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Ehrhart polynomials of lattice-face polytopes

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Abstract. There is a simple formula for the Ehrhart polynomial of a cyclic polytope. The purpose of this paper is to show that the same formula holds for a more general class of polytopes, lattice-face polytopes. We develop a way of decomposing any d dimensional simplex in general position into $d!$ signed sets, each of which corresponds to a permutation in the symmetric group S_d, and reduce the problem of counting lattice points in a polytope in general position to counting lattice points in these special signed sets. Applying this decomposition to a lattice-face simplex, we obtain signed sets with special properties that allow us to count the number of lattice points inside them. We are thus able to conclude the desired formula for the Ehrhart polynomials of lattice-face polytopes.

Résumé. Il y a une formule simple pour le polynôme d'Ehrhart d'un polytope cyclique. Le but de cet article est de prouver que la même formule est vraie pour une classe plus générale de polytope, les polytopes "treillis-faces". Nous donnons une manière de décomposer n'importe quel simplexe de dimension d en position générale en $d!$ ensembles signés. Chacun de ces ensembles correspond à une permutation dans le groupe symétrique S_d, et ramène le problème de compter des points de treillis dans un polytope en position générale à compter des points de treillis dans ces ensembles signés particuliers. Appliquant cette décomposition à un simplexe de treillis-faces, nous obtenons des ensembles signés dont les propriétés nous permettent de compter le nombre de points de treillis qu'ils contiennent. Nous obtenons ainsi la formule désirée pour les polynômes d'Ehrhart des polytopes de treillis-faces.

1. Introduction

A d-dimensional *lattice* $\mathbb{Z}^d = \{x = (x_1, \dots, x_d) \mid \forall x_i \in \mathbb{Z}\}$ is the collection of all points with integer coordinates in $\mathbb{R}^d$. Any point in a lattice is called a *lattice point*.

A *convex polytope* is a convex hull of a finite set of points. We often omit convex and just say polytope. For any polytope P and some positive integer $m \in \mathbb{N}$, we use $i(m, P)$ to denote the number of lattice points in mP, where $mP = \{mx | x \in P\}$ is the *mth dilated polytope* of P.

An *integral* or *lattice* polytope is a convex polytope whose vertices are all lattice points. Eugène Ehrhart [**4**] showed that for any d-dimensional integral polytope, $i(P, m)$ is a polynomial in m of degree d. Thus, we call $i(P, m)$ the Ehrhart polynomial of P when P is an integral polytope. Please see [**2, 3**] for more reference to the literature of lattice point counting. Although Ehrhart's theory was developed in 1960's, we still do not know much about the coefficients of Ehrhart polynomials for general polytopes except that the leading, second and last coefficients of $i(P, m)$ are the normalized volume of P, one half of the normalized volume of the boundary of P and 1, respectively.

In [**6**], the author showed that for any d-dimensional cyclic polytope P, we have that

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{(d-k)}(P)) m^k, \tag{1.1}$$

where $\pi^{(k)}$ is the map which ignores the last k coordinates of a point, and asked whether there are other integral polytopes that have the the same form of Ehrhart polynomials.

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In this paper, we define a new family of integral polytopes, *lattice-face* polytopes, and show (Theorem 3.4) that their Ehrhart polynomials are in the form of (1.1).

The main method of [**6**] is a decomposition of an arbitrary d dimensional simplex cyclic polytope into $d!$ signed sets, each of which corresponds to a permutation in the symmetric group S_d and has the same sign as the corresponding permutation. However, for general polytopes, such a decomposition does not work.

In this paper, we develop a way of decomposing any d dimensional simplex in *general position* into $d!$ signed sets, where the sign of each set is not necessarily the same as the corresponding permutation. Applying the new decomposition to a lattice-face simplex, we are able to show (Theorem 3.5) that the number of lattice points in terms of a formula (6.1) involving Bernoulli polynomials, signs of permutations, and determinants, and then to analyze this formula further to derive the theorem. Theorem 3.5, together with some simple observation in section 2 and 3, imply Theorem 3.4.

2. Preliminaries

We first give some definitions and notations, most of which follows [**6**].

All polytopes we will consider are full-dimensional, so for any convex polytope P, we use d to denote both the dimension of the ambient space $\mathbb{R}^d$ and the dimension of P. We call a d-dimensional polytope a d-polytope. Also, We use ∂P and $I(P)$ to denote the boundary and the interior of P, respectively.

For any set S, we use $\text{conv}(S)$ to denote the convex hull of all of points in S.

Recall that the projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the map that forgets the last coordinate. For any set $S \subset \mathbb{R}^d$ and any point $y \in \mathbb{R}^{d-1}$, let $\rho(y, S) = \pi^{-1}(y) \cap S$ be the intersection of S with the inverse image of y under π. Let $p(y, S)$ and $n(y, S)$ be the point in $\rho(y, S)$ with the largest and smallest last coordinate, respectively. If $\rho(y, S)$ is the empty set, i.e., $y \notin \pi(S)$, then let $p(y, S)$ and $n(y, S)$ be empty sets as well. Clearly, if S is a d-polytope, $p(y, S)$ and $n(y, S)$ are on the boundary of S. Also, we let $\rho^+(y, S) = \rho(y, S) \setminus n(y, S)$, and for any $T \subset \mathbb{R}^{d-1}$, $\rho^+(T, S) = \cup_{y \in T} \rho^+(y, S)$.

DEFINITION 2.1. Define $PB(P) = \bigcup_{y \in \pi(P)} p(y, P)$ to be the *positive boundary* of P; $NB(P) = \cup_{y \in \pi(P)} n(y, P)$ to be the *negative boundary* of P and $\Omega(P) = P \setminus NB(P) = \rho^+(\pi(P), P) = \cup_{y \in \pi(P)} \rho^+(y, P)$ to be the *non-negative part* of P.

DEFINITION 2.2. For any facet F of P, if F has an interior point in the positive boundary of P, then we call F a *positive facet* of P and define the sign of F as $+1$: $\text{sign}(F) = +1$. Similarly, we can define the *negative facets* of P with associated sign -1. For the facets that are neither positive nor negative, we call them *neutral facets* and define the sign as 0.

It's easy to see that $F \subset PB(P)$ if F is a positive facet and $F \subset NB(P)$ if F is a negative facet.

Because the usual set union and set minus operation do not count the number of occurrences of an elements, which is important in our paper, from now on we will consider any polytopes or sets as *multisets* which allow *negative multiplicities*. In other words, we consider any element of a multiset as a pair (x, m), where m is the multiplicity of element x. Then for any multisets M_1, M_2 and any integers m, n and i, we define the following operators:

a) Scalar product: $iM_1 = i \cdot M_1 = \{(x, im) \mid (x, m) \in M_1\}$.
b) Addition: $M_1 \oplus M_2 = \{(x, m + n) \mid (x, m) \in M_1, (x, n) \in M_2\}$.
c) Subtraction: $M_1 \ominus M_2 = M_1 \oplus ((-1) \cdot M_2)$.

It's clear the following holds:

LEMMA 2.3. *For any polytope* $P \subset \mathbb{R}^d$, $\forall R_1, \dots, R_k \subset \mathbb{R}^{d-1}$, $\forall i_1, \dots, i_k \in \mathbb{Z}$:

$$\rho^+\left(\bigoplus_{j=1}^{k} i_j R_j,\ P\right) = \bigoplus_{j=1}^{k} i_j \rho^+(R_j,\ P).$$

DEFINITION 2.4. We say a set S has *weight* w, if each of its elements has multiplicity either 0 or w. And S is a *signed* set if it has weight 1 or -1.

Let P be a convex polytope. For any y an interior point of $\pi(P)$, since π is a continuous open map, the inverse image of y contains an interior point of P. Thus $\pi^{-1}(y)$ intersects the boundary of P exactly twice. For any y a boundary point of $\pi(P)$, again because π is an open map, we have that $\rho(y, P) \subset \partial P$, so

$\rho(y, P) = \pi^{-1}(y) \cap \partial P$ is either one point or a line segment. The polytopes P we will be interested in are those satisfying $\rho(y, P)$ has has only one point.

LEMMA 2.5. *If a polytope P satisfies:*

$$|\rho(y, P)| = 1, \forall y \in \partial\pi(P), \tag{2.1}$$

then P has the following properties:

(i) *For any $y \in I(\pi(P))$, $\pi^{-1}(y) \cap \partial P = \{p(y, P), n(y, P)\}$.*
(ii) *For any $y \in \partial\pi(P)$, $\pi^{-1}(y) \cap \partial P = \rho(y, P) = p(y, P) = n(y, P)$, so $\rho^+(y, P) = \emptyset$.*
(iii) *If $P = \bigsqcup_{i=1}^k P_i$, where the P_i's all satisfy (2.1), then $\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i)$. ($P = \bigsqcup_{i=1}^k P_i$ means that P_i's give a decomposition of P, i.e., $P = \bigcup_{i=1}^k P_i$, and for any $i \neq j$, $P_i \cap P_j$ is contained in their boundaries.)*
(iv) *The set of facets of P are partitioned into the set of positive facets and the set of negative facets, i.e., there is no neutral facets.*

The proof of this lemma is straightforward, so we won't include it here.

The main purpose of this paper is to discuss the number of lattice points in a polytope. Therefore, for simplicity, for any set $S \in \mathbb{R}^d$, we denote by $\mathcal{L}(S) = S \cap \mathbb{Z}^d$ the set of lattice points in S. It's not hard to see that $\mathcal{L}$ commutes with some of the operations we defined earlier, e.g. ρ, ρ^+, Ω.

3. Lattice-face polytopes

A *d-simplex* is a polytope given as the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^d$.

DEFINITION 3.1. We define *lattice-face* polytopes recursively. We call a one dimensional polytope a *lattice-face* polytope if it is integral.

For $d \geq 2$, we call a d-dimensional polytope P with vertex set V a *lattice-face* polytope if for any d-subset $U \subset V$,

a) $\pi(\text{conv}(U))$ is a lattice-face polytope, and
b) $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{d-1}$, where H_U is the affine space spanned by U. In other words, after dropping the last coordinate of the lattice of H_U, we get the $(d-1)$-dimensional lattice.

To understand the definition, let's look at examples of 2-polytopes.

EXAMPLE 3.2. Let P_1 be the polytope with vertices $v_1 = (0, 0), v_2 = (2, 0)$ and $v_3 = (2, 1)$. Clearly, for any 2-subset U, condition *a*) is always satisfied. When $U = \{v_1, v_2\}$, H_U is $\{(x, 0) \mid x \in \mathbb{R}\}$. So $\pi(\mathcal{L}(H_U)) = \mathbb{Z}$, i.e., *b*) holds. When $U = \{v_1, v_3\}$, H_U is $\{(x, y) \mid x = 2y\}$. Then $\mathcal{L}(H_U) = \{(2z, z) \mid z \in \mathbb{Z}\} \Rightarrow \pi(\mathcal{L}(H_U)) = 2\mathbb{Z} \neq \mathbb{Z}$. When $U = \{v_2, v_3\}$, H_U is $\{(2, y) \mid y \in \mathbb{R}\}$. Then $\pi(\mathcal{L}(H_U)) = \{2\} \neq \mathbb{Z}$. Therefore, P_1 is not a lattice-face polytope.

Let P_2 be the polytope with vertices $(0, 0), (1, 1)$ and $(2, 0)$. One can check that P_2 is a lattice-face polytope.

LEMMA 3.3. *Let P be a lattice-face d-polytope with vertex set V, then we have:*

(i) *$\pi(P)$ is a lattice-face $(d-1)$-polytope.*
(ii) *mP is a lattice-face d-polytope, for any positive integer m.*
(iii) *π induces a bijection between $\mathcal{L}(NB(P))$ and $\mathcal{L}(\pi(P))$.*
(iv) *Any d-subset U of V forms a $(d-1)$-simplex. Thus $\pi(\text{conv}(U))$ is a $(d-1)$-simplex.*
(v) *Let H be the hyperplane determined by some d-subset of V. Then for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\rho(y, H)$ is a lattice point.*
(vi) *P is an integral polytope.*

PROOF. (i), (ii), (iii), (iv) and (v) are easy to prove. We prove (vi) by induction on d.

Any 1-dimensional lattice-face polytope is integral by definition.

For $d \geq 2$, suppose any $(d-1)$ dimensional lattice-face polytope is an integral polytope. Let P be a d dimensional lattice-face polytope with vertex set V. For any vertex $v_0 \in V$, let U be a subset of V that contains v_0. Let $U = \{v_0, v_1, \ldots, v_{d-1}\}$. We know that $P' = \pi(\text{conv}(U))$ is a lattice-face $(d-1)$-simplex with vertices $\{\pi(v_0), \ldots, \pi(v_{d-1})\}$. Thus, by the induction hypothesis, P' is an integral polytope. In particular, $\pi(v_0)$ is a lattice point. Therefore, $v_0 = \rho(\pi(v_0), H)$ is a lattice point. $\square$

The main theorem of this paper is to describe all of the coefficients of the Ehrhart polynomial of a lattice-face polytope.

Theorem 3.4. *Let P be a lattice-face d-polytope, then*

$$i(P,m) = \mathrm{Vol}(mP) + i(\pi(P),m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{(d-k)}(P))m^k. \tag{3.1}$$

However, by Lemma 3.3/(ii),(iii), we have that

$$i(P,m) = |\mathcal{L}(\Omega(mP))| + i(\pi(P),m).$$

Therefore, to prove Theorem 3.4 it is sufficient to prove the following theorem:

Theorem 3.5. *For any P a lattice-face polytope,*

$$|\mathcal{L}(\Omega(P))| = \mathrm{Vol}(P).$$

Remark 3.6. We have an alternative definition of lattice-face polytopes, which is equivalent to Definition 3.1. Indeed, a d-polytope on a vertex set V is a lattice-face polytope if and only if for all k with $0 \le k \le d-1$,

$$\text{for any } (k+1)\text{-subset } U \subset V,\ \pi^{d-k}(\mathcal{L}(H_U)) \cong \mathbb{Z}^k, \tag{3.2}$$

where H_U is the affine space determined by U. In other words, after dropping the last $d-k$ coordinates of the lattice of H_U, we get the k-dimensional lattice.

4. A signed decomposition of the nonnegative part of a simplex in general position

The volume of a polytope is not very hard to characterize. So our main problem is to find the a way to describe the number of lattice points in the nonnegative part of a lattice-face polytope. We are going to do this via a signed decomposition.

4.1. Polytopes in general position. For the decomposition, we will work with a more general type of polytope (which contains the family of lattice-face polytopes).

Definition 4.1. We say that a d-polytope P with vertex set V is in *general position* if for any $k : 0 \le k \le d-1$, and any $(k+1)$-subset $U \subset V$, $\pi^{d-k}(\mathrm{conv}(U))$ is a k-simplex, where $\mathrm{conv}(U)$ is the convex hull of all of points in U.

It's easy to see that a lattice-face polytope is a polytope in general position. Therefore, the following discussion can be applied to lattice-face polytopes.

The following lemma states some properties of a polytope in general position. The proof is omitted.

Lemma 4.2. *Given a d-polytope P in general position with vertex set V, then*

(i) *P satisfies (2.1).*

(ii) *For any nonempty subset U of V, let $Q = \mathrm{conv}(U)$. If U has dimension $k (0 \le k \le d)$, then $\pi^{d-k}(Q)$ is a k-polytope in general position. In particular, for any facet F of P, $\pi(F)$ is a $(d-1)$-polytope in general position.*

(iii) *For any triangulation of $P = \bigsqcup_{i=1}^k P_i$ without introducing new vertices, $\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i)$. Thus, $\mathcal{L}(\Omega(P)) = \bigoplus_{i=1}^k \mathcal{L}(\Omega(P_i))$.*

(iv) *For any hyperplane H determined by one facet of P and any $y \in \mathbb{R}^{d-1}$, $\rho(y,H)$ is one point.*

Remark 4.3. By (iii), and because $\mathrm{Vol}(\bigsqcup_{i=1}^k P_i) = \sum_{i=1}^k \mathrm{Vol}(P_i)$, to prove Theorem 3.5 it is sufficient to prove the case when P is a lattice-face simplex.

Therefore, we will only construct our decomposition in the case of simplices in general position. However, before the construction, we need one more proposition about the nonnegative part of a polytope in general position.

Proposition 4.4. *Let P be a d-polytope in general position with facets $F_1, F_2 \dots F_k$. Let H be the hyperplane determined by F_k. For $i : 1 \le i \le k$, let $F_i' = \pi^{-1}(\pi(F_i)) \cap H$ and $Q_i = \mathrm{conv}(F_i \cup F_i')$.*

Then

$$\Omega(P) = -\,\mathrm{sign}(F_k) \bigoplus_{i=1}^{k-1} \mathrm{sign}(F_i)\rho^+(\Omega(\pi(F_i)), Q_i). \tag{4.1}$$

The proof of Proposition 4.4 is similar to the proof of Proposition 2.6 in [**6**], so we do not include it here.

Now, we can use this proposition to inductively construct a decomposition of the nonnegative part $\Omega(P)$ of a d-simplex P in general position into $d!$ signed sets.

Decomposition of $\Omega(P)$:

- If $d = 1$, we do nothing: $\Omega(P) = \Omega(P)$.
- If $d \geq 2$, then by applying Proposition 4.4 to P and letting $k = d+1$, we have

$$\Omega(P) = -\operatorname{sign}(F_{d+1}) \bigoplus_{i=1}^{d} \operatorname{sign}(F_i)\rho^+(\Omega(\pi(F_i)), Q_i).$$

However, by Lemma 4.2/(ii), each $\pi(F_i)$ is a $(d-1)$-simplex in general position. By the induction hypothesis, $\Omega(\pi(F_i)) = \bigoplus_{j=1}^{(d-1)!} S_{i,j}$, where $S_{i,j}$'s are signed sets.

$$\rho^+(\Omega(\pi(F_i)), Q_i) = \rho^+(\bigoplus_{j=1}^{(d-1)!} S_{i,j}, Q_i) = \bigoplus_{j=1}^{(d-1)!} \rho^+(S_{i,j}, Q_i).$$

Since each $\rho^+(S_{i,j}, Q_i)$ is a signed set, we have decomposed $\Omega(P)$ into $d!$ signed sets.

Now we know that we can decompose $\Omega(P)$ into $d!$ signed sets. But we still need to figure out what these sets are and which signs they have. In the next subsection, we are going to discuss the sign of a facet of a d-simplex, which is going to help us determine the signs in our decomposition.

4.2. The sign of a facet of a d-simplex. From now on, we will always use the following setup for a d-simplex unless otherwise stated:

Suppose P is a d-simplex in general position with vertex set $V = \{v_1, v_2, \ldots, v_{d+1}\}$, where the coordinates of v_i are $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})$.

For any i, we denote by F_i the facet determined by vertices in $V \setminus \{v_i\}$ and H_i the hyperplane determined by F_i.

For any $\sigma \in S_d$ and $k : 1 \leq k \leq d$, we define matrices $X(\sigma, k)$ and $Y(\sigma, k)$ to be the matrices

$$X(\sigma,k) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k} \\ 1 & x_{d+1,1} & x_{d+1,2} & \cdots & x_{d+1,k} \end{pmatrix}_{(k+1)\times(k+1)},$$

$$Y(\sigma,k) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k-1} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k-1} \end{pmatrix}_{k\times k}.$$

We also define $z(\sigma, k)$ to be

$$z(\sigma,k) = \det(X(\sigma,k))/\det(Y(\sigma,k)),$$

where $\det(M)$ is the determinant of a matrix M.

Now we can determine the sign of a facet F_i of P by looking at the determinants of these matrices, denoting by $\operatorname{sign}(x)$ the usual definition of sign of a real number x.

Lemma 4.5. (i) *$\forall i : 1 \leq i \leq d$ and $\forall \sigma \in S_d$ with $\sigma(d) = i$,*

$$\operatorname{sign}(F_i) = \operatorname{sign}(\det(X(\sigma,d))/\det(X(\sigma,d-1))). \tag{4.2}$$

(ii) *When $i = d+1$ and for $\forall \sigma \in S_d$,*

$$\operatorname{sign}(F_{d+1}) = -\operatorname{sign}(\det(X(\sigma,d))/\det(Y(\sigma,d))) = -\operatorname{sign}(z(\sigma,d)). \tag{4.3}$$

PROOF. For any $i : 1 \leq i \leq d+1$, let $v_i' = \rho(\pi(v_i), H_i)$, i.e. v_i' is the unique point of the hyperplane spanned by F_i which has the same coordinates as v_i except for the last one. Suppose the coordinates of v_i' are $(x_{i,1}, \dots, x_{i,d-1}, x_{i,d}')$. Then

$$\operatorname{sign}(F_i) = -\operatorname{sign}(x_{i,d} - x_{i,d}').$$

$\forall i : 1 \leq i \leq d$ and $\forall \sigma \in S_d$ with $\sigma(d) = i$, because v_i' is in the hyperplane determined by F_i, we have that

$$\det\left(\begin{pmatrix} 1 & x_{\sigma(1),1} & \cdots & x_{\sigma(1),d-1} & x_{\sigma(1),d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{\sigma(d-1),1} & \cdots & x_{\sigma(d-1),d-1} & x_{\sigma(d-1),d} \\ 1 & x_{\sigma(d),1} & \cdots & x_{\sigma(d),d-1} & x_{\sigma(d),d}' \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d-1} & x_{d+1,d} \end{pmatrix}\right) = 0.$$

Therefore,

$$\det(X(\sigma, d)) = (-1)^{2d+1}(x_{i,d} - x_{i,d}') \det(X(\sigma, d-1)).$$

Thus,

$$\operatorname{sign}(\det(X(\sigma, d))/\det(X(\sigma, d-1))) = -\operatorname{sign}(x_{i,d} - x_{i,d}') = \operatorname{sign}(F_i).$$

We can similarly prove the formula for $i = d+1$. □

4.3. Decomposition formulas. The following theorem describes the signed sets in our decomposition.

THEOREM 4.6. *Let P be a d-simplex with vertex set $V = \{v_1, v_2, \dots, v_{d+1}\}$, where the coordinates of v_i are $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d})$. For any $\sigma \in S_d$, and $k : 0 \leq k \leq d-1$, let $v_{\sigma,k}$ be the point with first k coordinates the same as v_{d+1} and affinely dependent with $v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}, v_{\sigma(k+1)}$. (Because P is in general position, one sees that there exists one and only one such point.) We also let $v_{\sigma,d} = v_{d+1}$. Then*

$$\Omega(P) = \bigoplus_{\sigma \in S_d} \operatorname{sign}(\sigma, P) S_\sigma,$$

where

$$\operatorname{sign}(\sigma, P) = \operatorname{sign}(\det(X(\sigma, d))) \prod_{i=1}^{d} \operatorname{sign}(z(\sigma, i)), \tag{4.4}$$

and

$$S_\sigma = \{s \in \mathbb{R}^d \mid \pi^{d-k}(s) \in \Omega(\pi^{d-k}(\operatorname{conv}(\{v_{\sigma,0}, \dots, v_{\sigma,k}\})))\forall 1 \leq k \leq d\} \tag{4.5}$$

is a set of weight 1, i.e. a regular set.

Hence,

$$\mathcal{L}(\Omega(P)) = \bigoplus_{\sigma \in S_d} \operatorname{sign}(\sigma, P) \mathcal{L}(S_\sigma).$$

PROOF. Proof by induction. □

COROLLARY 4.7. *If P is a d-simplex in general position, then*

$$|\mathcal{L}(\Omega(P))| = \sum_{\sigma \in S_d} \operatorname{sign}(\sigma, P) |\mathcal{L}(S_\sigma)|. \tag{4.6}$$

Therefore, if we can calculate the number of lattice points in S_σ's, then we can calculate the number of lattice points in the nonnegative part of a d-simplex in general position. However, it's not so easy to find $|\mathcal{L}(S_\sigma)|$'s for an arbitrary polytope. But we can do it for any lattice-face d-polytope.

5. Lattice enumeration in S_σ and Bernoulli polynomials

In this section, we will count the number of lattice points in S_σ's when P is a lattice-face d-simplex. This calculation involves Bernoulli polynomials.

5.1. Counting lattice points in S_σ. We say a map from $\mathbb{R}^d \to \mathbb{R}^d$ is *lattice preserving* if it is invertible and it maps lattice points to lattice points. Clearly, given a lattice preserving map f, for any set $S \in \mathbb{R}^d$ we have that $|\mathcal{L}(S)| = |\mathcal{L}(f(S))|$.

Let P be a lattice face d-simplex with vertex set $V = \{v_1, \dots, v_{d+1}\}$, where we use the same setup as before for d-simplices.

Given any $\sigma \in S_d$, recall that S_σ is defined as in (4.5). To count the number of lattice points in S_σ, we want to find a lattice preserving affine transformation which simplifies the form of S_σ.

Before trying to find such a transformation, we will define more notations.

For any $\sigma \in S_d$, $k : 1 \le k \le d$ and $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we define matrix $\tilde{X}(\sigma, k; x)$ as

$$\tilde{X}(\sigma,k;x) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k} \\ 1 & x_1 & x_2 & \cdots & x_k \end{pmatrix}_{(k+1)\times(k+1)},$$

and for $j : 0 \le j \le k$, let $M(\sigma, k; j)$ be the minor of the matrix $\tilde{X}(\sigma, k; x)$ obtained by omitting the last row and the $(j+1)$th column. Then

$$\det(\tilde{X}(\sigma,k;x)) = (-1)^k (M(\sigma,k;0) + \sum_{j=1}^{k} (-1)^j M(\sigma,k;j) x_j). \tag{5.1}$$

Note that $M(\sigma, k; k) = \det(Y(\sigma, k))$. Therefore,

$$\frac{\det(\tilde{X}(\sigma,k;x))}{\det(Y(\sigma,k))} = (-1)^k \frac{M(\sigma,k;0)}{\det(Y(\sigma,k))} + \sum_{j=1}^{k-1} (-1)^{k+j} \frac{M(\sigma,k;j)}{\det(Y(\sigma,k))} x_j + x_k. \tag{5.2}$$

Lemma 5.1. *Suppose P is a lattice-face d-simplex. $\forall \sigma \in S_d, \forall k : 1 \le k \le d$, and $\forall j : 0 \le j \le k-1$, we have that*

$$\frac{M(\sigma,k;j)}{\det(Y(\sigma,k))} \in \mathbb{Z}.$$

This lemma, as well as Lemma 5.6, can be directly derived from the definition of the lattice-face polytopes. We omit the proofs here.

Given this lemma, we have the following proposition.

Proposition 5.2. *There exist a lattice-preserving affine transformation T_σ which maps $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ to*

$$(\frac{\det(\tilde{X}(\sigma,1;x))}{\det(Y(\sigma,1))}, \frac{\det(\tilde{X}(\sigma,2;x))}{\det(Y(\sigma,2))}, \dots, \frac{\det(\tilde{X}(\sigma,d;x))}{\det(Y(\sigma,d))}).$$

Proof. Let $\alpha_\sigma = (-\frac{M(\sigma,1;0)}{\det(Y(\sigma,1))}, \frac{M(\sigma,2;0)}{\det(Y(\sigma,2))}, \dots, (-1)^d \frac{M(\sigma,d;0)}{\det(Y(\sigma,d))})$ and $M_\sigma = (m_{\sigma,j,k})_{d\times d}$, where

$$m_{\sigma,j,k} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j > k \\ (-1)^{k+j} \frac{M(\sigma,k;j)}{\det(Y(\sigma,k))} & \text{if } j < k \end{cases}.$$

We define $T_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ by mapping x to $\alpha_\sigma + x M_\sigma$. By (5.2),

$$\alpha_\sigma + x M_\sigma = (\frac{\det(\tilde{X}(\sigma,1;x))}{\det(Y(\sigma,1))}, \frac{\det(\tilde{X}(\sigma,2;x))}{\det(Y(\sigma,2))}, \dots, \frac{\det(\tilde{X}(\sigma,d;x))}{\det(Y(\sigma,d))}).$$

Also, because all of the entries in M_σ and α_σ are integers and the determinant of M_σ is 1, T_σ is lattice preserving. $\square$

Corollary 5.3. *Given P a lattice-face polytope with vertex set $V = \{v_1, v_2, \dots, v_{d+1}\}$, we have that*

(i) *$\forall i : 1 \le i \le d$, the last $d+1-i$ coordinates of $T_\sigma(v_{\sigma(i)})$ are all zero.*

(ii) *$T_\sigma(v_{d+1}) = (z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d))$.*

(iii) *Recall that for $k : 0 \leq k \leq d-1$, $v_{\sigma,k}$ is the point with first k coordinates the same as v_{d+1} and affinely dependent with $v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}, v_{\sigma(k+1)}$. Then the first k coordinates of $T_\sigma(v_{\sigma,k})$ are the same as $T_\sigma(v_{d+1})$ and the rest of the coordinates are zero.*

PROOF. (i) This follows from that fact that $\det(\tilde{X}(\sigma, k; x_{\sigma(i)})) = 0$ if $i \leq k \leq d$.

(ii) This follows from the fact that $\tilde{X}(\sigma, k; x_{d+1}) = X(\sigma, k)$ and $z(\sigma, k) = \det(X(\sigma, k))/\det(Y(\sigma, k))$.

(iii) Because for any $x \in \mathbb{R}^d$, the kth coordinate of T_σ only depends on the first k coordinates of x, $T_\sigma(v_{\sigma,k})$ has the same first k coordinates as $T_\sigma(v_{d+1})$. T_σ is an affine transformation. So $T_\sigma(v_{\sigma,k})$ is affinely dependent with $T_\sigma(v_{\sigma(1)}), T_\sigma(v_{\sigma(2)}), \dots, T_\sigma(v_{\sigma(k)}), T_\sigma(v_{\sigma(k+1)})$, the last $d-k$ coordinates of which are all zero. Therefore the last $d-k$ coordinates of $T_\sigma(v_{\sigma,k})$ are all zero as well.

□

Recalling that $v_{\sigma,d} = v_{d+1}$, we are able to describe $T_\sigma(S_\sigma)$ now.

PROPOSITION 5.4. *Let $\widehat{S}_\sigma = T_\sigma(S_\sigma)$. Then*

$$s = (s_1, s_2, \dots, s_d) \in \widehat{S}_\sigma \Leftrightarrow \forall 1 \leq k \leq d, s_k \in \Omega(\operatorname{conv}(0, \frac{z(\sigma,k)}{z(\sigma,k-1)} s_{k-1})), \tag{5.3}$$

where by convention we let $z(\sigma, 0) = 1$ and $s_0 = 1$.

PROOF. This can be deduced from the fact that

$$\widehat{S}_\sigma = \{s \in \mathbb{R}^d \mid \pi^{d-k}(s) \in \Omega(\pi^{d-k}(\operatorname{conv}(\{\widehat{v}_{\sigma,0}, \dots, \widehat{v}_{\sigma,k}\})))\forall 1 \leq k \leq d\},$$

where $\widehat{v}_{\sigma,i} = (z(\sigma, 1), \dots, z(\sigma, i), 0, \dots, 0)$, for $0 \leq i \leq d$.

□

Because T_σ is a lattice preserving map, $|\mathcal{L}(S_\sigma)| = |\mathcal{L}(\widehat{S}_\sigma)|$. Hence, our problem becomes to find the number of lattice points in $\widehat{S}_\sigma$. However, $\widehat{S}_\sigma$ is much nicer than S_σ. Actually, we can give a formula to calculate all of the sets having the same shape as $\widehat{S}_\sigma$.

LEMMA 5.5. *Given real nonzero numbers $b_0 = 1, b_1, b_2, \dots, b_d$, let $a'_k = b_k/b_{k-1}$ and $a_k = b_k/|b_{k-1}|, \forall k : 1 \leq k \leq d$. Let S be the set defined by the following:*

$$s = (s_1, s_2, \dots, s_d) \in S \Leftrightarrow \forall 1 \leq k \leq d, s_k \in \Omega(\operatorname{conv}(0, a'_k s_{k-1})),$$

where s_0 is set to 1. Then

$$|\mathcal{L}(S)| = \sum_{s_1 \in \mathcal{L}(\Omega(\operatorname{conv}(0,a'_1)))} \sum_{s_2 \in \mathcal{L}(\Omega(\operatorname{conv}(0,a'_2 s_1)))} \cdots \sum_{s_d \in \mathcal{L}(\Omega(\operatorname{conv}(0,a'_d s_{d-1})))} 1.$$

In particular, if $b_d > 0$, then

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{\overline{\lfloor a_1 \rfloor}} \sum_{s_2=1}^{\overline{\lfloor a_2 s_1 \rfloor}} \cdots \sum_{s_d=1}^{\overline{\lfloor a_d s_{d-1} \rfloor}} 1,$$

where for any real number x, $\lfloor x \rfloor$ is the largest integer no greater than x and $\overline{x}$ is defined as

$$\overline{x} = \begin{cases} x, & \text{if } x \geq 0 \\ -x-1, & \text{if } x < 0 \end{cases}.$$

PROOF. The first formula is straightforward. The second formula follows from the facts that for any real numbers x,

$$\mathcal{L}(\Omega(\operatorname{conv}(0, x))) = \begin{cases} \{z \in \mathbb{Z} \mid 1 \leq z \leq \overline{\lfloor x \rfloor}\} & \text{if } x \geq 0 \\ \{z \in \mathbb{Z} \mid -\overline{\lfloor x \rfloor} \leq z \leq 0\} & \text{if } x < 0 \end{cases},$$

the sign of s_i is the same as the sign of b_i, and because $b_d > 0$, all the s_i's are non-zero. □

However, for lattice-polytopes, we have another good property of the $z(\sigma, k)$'s.

LEMMA 5.6. *If P is a lattice-polytope d-simplex, then*

$$z(\sigma, k)/z(\sigma, k-1) \in \mathbb{Z}.$$

For any lattice-face d-simplex P, we can always find a way to order its vertices into $V = \{v_1, v_2, \dots, v_{d+1}\}$, so that the corresponding $\det(X(1,d))$ and $\det(Y(1,d))$ are positive, where 1 stands for the identity permutation in S_d. Note $z(\sigma, d)$ is independent of σ. So it is positive. Therefore, by Lemma 5.5 and Lemma 5.6, we have the following result.

PROPOSITION 5.7. *Let P be a lattice-face d-simplex with vertex set V, where the order of vertices makes both $\det(X(1,d))$ and $\det(Y(1,d))$ positive. Define*

$$a(\sigma, k) = \frac{z(\sigma,k)}{|z(\sigma,k-1)|}, \forall k : 1 \le k \le d.$$

Then

$$|\mathcal{L}(S_\sigma)| = \sum_{s_1=1}^{\overline{a(\sigma,1)}} \sum_{s_2=1}^{\overline{a(\sigma,2)s_1}} \cdots \sum_{s_d=1}^{\overline{a(\sigma,d)s_{d-1}}} 1. \tag{5.4}$$

Because of (5.4), it's natural for us to define

$$f_d(a_1, a_2, \dots, a_d) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1, \tag{5.5}$$

for any positive integers $a_1, a_2, \dots, a_d$. However, since f_d is just a polynomial in the a_i's, we can extend the domain of f_d from $\mathbb{Z}^d_{>0}$ to $\mathbb{Z}^d$ or even $\mathbb{R}^d$. And for convenience, we still use the form of (5.5) to write $f_d(a_1, \dots, a_n)$ even when a_i's are not all positive integers.

Also, fixing $b_0 = 1$, we define

$$g_d(b_1, b_2, \dots, b_d) = f_d(b_1/b_0, b_2/b_1, \dots, b_d/b_{d-1}),$$

for any $(b_1, b_2, \dots, b_d) \in (\mathbb{R} \setminus \{0\})^d$.

f_d and g_d are closely related to formula (5.4). In next subsection, we will discuss Bernoulli polynomials and power sums, which are connected to f_d and g_d, and then rewrite (5.4) in terms of g_d. Please refer to [**3**, Section 2.4] for other examples about Bernoulli polynomials and their relation to lattice polytopes.

5.2. Power sums and Bernoulli polynomials. The *kth Bernoulli polynomials*, $B_k(x)$, is defined as [**1**, p. 804]

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!},$$

The Bernoulli polynomials satisfy [**5**]

$$B_k(1-x) = (-1)^k B_k(x), \forall k \ge 0, \tag{5.6}$$

as well as the relation [**8**, p. 127]

$$B_k(x+1) - B_k(x) = kx^{k-1}, \forall k \ge 1. \tag{5.7}$$

We call $B_k = B_k(0)$ a *Bernoulli number*. It satisfies [**7**] that

$$B_k(0) = 0, \text{for any odd number } k \ge 3. \tag{5.8}$$

For $k \ge 0$, let

$$S_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}}{k+1}.$$

Given any n a nonnegative integer, by (5.7), we have that

$$S_k(n) = \sum_{i=0}^{n} i^k = \begin{cases} \sum_{i=1}^{n} i^k & \text{if } k \ge 1 \\ n+1 & \text{if } k = 0 \end{cases}.$$

Therefore, we call $S_k(x)$ the *kth power sum polynomial.*

LEMMA 5.8. *For any $k \ge 1$, the constant term of $S_k(x)$ is 0, i.e., x is a factor of $S_k(x)$, and*

$$S_k(x) = (-1)^{k+1} S_k(-x-1). \tag{5.9}$$

PROOF. The constant term of $S_k(x)$ is $S_k(0) = 0$. The formula follows from (5.6) and (5.8). $\square$

Lemma 5.9. *$f_d(a_1, \dots, a_d)$ is a polynomial in a_1 of degree d. And $\prod_{i=1}^d a_i$ is a factor of it. In particular, f_d can be written as*

$$f_d(a_1, \dots, a_d) = \sum_{k=1}^{d} f_{d,k}(a_2, \dots, a_d) a_1^k, \tag{5.10}$$

where $f_{d,k}(a_2, \dots, a_d)$ is a polynomial in $a_2, \dots, a_d$ with a factor $\prod_{i=2}^d a_i$.

Proof. This can be proved by induction on d, using the fact that $S_k(x)$ has a factor x. □

Proposition 5.10. *Given $s_0 = 1, a = (a_1, a_2, \dots, a_d) \in \mathbb{R}^d$, for any $j : 1 \leq j \leq d-1$,*

$$f_d(a_1, a_2, \dots, a_d) = -\sum_{s_1=1}^{a_1 s_0} \cdots \sum_{s_{j-1}=1}^{a_{j-1}s_{j-2}} \sum_{s_j=1}^{-a_j s_{j-1}-1} \sum_{s_{j+1}=1}^{-a_{j+1}s_j} \sum_{s_{j+2}=1}^{a_{j+2}s_{j+1}} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1.$$

Given $b = (b_1, b_2, \dots, b_d) \in (\mathbb{R} \setminus \{0\})^d$ with $b_d > 0$, let $a_k = b_k/|b_{k-1}|$, then

$$g_d(b_1, b_2, \dots, b_d) = \operatorname{sign}\left(\prod_{i=1}^d b_i\right) \sum_{s_1=1}^{\overline{a_1}} \sum_{s_2=1}^{\overline{a_2 s_1}} \cdots \sum_{s_d=1}^{\overline{a_d s_{d-1}}} 1. \tag{5.11}$$

Proof. This follows from (5.9), (5.10) and an inductive argument. □

Proposition 5.11. *Let P be a lattice-face d-simplex with vertex set V, where the order of vertices makes both $\det(X(1,d))$ and $\det(Y(1,d))$ positive. Then*

$$|\mathcal{L}(S_\sigma)| = \operatorname{sign}\left(\prod_{i=1}^d z(\sigma, i)\right) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)). \tag{5.12}$$

Therefore,

$$|\mathcal{L}(\Omega(P))| = \sum_{\sigma \in S_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)). \tag{5.13}$$

Proof. We can get (5.12) by comparing (5.4) and (5.11). And (5.13) follows from (4.6), (4.4), (5.12) and the fact that $\det(X(\sigma,d)) = \operatorname{sign}(\sigma)\det(X(1,d))$. □

6. Proof of the Main Theorems

We now have all the ingredients but one to prove Theorem 3.5. The missing one is stated as the following proposition.

Proposition 6.1. *Let $V = \{v_1, v_2, \dots, v_{d+1}\}$ be the vertex set of a d-simplex in general position, where the coordinates of v_i are $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d})$. Then*

$$\sum_{\sigma \in S_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)) = \frac{1}{d!}\det(X(1,d)). \tag{6.1}$$

Given this proposition, we can prove Theorem 3.5.

Proof of Theorem 3.5. As we mentioned in Remark 4.3, to prove Theorem 3.5, it is sufficient to prove the case when P is a lattice-face simplex.

When P is a lattice-face d-simplex, we still assume that the order of the vertices of P makes both $\det(X(1,d))$ and $\det(Y(1,d))$ positive. Thus, (5.13), (6.1) and the fact that the volume of P is $\frac{1}{d!}|\det(X(1,d))|$ imply that

$$|\mathcal{L}(\Omega(P))| = \operatorname{Vol}(P).$$

□

As we mentioned earlier, Theorem 3.4 follows from Theorem 3.5.

The proof of Proposition 6.1 is lengthy and self-contained, so we do not include it here.

7. Examples and Further discussion

7.1. Examples of lattice-face polytopes. In this subsection, we use a fixed family of lattice-face polytopes to illustrate our results. Let $d = 3$, and for any positive integer k, let P_k be the polytope with the vertex set $V = \{v_1 = (0,0,0), v_2 = (4,0,0), v_3 = (3,6,0), v_4 = (2,2,10k)\}$. One can check that P_k is a lattice-face polytope.

EXAMPLE 7.1 (Example of Theorem 3.4). The volume of P_k is $40k$, and

$$i(P_k, m) = 40km^3 + 12m^2 + 4m + 1.$$

$\pi(P_k) = \mathrm{conv}\{(0,0), (4,0), (3,6)\}$, where

$$i(\pi(P_k), m) = 12m^2 + 4m + 1.$$

So

$$i(P_k, m) = 40km^3 + i(\pi(P_k), m),$$

which agrees with Theorem 3.4.

EXAMPLE 7.2 (Example of Formula (4.1)). $F_4 = \mathrm{conv}(v_1, v_2, v_3)$ is a negative facet. The hyperplane determined by F_4 is $H = \{(x_1, x_2, x_3) \mid x_3 = 0\}$. Thus, $v_4' = \pi^{-1}(\pi(v_4)) \cap H = (2,2,0)$.

$F_3 = \mathrm{conv}(v_1, v_2, v_4)$ is a positive facet. $\pi(F_3) = \mathrm{conv}((0,0),(4,0),(2,2))$. $\Omega(\pi(F_3)) = \pi(F_3) \backslash \mathrm{conv}((0,0),(4,0))$. $F_3' = \pi^{-1}(\pi(F_3)) \cap H = \mathrm{conv}(v_1, v_2, v_4')$. So

$$Q_3 = \mathrm{conv}(F_3 \cup F_3') = \mathrm{conv}(v_1, v_2, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_3)), Q_3) = Q_3 \setminus F_3'.$$

$F_2 = \mathrm{conv}(v_1, v_3, v_4)$ is a positive facet. $\pi(F_2) = \mathrm{conv}((0,0),(3,6),(2,2))$. $\Omega(\pi(F_2)) = \pi(F_2) \backslash (\mathrm{conv}((0,0),(2,2)) \cup \mathrm{conv}((2,2),(3,6)))$. $F_2' = \pi^{-1}(\pi(F_2)) \cap H = \mathrm{conv}(v_1, v_3, v_4')$. So

$$Q_2 = \mathrm{conv}(F_2 \cup F_2') = \mathrm{conv}(v_1, v_3, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_2)), Q_2) = Q_2 \setminus (F_2' \cup \mathrm{conv}(v_1, v_4, v_4') \cup \mathrm{conv}(v_3, v_4, v_4')).$$

$F_1 = \mathrm{conv}(v_2, v_3, v_4)$ is a positive facet. $\pi(F_1) = \mathrm{conv}((4,0),(3,6),(2,2))$. $\Omega(\pi(F_1)) = \pi(F_1) \backslash \mathrm{conv}((4,0),(2,2))$. $F_1' = \pi^{-1}(\pi(F_1)) \cap H = \mathrm{conv}(v_2, v_3, v_4')$. So

$$Q_1 = \mathrm{conv}(F_1 \cup F_1') = \mathrm{conv}(v_2, v_3, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_1)), Q_1) = Q_1 \setminus (F_1' \cup \mathrm{conv}(v_2, v_4, v_4')).$$

Therefore,

$$\Omega(P_k) = P_k \setminus F_4 = -\,\mathrm{sign}(F_4) \bigoplus_{i=1}^{3} \mathrm{sign}(F_i) \rho^+(\Omega(\pi(F_i)), Q_i),$$

which agrees with Proposition 4.4.

EXAMPLE 7.3 (Example of Decomposition). In this example, we decompose P_k into $3!$ sets, where 5 of them have positive signs and one has negative sign, which is different from the cases for cyclic polytopes, where half of the sets have positive signs and the other half have negative signs.

Recall that $v_{\sigma,3} = v_4 = (2,2,10k)$, for any $\sigma \in S_3$.

When $\sigma = 123 \in S_3$, $v_{123,2} = v_4' = (2,2,0)$, $v_{123,1} = (2,0,0)$ and $v_{123,0} = v_1 = (0,0,0)$. Then

$$S_{123} = \mathrm{conv}(\{v_{123,i}\}_{0 \le i \le 3}) \setminus \mathrm{conv}(\{v_{123,i}\}_{0 \le i \le 2}),$$

with $\mathrm{sign}(123, P_k) = +1$.

When $\sigma = 213 \in S_3$, $v_{213,2} = v_4' = (2,2,0)$, $v_{213,1} = (2,0,0)$ and $v_{213,0} = v_2 = (4,0,0)$. Then

$$S_{213} = \mathrm{conv}(\{v_{213,i}\}_{0 \le i \le 3}) \setminus (\mathrm{conv}(\{v_{213,i}\}_{0 \le i \le 2}) \cup \mathrm{conv}(\{v_{213,i}\}_{1 \le i \le 3})),$$

with $\mathrm{sign}(213, P_k) = +1$.

One can check that

$$S_{123} \oplus S_{213} = \rho^+(\Omega(\pi(F_3)), Q_3).$$

When $\sigma = 231 \in S_3$, $v_{231,2} = v_4' = (2,2,0)$, $v_{231,1} = (2,12,0)$ and $v_{231,0} = v_2 = (4,0,0)$. Then

$$S_{231} = \mathrm{conv}(\{v_{231,i}\}_{0 \le i \le 3}) \setminus (\mathrm{conv}(\{v_{231,i}\}_{0 \le i \le 2}) \cup \mathrm{conv}(\{v_{231,i}\}_{i=0,2,3} \cup \mathrm{conv}(\{v_{231,i}\}_{1 \le i \le 3})),$$

with $\mathrm{sign}(231, P_k) = +1$.

When $\sigma = 321 \in S_3$, $v_{321,2} = v'_4 = (2,2,0)$, $v_{321,1} = (2,12,0)$ and $v_{321,0} = v_3 = (3,6,0)$. Then

$$S_{321} = \mathrm{conv}(\{v_{321,i}\}_{0\le i\le 3}) \setminus (\mathrm{conv}(\{v_{321,i}\}_{0\le i\le 2}) \cup \mathrm{conv}(\{v_{321,i}\}_{i=0,2,3} \cup \mathrm{conv}(\{v_{321,i}\}_{1\le i\le 3})),$$

with $\mathrm{sign}(321, P_k) = -1$.

One can check that

$$S_{231} \ominus S_{321} = \rho^+(\Omega(\pi(F_1)), Q_1).$$

Similarly, we have that

$$S_{132} \oplus S_{312} = \rho^+(\Omega(\pi(F_2)), Q_2).$$

Therefore, $\Omega(P_k) = \bigoplus_{\sigma\in S_3} \mathrm{sign}(\sigma, P_k) S_\sigma$, which coincides with Theorem 4.6.

7.2. Further discussion. Recall that Remark 3.6 gives an alternative definition for lattice-face polytopes. Note in this definition, when $k = 0$, satisfying (3.2) is equivalent to say that P is an integral polytope, which implies that the last coefficient of the Ehrhart polynomial of P is 1. Therefore, one may ask

QUESTION 7.4. If P is a polytope that satisfies (3.2) for all $k \in K$, where K is a fixed subset of $\{0, 1, \ldots, d-1\}$, can we say something about the Ehrhart polynomials of P?

A special set K can be chosen as the set of consecutive integers from 0 to d', where d' is an integer no greater than $d - 1$. Based on some examples in this case, the Ehrhart polynomials seems to follow a certain pattern, so we conjecture the following:

CONJECTURE 7.5. *Given $d' \le d-1$, if P is a d-polytope with vertex set V such that $\forall k : 0 \le k \le d'$, (3.2) is satisfied, then for $0 \le k \le d'$, the coefficient of m^k in $i(P,m)$ is the same as in $i(\pi^{d-d'}(P), m)$. In other words,*

$$i(P,m) = i(\pi^{d-d'}(P), m) + \sum_{i=d'+1}^{d} c_i m^i.$$

When $d' = 0$, the condition on P is simply that it is integral. And when $d' = d-1$, we are in the case that P is a lattice-face polytope. Therefore, for these two cases, this conjecture is true.

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