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A Labelling of the Faces in the Shi Arrangement

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Abstract. Let $\mathcal{F}_n$ be the face poset of the n-dimensional Shi arrangement, and let $\mathcal{P}_n$ be the poset of parking functions of length n with the order defined by $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if $a_i \leq b_i$ for all i. Pak and Stanley constructed a labelling of the regions in $\mathcal{F}_n$ by elements of $\mathcal{P}_n$. We extend this in a natural way to a labelling of all faces in $\mathcal{F}_n$ by closed intervals of $\mathcal{P}_n$, and explore some interesting and unexpected properties of this bijection. We give some results that contribute to characterize the intervals that appear as labels and consequently to a better comprehension of $\mathcal{F}_n$.

Résumé. Soit $\mathcal{F}_n$ l'ensemble partiellement ordonné des faces de l'arrangement de Shi en dimension n, et soit $\mathcal{P}_n$ l'ensemble partiellement ordonné des fonctions de parking de longueur n dont l'ordre est défini par $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ si $a_i \leq b_i$ pout tout i. Pak et Stanley ont construit un étiquetage des regions de $\mathcal{F}_n$ avec des éléments de $\mathcal{P}_n$. On généralise cette étude de manière naturelle à un étiquetage à toutes les faces de $\mathcal{F}_n$ en utilisant des intervalles fermés de $\mathcal{P}_n$ et on éxamine quelques curieuses et inattendues propriétés de cette bijection. On donne des resultats qui contribuent à caractériser les intervalles qui apparaissent comme étiquèttes et ainsi une meilleure compréhension de $\mathcal{F}_n$.

1. Preliminaries

1.1. The Shi arrangement. A n-dimensional (real) hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{R}^n$. Any hyperplane arrangement $\mathcal{A}$ cuts $\mathbb{R}^n$ into open regions that are polyhedra (called the regions of $\mathcal{A}$), so they have faces. More specifically, faces of $\mathcal{A}$ are nonempty intersections between the closure of a region and some or none hyperplanes in $\mathcal{A}$. The poset consisting of all these faces ordered by inclusion is called the face poset of $\mathcal{A}$.

The n-dimensional Shi arrangement $\mathcal{S}_n$ consists of the $n(n-1)$ hyperplanes

$$\mathcal{S}_n: \quad x_i - x_j = 0, 1 \qquad \text{for } 1 \leq i < j \leq n.$$

Let $\mathcal{F}_n$ be the face poset of $\mathcal{S}_n$, and let $\mathcal{R}_n$ be the set of n-dimensional faces in $\mathcal{F}_n$. Then $\mathcal{R}_n$ is the set of closures of the regions of $\mathcal{S}_n$. However we will identify the regions of $\mathcal{S}_n$ with their closure, so we will make no distinction between the elements of $\mathcal{R}_n$ and the regions of $\mathcal{S}_n$. This arrangement was first considered by Shi [**4**], who showed that $|\mathcal{R}_n| = (n+1)^{n-1}$.

Faces of any hyperplane arrangement $\mathcal{A}$ can be described by specifying for every $H \in \mathcal{A}$, which side of H contains the face. That is, for any $H \in \mathcal{A}$ define H^+ and H^- as the two closed halfspaces determined by H (the choice of which one is H^+ is arbitrary), and let $H^0 = H$. Then the elements in the face poset of $\mathcal{A}$ are precisely the nonempty intersections of the form

$$F = \bigcap_{H \in \mathcal{A}} H^{\sigma_H}$$

where $\sigma_H \in \{+, -, 0\}$. So every face F is encoded by its sign sequence $(\sigma_H)_{H \in \mathcal{A}}$, where $\sigma_H \neq 0$ if and only if $F \subseteq H^{\sigma_H}$ and $F \nsubseteq H$.

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For the Shi arrangement it is useful to represent this sequence as a matrix. We will assume as convention that for $i < j$ if $H : x_i = x_j$ then $H^- : x_i \geq x_j$, and if $H : x_i = x_j + 1$ then $H^- : x_i \leq x_j + 1$. First define $\mathcal{M}_n$ as the set of all $n \times n$ matrices whose entries belong to $\{+, -, 0\}$. Then for any $F \in \mathcal{F}_n$ consider its sign sequence $(\sigma_H)_{H \in \mathcal{S}_n}$, and define its *associated matrix* $M_F \in \mathcal{M}_n$ as follows:

$$(M_F)_{i,j} = \begin{cases} \sigma_H & \text{if } j < i, \text{ where } H : x_j = x_i \\ \sigma_H & \text{if } i < j, \text{ where } H : x_i = x_j + 1 \\ 0 & \text{if } i = j. \end{cases}$$

For example, the matrix associated to the region defined by $x_n \leq x_{n-1} \leq \ldots \leq x_1 \leq x_n + 1$ has all entries equal to $-$, except for diagonal ones which are 0. In general, if $F \in \mathcal{F}_n$ then F is a region if and only if all non-diagonal entries of M_F are different from zero. And if $F, G \in \mathcal{F}_n$ then $F \subseteq G$ if and only if M_G has the same entries as M_F except for some non-diagonal zero entries of M_F which become $-$ or $+$ in M_G.

However, there is another way of representing a face that will be very useful for us. For notation simplicity, if n is a positive integer let $[n] = \{1, 2, \ldots, n\}$. Now, if $F \in \mathcal{F}_n$, we will say a function $X : [n] \to \mathbb{R}$ is an *interval representation* of F if the point $\big(X(1), X(2), \ldots, X(n)\big) \in \mathbb{R}^n$ belongs to F and not to any other face properly contained in F. We will denote by $\mathcal{X}_n$ the set of all functions from $[n]$ to $\mathbb{R}$. Two interval representations $X, X' \in \mathcal{X}_n$ will be called *equivalent* if they represent the same face. We can imagine these interval representations as ways in which n numbered intervals of length 1 can be placed on the real line: any $X \in \mathcal{X}_n$ can be thought as the collection of the n intervals $[X(i), X(i) + 1]$ for $i \in [n]$. Interval $[X(i), X(i) + 1]$ will be refered as the i-th interval of X. So the face represented by X is determined only by the relative position of the endpoints of the intervals of X.

1.2. Parking functions. A parking function of length n is a sequence $P = (P_1, P_2, \ldots, P_n) \in [n]^n$ such that if $Q_1 \leq Q_2 \leq \ldots \leq Q_n$ is the increasing rearrangement of the terms of P, then $Q_i \leq i$. Parking functions were first considered by Konheim and Weiss [**3**] under a slightly different definition, but equivalent to ours. Let $\mathcal{P}_n$ be the poset of the parking functions of length n with the order defined by $(P_1, P_2, \ldots, P_n) \leq (Q_1, Q_2, \ldots, Q_n)$ if $P_i \leq Q_i$ for all $i \in [n]$.

Pak and Stanley constructed a bijection between $\mathcal{R}_n$ and the parking functions of length n as follows [**5**]: Let $R_0 \in \mathcal{R}_n$ be the region defined by $x_n \leq x_{n-1} \leq \ldots \leq x_1 \leq x_n + 1$, and define its label $\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n$. Suppose that $R, R' \in \mathcal{R}_n$, R is labelled and R' is unlabelled, R and R' are only separated by the hyperplane $H : x_i = x_j$ $(i < j)$, and R_0 and R are on the same side of H; then define $\lambda(R') = \lambda(R) + e_i$ ($e_i \in \mathbb{Z}^n$ is the i-th vector of the canonical basis). If under the same hypothesis R and R' are only separated by the hyperplane $H : x_i = x_j + 1$ $(i < j)$ and R_0 and R are on the same side of H; then define $\lambda(R') = \lambda(R) + e_j$.

Figure 1 shows the projection of the arrangement $\mathcal{S}_3$ on the plane defined by $x + y + z = 0$, and the labelling of the regions in a simplified notation.

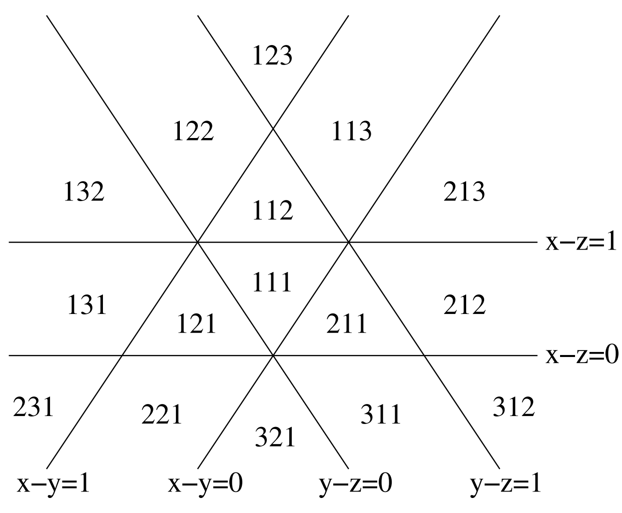

FIGURE 1. Arrangement $\mathcal{S}_3$ and the labelling λ

Notice that in our convention, if $R \in \mathcal{R}_n$ then we have that $\lambda(R) = (a_1+1, a_2+1, \ldots, a_n+1)$ where a_i is the number of $+$ entries in the i-th column of M_R. Stanley showed that this labelling was in fact a bijection between $\mathcal{R}_n$ and the elements of $\mathcal{P}_n$, that is, he showed that all these labels were parking functions, each appearing once.

2. The labelling of $\mathcal{F}_n$

We will now extend this labelling to all faces in $\mathcal{F}_n$. First we prove a lemma that allows us to define the labelling.

Lemma 2.1. *Let $F \in \mathcal{F}_n$. Then there exist two unique regions $F^-, F^+ \in \mathcal{R}_n$ such that $F \subseteq F^-$, $F \subseteq F^+$ and for any region $R \in \mathcal{R}_n$, if $F \subseteq R$ then $\lambda(F^-) \leq \lambda(R) \leq \lambda(F^+)$ in $\mathcal{P}_n$. Moreover, $F^- \cap F^+ = F$.*

Proof. Consider an interval representation $X \in \mathcal{X}_n$ of F. Clearly the lemma is true if $F \in \mathcal{R}_n$, that is, if there are no equalities in X of the form $X(i) = X(j)$ or $X(i) = X(j)+1$ with $i < j$, because in this case $F^- = F^+ = F$. In other case, let r be the maximum $X(i)$ for which there exists a $j > i$ such that $X(i) = X(j)$ or $X(i) = X(j)+1$. Take k as the maximum i such that $X(i) = r$. Define a new interval representation $X' \in \mathcal{X}_n$ by

$$X'(i) = \begin{cases} X(i) & \text{if } i \neq k \\ X(i)+\epsilon & \text{if } i = k \end{cases}$$

where ϵ is a sufficiently small positive real number so that for all j, if $X(k) < X(j)$ then $X(k)+\epsilon < X(j)$, and if $X(k) < X(j)+1$ then $X(k)+\epsilon < X(j)+1$. So X' is the same interval representation as X, but its k-th interval is moved a little bit to the right. Let $F' \in \mathcal{F}_n$ be the face represented by X'.

By the definition of X' it is clear that inequalities in X remain unchanged in X', and also equalities that do not involve $X(k)$. That is, if $X(i) < X(j)$ then $X'(i) < X'(j)$, if $X(i) < X(j)+1$ with $i < j$ then $X'(i) < X'(j)+1$, and if $X(i) > X(j)+1$ with $i < j$ then $X'(i) > X'(j)+1$. Also if $X(i) = X(j)$ and $i, j \neq k$ then $X'(i) = X'(j)$, and if $X(i) = X(j)+1$ with $i < j$ and $i, j \neq k$ then $X'(i) = X'(j)+1$. Notice as well that there are no equalities in X of the form $X(i) = X(k)+1$ with $i < k$ because it imply be a contradiction with the maximality of r, neither equalities of the form $X(k) = X(i)$ with $k < i$ because they contradict the choice of k. So all equalities in X involving $X(k)$ must be of the form $X(k) = X(i)+1$ with $k < i$, or $X(i) = X(k)$ with $i < k$. In the first case we have that $X'(k) > X(i)+1 = X'(i)+1$, so $(M_{F'})_{k,i} = +$. In the second case $X'(i) = X(i) < X'(k)$, so $(M_{F'})_{k,i} = +$. All this shows that $M_{F'}$ has the same entries as M_F except for the non-diagonal zero entries in the k-th row and k-th column of M_F, which become $+$ in $M_{F'}$.

If we repeat this construction starting with the face F' we obtain a face F'', satisfying that $M_{F''}$ has the same entries as $M_{F'}$ except for some non-diagonal zero entries in $M_{F'}$ that become $+$ in $M_{F''}$. And continuing with this process we finally get a face F^+, such that M_{F^+} is the same matrix as M_F but replacing all its non-diagonal zero entries by $+$.

Consider now the same construction, but defining X' by moving the k-th interval of X a little bit to the left. The non-diagonal zero entries in the k-th row and k-th column of M_F become now $-$ in $M_{F'}$, so repeating the process we finally get a face F^- such that M_{F^-} is the same matrix as M_F but replacing all its non-diagonal zero entries by $-$.

By this description of their associated matrices, it is easy to see that $F^+ \cap F^- = F$. Now, let $R \in \mathcal{R}_n$ be any region containing F. Remember that M_R must be the same matrix as M_F, but changing the non-diagonal zero entries in M_F by $-$ or $+$. Then for every $i \in [n]$ the number of $+$ entries in the i-th column of M_R must be at least the number of $+$ entries in the i-th column of M_{F^-}, and at most the number of $+$ entries in the i-th column of M_{F^+}. Hence $\lambda(R) \in \mathcal{P}_n$ must satisfy the relation $\big(\lambda(F^-)\big)_i \leq \big(\lambda(R)\big)_i \leq \big(\lambda(F^+)\big)_i$ for all i, that is, $\lambda(F^-) \leq \lambda(R) \leq \lambda(F^+)$ in $\mathcal{P}_n$. This property implies easily the uniqueness of F^- and F^+, so the proof is complete. $\square$

This lemma is interesting by itself, as the following result shows.

Corollary 2.2. *Let $R_1, R_2, \ldots, R_k \in \mathcal{R}_n$, and define $P^i = (P^i_1, P^i_2, \ldots, P^i_n) = \lambda(R_i)$ for $1 \leq i \leq k$. If*

$$Q = \Big(\max_i P^i_1, \max_i P^i_2, \ldots, \max_i P^i_n\Big)$$

is not a parking function then $\bigcap_{i=1}^k R_i = \emptyset$.

Proof. If $F = \bigcap_{i=1}^{k} R_i \neq \emptyset$ then $F \in \mathcal{F}_n$. Hence by the Lemma we have that $P^i \leq \lambda(F^+)$ for all i, but this implies that Q is a parking function. $\square$

We now define the labelling of the faces in $\mathcal{F}_n$. Denote by $\mathrm{Int}(\mathcal{P}_n)$ the set of all closed intervals of $\mathcal{P}_n$.

Definition 2.3. The labelling $\lambda : \mathcal{F}_n \to \mathrm{Int}(\mathcal{P}_n)$ is defined by $\lambda(F) = \big[\lambda(F^-), \lambda(F^+)\big]$.

We will use λ also for this labelling because it can be considered as an extension of the labelling we had for regions (by identifying $\lambda(R)$ with $\big\{\lambda(R)\big\}$).

Notice that different faces have different labels, because $F^- \cap F^+ = F$ for all $F \in \mathcal{F}_n$. Unfortunately, not all closed intervals of $\mathcal{P}_n$ appear as labels of some face.

3. Properties of the labelling

Clearly the main property of this labelling is stated in the following surprising theorem.

Theorem 3.1. *Let $F \in \mathcal{F}_n$. Then $\lambda(F) = \big\{\lambda(R) \mid R \in \mathcal{R}_n$ and $F \subseteq R\big\}$.*

Proof. Let $I(F) = \big\{\lambda(R) \mid R \in \mathcal{R}_n \text{ and } F \subseteq R\big\}$. Lemma 2.1 tells us that $I(F) \subseteq \lambda(F)$. Now, notice that

$$\Big|\lambda(F)\Big| = \prod_{i=1}^{n}\Big(\big(\lambda(F^+)\big)_i - \big(\lambda(F^-)\big)_i + 1\Big)$$

because $P = (P_1, P_2, \ldots, P_n)$ is a parking function in $\lambda(F)$ if and only if $\big(\lambda(F^-)\big)_i \leq P_i \leq \big(\lambda(F^+)\big)_i$ for all i. Now let $X \in \mathcal{X}_n$ be an interval representation of F, and define $A(F,i) = \big\{j \in [n] \mid j > i \text{ and } X(i) = X(j)\big\}$ and $B(F,i) = \big\{j \in [n] \mid j < i \text{ and } X(j) = X(i)+1\big\}$. Then $c(F,i) = |A_i| + |B_i|$ is the number of non-diagonal zero entries in the i-th column of M_F. So $c(F,i)$ is the difference between the number of $+$ entries in the i-th column of M_{F^+} and the number of $+$ entries in the i-th column of M_{F^-}. Hence $c(F,i) = \big(\lambda(F^+)\big)_i - \big(\lambda(F^-)\big)_i$, and

$$\Big|\lambda(F)\Big| = \prod_{i=1}^{n}\big(c(F,i)+1\big).$$

We will then prove that $\big|I(F)\big| \geq \prod_{i=1}^{n}\big(c(F,i)+1\big)$, which is equivalent to the equality between $I(F)$ and $\lambda(F)$ by a cardinality argument. Notice that $\big|I(F)\big|$ is the number of regions that contain F as a face. Then $\big|I(F)\big|$ is the number of ways (up to equivalence) in which the intervals of X can be moved a little bit, changing all equalities in X of the form $X(i) = X(j)$ or $X(i) = X(j)+1$ $(1 \leq i < j \leq n)$ to inequalities. So we will prove there are at least $\prod_{i=1}^{n}\big(c(F,i)+1\big)$ different ways of doing this.

The proof is by induction on n. If $n = 2$ there are 5 faces in $\mathcal{F}_2$, and it is easy to check that for each one of them the equality holds. Now assume the assertion is true for $n-1$. Consider $F \in \mathcal{F}_n$ and let $X \in \mathcal{X}_n$ be an interval representation of F. Let r be the minimum $X(i)$, and let k be the minimum i such that $X(i) = r$. By the choice of k there is no i such that $i < k$ and $X(i) = X(k)$, or $i > k$ and $X(k) = X(i)+1$. That is, for all $i \neq k$ we have that $k \notin A(F,i)$ and $k \notin B(F,i)$. Then, ignoring the k-th interval, by induction hypothesis there are at least $\prod_{i \neq k}\big(c(F,i)+1\big)$ different ways of moving (as explained before) all intervals of X except the k-th interval. Consider one of these ways in which these intervals can be moved, and for $i \neq k$ let $X'(i)$ be the new position of the i-th interval. We can assume without loss of generality that the intervals were moved very little, so that there exists an open interval U around $X(k)+1$ such that $X'(i)+1 \in U$ if and only if $X(k)+1 = X(i)+1$, and $X'(i) \in U$ if and only if $X(i) = X(k)+1$. Then the $c(F,k)$ points of $\big\{X'(i)+1 \mid i \in A(F,k)\big\} \cup \big\{X'(i) \mid i \in B(F,k)\big\}$ separate the interval U in $c(F,k)+1$ disjoint open intervals $U_0, U_1, \ldots, U_{c(F,k)}$. For every j such that $0 \leq j \leq c(F,k)$ let z_j be some point inside interval U_j, and define $Y_j \in \mathcal{X}_n$ as follows:

$$Y_j(i) = \begin{cases} X'(i) & i \neq k \\ z_j & \text{if } i = k. \end{cases}$$

So Y_j is an interval representation obtained by moving all intervals of X a little bit (as explained before). Because U was chosen sufficiently small, Y_j represents a region in $\mathcal{R}_n$ that contains F. Moreover, if $i \neq j$ then Y_i and Y_j represent different regions, because $Y_i(k) \in U_i$ and $Y_j(k) \in U_j$. So we have proved that for every way of moving all intervals of X except the k-th interval there are at least $c(F,k)+1$ different regions in $\mathcal{R}_n$ that contain F. Hence

$$\big|I(F)\big| \geq \big(c(F,k)+1\big)\prod_{i\neq k}\big(c(F,i)+1\big) = \prod_{i=1}^{n}\big(c(F,i)+1\big)$$

as we wanted, so the proof is complete. ☐

This Theorem can also be stated as follows.

Corollary 3.2. *Let $R_1, R_2, \ldots, R_k \in \mathcal{R}_n$ such that $F \subseteq \bigcap_{i=1}^{k} R_i \neq \emptyset$, and define $P^i = (P^i_1, P^i_2, \ldots, P^i_n) = \lambda(R_i)$ for $1 \leq i \leq k$. If $R \in \mathcal{R}_n$ is such that*

$$\Big(\min_i P^i_1, \min_i P^i_2, \ldots, \min_i P^i_n\Big) \leq \lambda(R) \leq \Big(\max_i P^i_1, \max_i P^i_2, \ldots, \max_i P^i_n\Big)$$

then $F \subseteq R$.

Another important consequence is stated in the next corollary.

Corollary 3.3. *Let $F, G \in \mathcal{F}_n$. Then $F \subseteq G$ if and only if $\lambda(F) \supseteq \lambda(G)$.*

So if we define $\mathcal{I}_n$ as the poset of all intervals of $\mathcal{P}_n$ appearing as labels, ordered by reverse inclusion, then λ is an isomorphism between $\mathcal{F}_n$ and $\mathcal{I}_n$. This means that the characterization of all intervals in $\mathcal{I}_n$ will give us a complete combinatorial description of $\mathcal{F}_n$. We already know that all intervals of $\mathcal{P}_n$ consisting of exactly one element appear in $\mathcal{I}_n$ as labels of some region.

Now, every $F \in \mathcal{F}_n$ has a dimension, which determines the rank of F in the poset $\mathcal{F}_n$. To see how this dimension is represented in $\mathcal{I}_n$ we need the following definition.

Let $X \in \mathcal{X}_n$. A *chain* of X is a k-tuple $(a_1, a_2, \ldots, a_k) \in [n]^k$ constructed as follows:

- Choose a_1 so that there is no $i < a_1$ such that $X(i) = X(a_1) + 1$, neither $i > a_1$ such that $X(a_1) = X(i)$.
- Once a_j has been chosen, if there exists some $i < a_j$ such that $X(i) = X(a_j)$ then $a_{j+1} = \max\big\{i < a_j \mid X(i) = X(a_j)\big\}$. If this i does not exist, but there exists some $l > a_j$ such that $X(a_j) = X(l)+1$, then $a_{j+1} = \max\big\{l > a_j \mid X(a_j) = X(l)+1\big\}$.
- The chain ends when there are no such i nor l as in the last step.

X can have several different chains, but the definition implies that all of them must be disjoint, and every $i \in [n]$ must belong to some chain of X. It is easy to see that chains represent sets of intervals that are binded one to another in X. That is, if we move a little bit the j-th interval to obtain a new interval representation $X' \in \mathcal{X}_n$, then for all i in the same chain as j we must also move the i-th interval in the same way if we want X' to represent the same face as X. Hence, the number of chains of X is the dimension of the face represented by X.

Proposition 3.1. *Let $F \in \mathcal{F}_n$, and $\lambda(F) = [P, Q]$. Then* $\dim(F) = \big|\{i \in [n] \mid P_i = Q_i\}\big|$.

Proof. Let $X \in \mathcal{X}_n$ be an interval representation of F. Remember the definitions of $A(F,i)$, $B(F,i)$ and $c(F,i)$ given in the proof of Theorem 3.1. Notice that if $H = (a_1, a_2, \ldots, a_k) \in [n]^k$ is a chain of X then $c(F, a_j) = 0$ if and only if $j = 1$, because $a_j \in A(F, a_{j+1}) \cup B(F, a_{j+1})$. So the number of chains of X is equal to the number of $i \in [n]$ such that $c(F,i) = 0$. But we had seen that $c(F,i) = Q_i - P_i$, so the proof is complete. ☐

Continuing with the same ideas we can prove the following proposition.

Proposition 3.2. *Let $F \in \mathcal{F}_n$, and $\lambda(F) = [P, Q]$. Then*

$$\{Q_1 - P_1, Q_2 - P_2, \ldots, Q_n - P_n\} = \{0, 1, 2, \ldots, m\}$$

for some $m \in \mathbb{N}$.

Proof. Let $X \in \mathcal{X}_n$ be an interval representation of F. Notice that if $H = (a_1, a_2, \ldots, a_k) \in [n]^k$ is a chain of X then $A(F, a_{j+1}) \cup B(F, a_{j+1}) \subseteq A(F, a_j) \cup B(F, a_j) \cup \{a_j\}$for all j, so $c(F, a_{j+1}) \leq c(F, a_j) + 1$. Then $Q_{a_{j+1}} - P_{a_{j+1}} \leq Q_{a_j} - P_{a_j} + 1$ for all j. Remembering that $Q_{a_1} - P_{a_1} = 0$ we have that $\{Q_{a_1} - P_{a_1}, Q_{a_2} - P_{a_2}, \ldots, Q_{a_k} - P_{a_k}\} = \{0, 1, \ldots, m_H\}$ for some $m_H \in \mathbb{N}$. Therefore, by taking the union over all chains of X, the proof is finished. ☐

Last proposition restricts a lot the possible intervals appearing as labels, and makes a step toward the characterization of the elements of $\mathcal{I}_n$.

We now characterize the possible sizes of the intervals that appear as labels of faces of a fixed dimension.

Proposition 3.3. *The set* $\{|\lambda(F)| \,\big|\, F \in \mathcal{F}_n \text{ and } \dim(F) = k\}$ *is the set of all positive numbers* d *such that* $d = 2^{a_1}3^{a_2}\ldots(m+1)^{a_m}$ *for some* $m \in \mathbb{N}$*, where* $a_i > 0$ *for all* $i \leq m$*, and* $a_1 + a_2 + \ldots + a_m = n - k$*.*

Proof. Let $F \in \mathcal{F}_n$ be a face such that $\dim(F) = k$, and let $\lambda(F) = [P, Q]$. Define $a_i = \big|\{j \mid Q_j - P_j = i\}\big|$. Proposition 3.2 tells us there exists $m \in \mathbb{N}$ such that $a_i > 0$ if and only if $i \leq m$. Then

$$\big|\lambda(F)\big| = \big|[P, Q]\big| = \prod_{i=1}^{n}(Q_i - P_i + 1) = 2^{a_1}3^{a_2}\ldots(m+1)^{a_m}.$$

It is clear that $a_0 + a_1 + \ldots + a_m = n$, so by Proposition 3.1 we have that $a_1 + a_2 + \ldots + a_m = n - k$.

On the other hand, if we take $a_0, a_1, \ldots, a_m$ such that $a_i > 0$ for all $i \leq m$ and $a_0 + a_1 + \ldots + a_m = n$ then it is easy to construct an interval representation X of a face $F \in \mathcal{F}_n$ satisfying $a_i = \big|\{j \mid c(F, j) = i\}\big|$. Therefore, remembering that if $\lambda(F) = [P, Q]$ then $c(F, j) = Q_j - P_j$, the proposition follows. □

Remember that if $F \in \mathcal{F}_n$ then $\big|\lambda(F)\big| = \big|\{R \in \mathcal{R}_n \mid F \subseteq R\}\big|$, so this proposition is also giving some geometrical information about the Shi arrangement.

Finally, we characterize the intervals appearing as labels of 1-dimensional faces.

Proposition 3.4. *Let* $I = [P, Q]$ *be an interval of* $\mathcal{P}_n$*. Then* I *is the label of a* 1*-dimensional face if and only if the following statements hold:*

- Q *is a permutation of* $[n]$*.*
- P *is determined by* Q *in the following way. Denote* $(a_1, a_2, \ldots, a_n) = (Q^{-1}(1), Q^{-1}(2), \ldots, Q^{-1}(n))$*, and let* $0 = i_0 < i_1 < i_2 < \ldots < i_k = n$ *be the numbers such that*

$$\{i_1, i_2, \ldots, i_{k-1}\} = \{j \in [n] \mid a_j < a_{j+1}\}.$$

Then for all $r \in [n]$*, if* j *is such that* $i_j < r \leq i_{j+1}$ *we have that*

$$P_{a_r} = i_{j-1} + \big|\{l \in [n] \mid i_{j-1} < l \leq i_j \text{ and } a_l > a_r\}\big| + 1,$$

where $i_{-1} = 0$*.*

Proof. To see that the conditions are necessary, let $F \in \mathcal{F}_n$ be a 1-dimensional face such that $\lambda(F) = [P, Q]$, and let $X \in \mathcal{X}_n$ be an interval representation of F. Then X consists only of one chain $H = (b_1, b_2, \ldots, b_n)$. Remember that $Q_i - 1$ is the number of non-diagonal + or 0 entries in the i-th column of M_F, that is,

$$Q_i = \big|\{j \in [n] \mid j > i \text{ and } X(j) \geq X(i)\}\big| + \big|\{j \in [n] \mid j < i \text{ and } X(j) \geq X(i) + 1\}\big| + 1.$$

But all intervals of X are on the same chain, so we have that for all i

$$\{j \in [n] \mid j > b_i \text{ and } X(j) \geq X(b_i)\} \cup \{j \in [n] \mid j < b_i \text{ and } X(j) \geq X(b_i) + 1\} = \{b_1, b_2, \ldots, b_{i-1}\},$$

hence $Q_{b_i} = i$. This shows that Q is a permutation of $[n]$, and that $a_i = b_i$ for all i.

Notice that the numbers $i_0, i_1, \ldots, i_k$ satisfy that for all m, $i_j < m \leq i_{j+1}$ if and only if $X(a_m) = X(a_1) - j$. Then $i_j = \big|\{l \in [n] \mid X(l) > X(a_1) - j\}\big|$. Remember also that $P_i - 1$ is the number of + entries in the i-th column of M_F, that is,

$$P_i = \big|\{j \in [n] \mid j > i \text{ and } X(j) > X(i)\}\big| + \big|\{j \in [n] \mid j < i \text{ and } X(j) > X(i) + 1\}\big| + 1.$$

Let $r \in [n]$ and j such that $i_j < r \leq i_{j+1}$, so $X(a_r) = X(a_1) - j$. Therefore, because X consists only of the chain H,

$$\begin{aligned}
P_{a_r} &= \big|\{l \mid l > a_r \text{ and } X(l) > X(a_r)\}\big| + \big|\{l \mid l < a_r \text{ and } X(l) > X(a_r) + 1\}\big| + 1 \\
&= \big|\{l \mid l > a_r \text{ and } X(l) = X(a_r) + 1\}\big| + \big|\{l \mid X(l) > X(a_r) + 1\}\big| + 1 \\
&= \big|\{l \mid l > a_r \text{ and } X(l) = X(a_1) - (j-1)\}\big| + \big|\{l \mid X(l) > X(a_1) - (j-1)\}\big| + 1 \\
&= \big|\{m \mid a_m > a_r \text{ and } i_{j-1} < m \leq i_j\}\big| + i_{j-1} + 1,
\end{aligned}$$

as we wanted.

On the other hand, it is easy to see that if $[P,Q]$ is an interval of $\mathcal{P}_n$ satisfying the previous conditions, then it appears as the label of a 1-dimensional face. In fact, the function $X \in \mathcal{X}_n$ defined by

$$X\left(Q^{-1}(i)\right) = -\left|\left\{l \in [n] \mid l < i \text{ and } Q^{-1}(l) < Q^{-1}(l+1)\right\}\right|$$

represents a 1-dimensional face F such that $\lambda(F) = [P,Q]$. □

This characterization has an interesting corollary.

Corollary 3.4. *Each region $R \in \mathcal{R}_n$ such that $\lambda(R)$ is a permutation of $[n]$ contains a unique 1-dimensional face $F \in \mathcal{F}_n$. Moreover, each 1-dimensional face $F \in \mathcal{F}_n$ is contained in a unique region $R \in \mathcal{R}_n$ such that $\lambda(R)$ is a permutation of $[n]$.*

Proof. Suppose R is a region such that $\lambda(R)$ is a permutation of $[n]$. By the last characterization we know that there exists a unique $P \in \mathcal{P}_n$ such that $[P,Q]$ is the label of a 1-dimensional face F. Theorem 3.1 implies that $F \subseteq R$. Moreover, if F' is a one dimensional face contained in R then $Q \in \lambda(F')$, and because Q is a maximal element of $\mathcal{P}_n$ we have that $\lambda(F') = [P',Q]$ for some $P' \in \mathcal{P}_n$. Therefore $P = P'$ and $F = F'$, so the face F is unique.

Now, if F is a 1-dimensional face then by the characterization $\lambda(F) = [P,Q]$, with Q a permutation of $[n]$. By Theorem 3.1, if R is the region such that $\lambda(R) = Q$ then $F \subseteq R$. Moreover, if R' is a region that contains F then $Q' = \lambda(R') \in [P,Q]$. Therefore, if Q' is a permutation of $[n]$ then $Q' = Q$, because Q' is a maximal element of $\mathcal{P}_n$. So $R = R'$, proving that the region R is unique. □

Corollary 3.5. *The number of 1-dimensional faces of $\mathcal{S}_n$ is $n!$.*

This is a particular example of a general result first stated by Athanasiadis [**1**]. However, this bijective proof allows a better comprehension of the geometrical organization of these faces.

4. Perspectives

After developing these results, it seems clear that there are still many aspects to understand about this labelling. We are now working on three main problems. In first place, we are trying to achieve a total and simple characterization of the intervals of $\mathcal{I}_n$. This would give a complete combinatorial description of the poset $\mathcal{F}_n$, thus a better comprehension of the geometry of the Shi arrangement. We are also trying to generalize to higher dimensions the way in which 1-dimensional faces were counted, obtaining this way a similar result to the one given by Athanasiadis [**1**]. Finally, we want to apply all these results to the theory of random walks on hyperplane arrangements, as defined by Brown and Diaconis in [**2**].

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