, Proposition 8.5 is only the very first observation that we can make from our understanding of the structure of $\mathcal{T}_{n,d}$. Our argument can be easily fine-tuned to explain all vanishing Schubert structure constants with $n \leq 5$. A systematic way of doing this in general would be very desirable.

## 9. Acknowledgments

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