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A Bijection for Unicellular Partitioned Bicolored Maps

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ABSTRACT. In the present paper we construct a bijection that relates a set $C_{N,p,q}$ of unicellular partitioned bicolored maps to a set of couples (t, σ) of ordered bicolored trees and partial permutations. This bijection allows us to derive an elegant formula for the enumeration of unicellular bicolored maps, an analogue of the well-known Harer-Zagier result for unicolored one-face maps.

RÉSUMÉ. Dans cet article nous construisons une bijection mettant en relation l'ensemble $C_{N,p,q}$ des cartes bicolores unicellulaires partitionnées et l'ensemble des couples (t, σ) d'arbres bicolores ordonnés et de permutations partielles. Cette bijection nous permet de dériver une formule élégante pour l'énumération des cartes bicolores unicellulaires, analogue au résultat de Harer et Zagier pour les cartes unicolores monofaces.

1. Introduction

Maps are graphs embedded in orientable surfaces. More precisely, a *map* is a $2-$cell decomposition of a compact, connected, orientable surface into vertices ($0-$cells), edges ($1-$cells) and faces ($2-cells$) homeomorphic to open discs. Loops and multiple edges are allowed. A detailed description of these objects as well as examples of their numerous applications in various branches of mathematics and physics can be found in the survey [**2**] and in [**8**]. One face (unicellular) maps represent an object of special interest. In particular, Harer and Zagier enumerated unicellular maps of genus g with prescribed number of edges in order to calculate the Euler characteristics of the moduli spaces (see [**6**]). Numerous proofs of this well-known formula have been proposed. As a rule they are technically complicated and up to recently no elementary proof was known. A first purely combinatorial method was given by Lass in [**7**]. Another one involving a direct bijection was developpped by Goulden and Nica in [**4**].

This paper is focused on unicellular bicolored maps, i.e. one-face maps with white and black vertices verifying the property that each edge is joining a black and a white vertices. Formally, a unicellular bicolored map of N edges, m white and n black vertices is equivalent to a couple of permutations $(\alpha, \beta) \in \Sigma_N$ such that α has m independent cycles, β has n independent cycles and $\alpha\beta = \gamma$ where γ is the long cycle $(1\,2\,3\,\ldots\,N)$. As a first approach to this question, in [**5**], Goupil and Schaeffer derived a formula to count the number of factorizations (α, β) of γ with α of cycle type λ and β of cycle type μ, for any pair (λ, μ) of partitions of N. Summing over all λ with m parts and μ with n parts allows to recover a complicated formula for this counting problem. Independently, a more elegant formula for the enumeration of these objects has been calculated by Adrianov in [**1**]. His method involves characters on the symmetric group and the resulting formula, expressed in terms of Gauss hypergeometric function, leaves little room for simple combinatorial interpretation. In this paper, we derive a new formula solving the same enumeration problem. To this end we construct a bijection having some aspects similar to the one of Goulden and Nica in [**4**] for unicolored maps.

Throughout this paper, we adopt the following notations. We denote by $BT(p, q)$ the set of ordered bicolored (black and white)trees with p white and q black vertices. We assume that all the trees in this set

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have a white root. The cardinality of $BT(p,q)$ (see e.g. [**3**]) is given by:

$$|BT(p,q)| = \frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2 \tag{1.1}$$

We also denote by $PP(X,Y,A)$ the set of partial permutations from any subset of X of cardinality A to any subset of Y (of the same cardinality). The cardinality of this set is given by:

$$|PP(X,Y,A)| = \binom{|X|}{A}\binom{|Y|}{A}A! \tag{1.2}$$

For the sake of simplicity, in all what follows, if X or Y is equal to $[M]$ (all the integers between 1 and M), we will note M instead of $[M]$. Now let us turn to our main result:

Theorem 1.1. *The numbers $B(m,n,N)$ of unicellular bicolored maps with m white vertices, n black vertices and N edges verify:*

$$\sum_{m,n\geq 1} B(m,n,N)y^m z^n = N!\sum_{p,q\geq 1}\binom{N-1}{p-1,q-1}\binom{y}{p}\binom{z}{q} \tag{1.3}$$

In the following section we give a bijective proof for this formula. To this extent we introduce a new class of objects, the *unicellular partitioned bicolored maps*.

2. Unicellular Partitioned Bicolored Maps

2.1. Definition. Let $C_{N,p,q}$ be the set of triples (π_1,π_2,α) such that π_1 et π_2 are partitions of $[N]$ into p and q blocks and such that α is a permutation of $[N]$ verifying the following properties :

- Each block of π_1 is the union of cycles of α.
- Each block of π_2 is the union of cycles of $\beta = \alpha^{-1}\gamma$, where $\gamma = (1\,2\,\ldots\,N)$.

2.2. Geometrical Interpretation. The set $C_{N,p,q}$ can be viewed as a set of unicellular partitioned bicolored maps. A triple (π_1,π_2,α) corresponds to a unicellular bicolored map with N edges where :

- The cycles of α describe the white vertices of the map.
- The cycles of $\beta = \alpha^{-1}\gamma$ describe the black vertices.
- π_1 partitions the white vertices into p subsets
- π_2 partitions the black vertices into q subsets

Example 2.1. Figure (1) gives a representation of the triple $(\pi_1,\pi_2,\alpha)\in C_{9,3,2}$, defined by $\alpha = (1)(24)(3)(57)(6)(89)$, $\beta = (1479)(23)(56)(8)$, $\pi_1 = \{\pi_1^{(1)},\pi_1^{(2)},\pi_1^{(3)}\}$, $\pi_2 = \{\pi_2^{(1)},\pi_2^{(2)}\}$ with :

$$\pi_1^{(1)} = \{2,4,6\},\quad \pi_1^{(2)} = \{8,9\},\quad \pi_1^{(3)} = \{1,3,5,7\},$$
$$\pi_2^{(1)} = \{2,3,5,6\},\quad \pi_2^{(2)} = \{1,4,7,8,9\}$$

where the numbering of the blocks is purely arbitrary.
To visualise it better we also assume that each block is associated with some particular shape: $\pi_1^{(1)}$ with square, $\pi_1^{(2)}$ with circle, $\pi_1^{(3)}$ with triangle, $\pi_2^{(1)}$ with rhombus and $\pi_2^{(2)}$ with pentagon. Therefore each vertex of our partitioned map will have a shape corresponding to its block.

2.3. Connection with Unicellular Bicolored Maps. Let $c_{N,p,q} = |\; C_{N,p,q}\;|$. Using the Stirling number of the second kind $S(a,b)$ enumerating the partitions of a set of a elements into b non-empty, unordered subsets, we have:

$$c_{N,p,q} = \sum_{m\geq p,n\geq q} S(m,p)S(n,q)B(m,n,N) \tag{2.1}$$

Then, since $\sum_{b=1}^{a} S(a,b)(x)_b = x^a$ (see e.g [**9**]) where the falling factorial $(x)_b = \prod_{i=0}^{b-1}(x-i)$:

$$\sum_{m,n\geq 1} B(m,n,N)y^m z^n = \sum_{p,q\geq 1} c_{N,p,q}(y)_p(z)_q \tag{2.2}$$

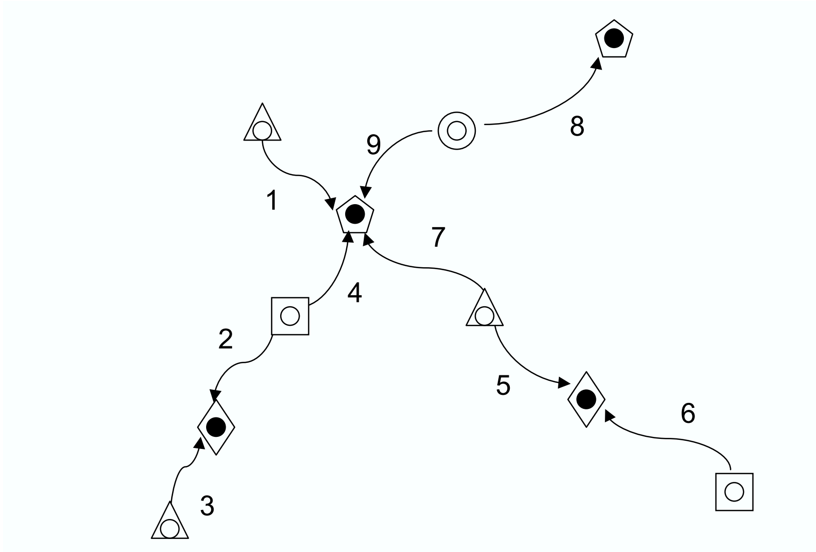

FIGURE 1. Example of a Partitioned Bicolored Map

3. Bijective Description of Unicellular Partitioned Bicolored Maps

3.1. Combinatorial Interpretation of the Main Formula. Combining equations 1.3 and 2.2 gives:

$$c_{N,p,q} = \frac{N!}{p!q!}\binom{N-1}{p-1,q-1} \tag{3.1}$$

Now if we rearrange the above formula, we get :

$$c_{N,p,q} = \left[\frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2\right]\left[\binom{N}{N+1-(p+q)}\binom{N-1}{N+1-(p+q)}(N+1-(p+q))!\right] \tag{3.2}$$

We finally have :

$$|C_{N,p,q}| = |BT(p,q)| \times |PP(N, N-1, N+1-(p+q))| \tag{3.3}$$

In order to prove our main theorem we simply need to show that the number of unicellular partitioned bicolored maps with N edges, p white blocks and q black blocks is equal to the number of bicolored trees with p white vertices and q black vertices (with a white root) times the number of partial permutations from any subset of $[N]$ containing $N+1-(p+q)$ elements to any subset of $[N-1]$ (containing $N+1-(p+q)$ elements). To this purpose we use a bijection between the appropriate sets.

3.2. Construction of the Bijection. In this section, we construct a bijective mapping that associates to a triple $(\pi_1, \pi_2, \alpha) \in C_{N,p,q}$ an ordered bicolored tree in $BT(p,q)$ and a partial permutation in $PP(N, N-1, N+1-(p+q))$.

Ordered Bicolored Tree

Let $\pi_1^{(1)}, \ldots, \pi_1^{(p)}$ and $\pi_2^{(1)}, \ldots, \pi_2^{(q)}$ be the blocks of the partitions π_1 and π_2 respectively. Denote by $m_1^{(i)}$ the maximal element of the block $\pi_1^{(i)}$ $(1 \le i \le p)$ and by $m_2^{(j)}$ the maximal element of $\pi_2^{(j)}$ $(1 \le j \le q)$. We attribute the index p to the block of partition π_1 containing the element 1. Suppose that the indexation of all other blocks is arbitrary and doesn't respect any supplementary constraints. We create a labelled ordered bicolored tree T on the set of p white and q black vertices, such that white vertices have black descendants and vice versa. The root of T is the white vertex p. For every $j = 1, \ldots, q$ we set that a black vertex j is a descendant of a white vertex i if the element $\beta(m_2^{(j)})$ belongs to the white block $\pi_1^{(i)}$. Similarly, for every

$i = 1, \ldots, p-1$ a white vertex i is a descendant of a black vertex j if the element $m_1^{(i)}$ belongs to the black block $\pi_2^{(j)}$. If black vertices j, k are both descendants of a white vertex i, then j is to the left of k when $\beta(m_2^{(j)}) < \beta(m_2^{(k)})$; if white vertices i, l are both descendants of a black vertex j, then i is to the left of l when $\beta^{-1}(m_1^{(i)}) < \beta^{-1}(m_1^{(l)})$. It can be proved that the previous construction allows to specify a unique path from any vertex i to the root vertex and thus, the tree T is well defined.

Removing the labels from T we obtain the bicolored ordered tree t.

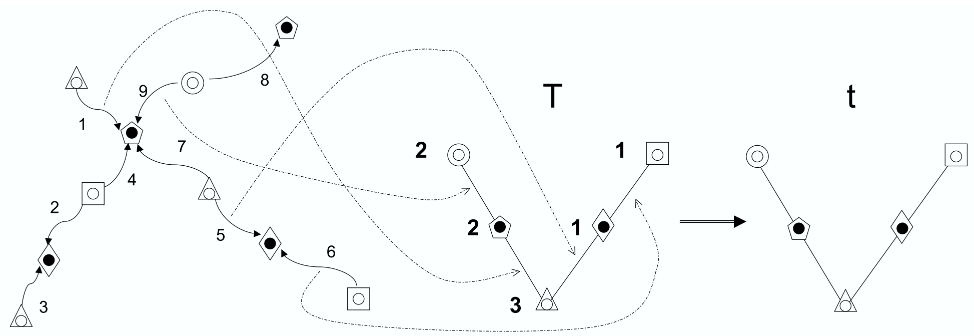

Figure 2. Construction of the Ordered Bicolored Tree

Example 3.1. Let us go back to example 2.1. We keep the previous numbering of the blocks since it verifies the condition $1 \in \pi_1^{(p)}$. For this example, $\beta(m_2^{(1)}) = \beta(6) = 5 \in \pi_1^{(3)}$ and $\beta(m_2^{(2)}) = \beta(9) = 1 \in \pi_1^{(3)}$ the black rhombus 1 and the black pentagon 2 are both descendants of the white triangle 3. Moreover, as $\beta(m_2^{(1)}) < \beta(m_2^{(2)})$ the vertex 2 is to the left of vertex 1. Further, $m_1^{(1)} = 6 \in \pi_2^{(1)}$, $m_1^{(2)} = 9 \in \pi_2^{(2)}$ and hence the white circle 1 is descendant of the black pentagon 1, while the white square 2 is descendant of the black rhombus 2. Thus, we construct first the tree T then, removing the labels, get the tree t (see Figure 3).

Partial Permutation

The construction of the partial permutation contains two main steps.

(i) *Relabelling permutations.* Consider the *reverse-labelled* bicolored tree t' resulting from the labelling of t, based on two independant reverse-labelling procedures for white and black vertices. The root is labelled p, the white vertices at level 2 are labelled from right to left, beginning with $p-1$, proceeding by labelling from right to left white vertices at level 4 and all the other even levels until reaching the leftmost white vertex at the top even level labelled by 1. The black vertices at level 1 are labelled from right to left, beginning with q, and following by labelling of the black vertices at all the other odd levels from left to right until reaching the leftmost vertex of the top odd level labelled by 1. Trees T and t' give two, possibly different, labellings of t. Suppose that the (black or white) vertex of t labelled i in T is labelled j in t'. Then define $\pi_1^j = \pi_1^{(i)}$ for white vertex and $\pi_2^j = \pi_2^{(i)}$ for black vertex, repeat this re-indexing for all white and all black vertices. We obtain a different indexing $\pi_1^1, \ldots, \pi_1^p$ of the white blocks of partition π_1; and a different indexing $\pi_2^1, \ldots, \pi_2^q$ of the black blocks of partition π_2. The reader can easily see, that $\pi_1^p = \pi_1^{(p)}$. Let ω^i and υ^j be the strings given by writing the elements of π_1^i and π_2^j in increasing order. Denote $\omega = \omega^1 \ldots \omega^p$ and $\upsilon = \upsilon^1 \ldots \upsilon^q$ concatenations of $\omega^1, \ldots, \omega^p$ and $\upsilon^1, \ldots, \upsilon^q$ respectively. We define $\lambda \in S_N$ by setting ω the first line and $[N]$ the second line of λ in the two-line representation of λ. Similarly, we define $\nu \in S_N$ by setting υ the first line and $[N]$ the second line of ν in the two-line representation of ν.

Example 3.2. Let us continue Example 3.1 by constructing relabelling permutations λ and ν. Figure 3 put the tree T and the reversed-labelled tree t' side by side that gives quite a natural

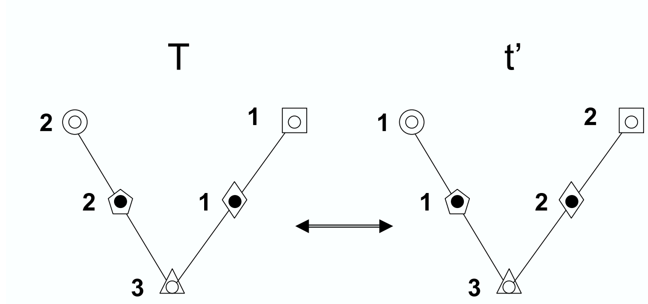

FIGURE 3. Relabelling of the Blocks

illustration of the block relabelling:

$$\begin{aligned} \pi_1^1 &= \pi_1^{(2)}, \pi_1^2 = \pi_1^{(1)}, \pi_1^3 = \pi_1^{(3)}, \\ \pi_2^1 &= \pi_2^{(2)}, \pi_2^2 = \pi_2^{(1)} \end{aligned}$$

The strings ω^i and υ^j are given by :

$$\begin{aligned} \omega^1 &= 89, \quad \omega^2 = 246, \;\; \omega^3 = 1357, \\ \upsilon^1 &= 14789, \upsilon^2 = 2356 \end{aligned}$$

We construct now the relabelling permutations λ and ν.

$$\lambda = \left(\begin{array}{cc|ccc|cccc} 8 & 9 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \right) \qquad \nu = \left(\begin{array}{ccccc|cccc} 1 & 4 & 7 & 8 & 9 & 2 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \right)$$

Figures 4 depicts this two new labellings on our example.

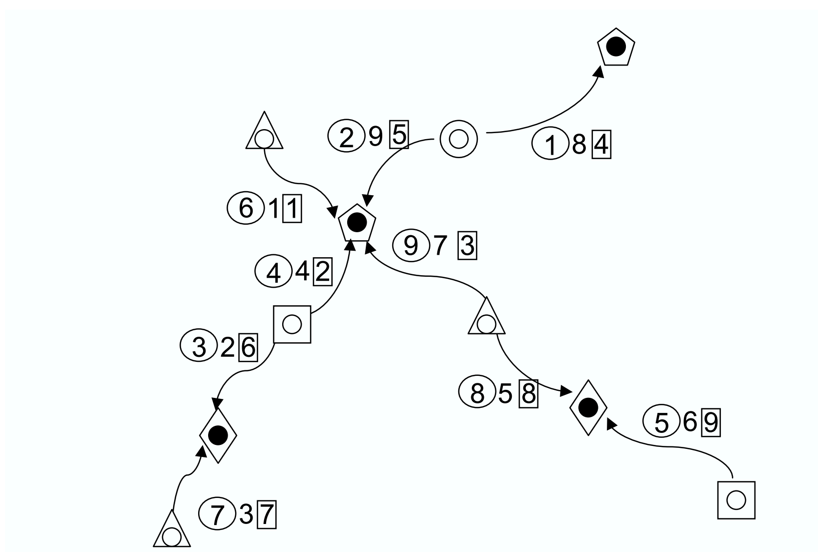

FIGURE 4. Relabellings of the Partitioned Bicolored Map

(ii) Partial permutation We can now introduce a partial permutation that gives an insight both on the connexion between the λ and ν relabelling and on the structure of the partitioned bicolored map. Let S be the subset of $[N]$ containing all the edges of the map that were not used to construct the bicolored tree. Namely :

$$S = [N] \setminus \{m_1^1, m_1^2, \ldots, m_1^{p-1}, \beta(m_2^1), \ldots, \beta(m_2^q)\} \tag{3.4}$$

We define the partial permutation σ on set $[N]$ as the composition of previously defined permutations applied to relabelling of S by λ:

$$\sigma = \nu \circ \beta^{-1} \circ \lambda^{-1} |_{\lambda(S)} \tag{3.5}$$

In Lemmae 3.4 and 3.5 we show that σ is a bijection between two subsets of $N+1-(p+q)$ elements and that its image set is included in $[N-1]$.

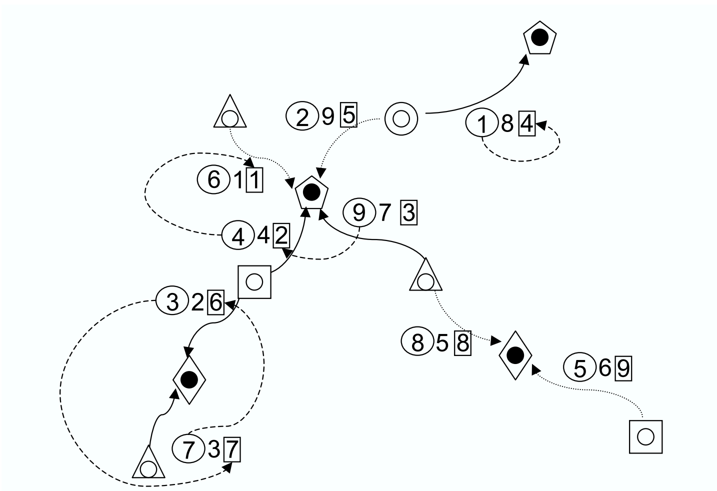

FIGURE 5. Connections through σ between λ and ν relabelling

EXAMPLE 3.3. On the example previously described the set S is equal to :

$$S = \{2, 3, 4, 7, 8\}$$

The partial permutation σ is defined by :

$$\sigma = \begin{pmatrix} 1 & 3 & 4 & 7 & 9 \\ 4 & 7 & 1 & 6 & 2 \end{pmatrix}$$

The set of vertices that were not used to construct the tree and their connections to the map through σ can be viewed on Figure 5.

LEMMA 3.4. *The cardinal of the set S defined above verifies $| S |= N+1-(p+q)$*

PROOF. To prove the assertion of this lemma we will show the equivalent statement :

$$\{m_1^1, m_1^2, \ldots, m_1^{p-1}\} \cap \{\beta(m_2^1), \ldots, \beta(m_2^q)\} = \emptyset$$

Assume that there exist i, j, $i = 1, \ldots, p-1$, $j = 1, \ldots, q$, such that

$$\beta(m_2^j) = m_1^i \tag{3.6}$$

Then as the blocks of π_2 are stable by β we have $m_1^i \in \pi_2^j$ and $m_1^i \leq m_2^j$. As the blocks of π_1 are stable by α, the assumption (3.6) also implies that $\alpha\beta(m_2^j) = \gamma(m_2^j) \in \pi_1^i$. Hence, $\gamma(m_2^j) \leq m_1^i$. Combining these two inequalities, we have $\gamma(m_2^j) \leq m_2^j$ that occurs only if $m_2^j = N$. In this case, $\gamma(m_2^j) = 1$ and $1 \in \pi_1^i$, i.e. $i = p$ which is a contradiction ■

LEMMA 3.5. *The element N does not belong to the image of permutation σ.*

PROOF. Let us remark that according to the construction of the relabelling permutation ν, we have $N = \nu(m_2^q)$. Besides,

$$\nu(m_2^q) = \nu \circ \beta^{-1} \circ \lambda^{-1}(\lambda(\beta(m_2^q)))$$

Thus, as $\lambda(\beta(m_2^q))$ does not belong to $\lambda(S)$, the element N does not belong to the image of permutation σ ■

Bijective Mapping

Let us denote by $\Theta_{N,p,q}$ the mapping defined by :

$$\begin{array}{lcccl} \Theta_{N,p,q} & : & C_{N,p,q} & \longrightarrow & BT(p,q) \times PP(N, N-1, N+1-(p+q)) \\ & & (\pi_1, \pi_2, \alpha) & \longmapsto & (t, \sigma) \end{array} \tag{3.7}$$

We prove in the following section that the mapping $\Theta_{N,p,q}$ is actually a bijection.

4. Proof of the Bijection

4.1. Injectivity. Let (t, σ) be the image of some triple $(\pi_1, \pi_2, \alpha) \in C_{N,p,q}$ by $\Theta_{N,p,q}$. We show in a constructive fashion that (π_1, π_2, α) is actually uniquely determined by (t, σ).

First we use the tree t and the integers lacking in the two lines of σ to find out the number of elements in each block and the extension of σ to the whole set $[N]$. By construction of σ, the integers lacking in its first line $\lambda(S)$ are the elements

$$\lambda(m_1^1), \ldots, \lambda(m_1^{p-1}), \lambda(\beta(m_2^1)), \ldots, \lambda(\beta(m_2^q))$$

Now, if black vertex j is a descendant of a white vertex i in the reversed-labelled tree t' then $\beta(m_2^j) \in \pi_1^i$ for $j = 1, \ldots, q$. Due to this property we know exactly how the elements $\beta(m_2^1), \ldots, \beta(m_2^q)$ are distributed between the white blocks. Moreover, by definition of λ any element of $\lambda(\pi_1^i)$ is strictly less than any element of $\lambda(\pi_1^j)$ for all $i < j$. Thus, to recover the exact order on the elements lacking in the first line of σ, it remains to establish the order on the lacking elements belonging to the same block $\lambda(\pi_1^i)$ that are not the maximum one (obviously the greatest) if any. These elements correspond to the set of descendants of the white vertex i in t'. As we have defined that a black vertex j_1 is on the left of a black vertex j_2, descendant of the same vertex i, if and only if $\beta(m_2^{j_1}) \leq \beta(m_2^{j_2})$ and the restriction of λ to any block of π_1 is an increasing function, their order is naturally induced by the left to right order on the set of descendants of i in the reversed-labelled tree t'.

Consider the set $\nu \circ \beta^{-1}(S)$ in the second line of σ. The lacking elements are

$$\nu(m_2^1), \ldots, \nu(m_2^q), \nu(\beta)^{-1}(m_1^1), \ldots, \nu(\beta)^{-1}(m_1^{p-1})$$

Similarly to the first line of σ, we use the structure of t', the relation between ν and t' as well as the fact that $\nu(\beta)^{-1}(m_1^{i_1}) \leq \nu(\beta)^{-1}(m_1^{i_2})$ if i_1 and i_2 are descendant of the same black vertex and i_1 is on the left of i_2 to order these elements. Once the order on both of the sets of lacking elements is established, the lacking integers can be uniquely identified with these elements. Hence, the extension $\overline{\sigma} = \nu \circ \beta^{-1} \circ \lambda^{-1}$ of the partial permutation σ to the whole set $[N]$ is uniquely determined since

$$\forall i \in [p-1], \qquad \overline{\sigma}(\lambda(m_1^i)) = \nu(\beta^{-1}(m_1^i)) \tag{4.1}$$

$$\forall j \in [q], \qquad \overline{\sigma}(\lambda(\beta(m_2^j))) = \nu(m_2^j) \tag{4.2}$$

Now, the knowledge of $\lambda(m_1^1), \ldots, \lambda(m_1^{p-1})$ and $\nu(m_2^1), \ldots, \nu(m_2^q)$ allows us to determine the number of elements in each of the blocks of partitions $\lambda(\pi_1) = \lambda(\pi_1^1), \ldots, \lambda(\pi_1^p)$ and $\nu(\pi_2) = \nu(\pi_2^1), \ldots, \nu(\pi_2^q)$. Indeed, the blocks of the above partitions are intervals:

$$\begin{array}{rcl} \lambda(\pi_1^1) & = & [\lambda(m_1^1)] \\ \lambda(\pi_1^i) & = & [\lambda(m_1^i)] \setminus [\lambda(m_1^{i-1})] \text{ for } 2 \leq i \leq p-1 \\ \lambda(\pi_1^p) & = & [N] \setminus [\nu(m_1^{p-1})] \\ \nu(\pi_2^1) & = & [\nu(m_1^1)] \\ \nu(\pi_2^i) & = & [\nu(m_1^i)] \setminus [\lambda(m_1^{i-1})] \text{ for } 2 \leq i \leq q \end{array}$$

Hence, $\lambda(\pi_1)$ and $\nu(\pi_2)$ are uniquely determined by (t, σ). Besides, since π_2 is stable by β, we can use $\overline{\sigma}$ to recover $\lambda(\pi_2)$. Indeed:

$$\begin{aligned}
\overline{\sigma}^{-1}(\nu(\pi_2)) &= \lambda \circ \beta \circ \nu^{-1}(\nu(\pi_2)) \\
\overline{\sigma}^{-1}(\nu(\pi_2)) &= \lambda \circ \beta(\pi_2) \\
\overline{\sigma}^{-1}(\nu(\pi_2)) &= \lambda(\pi_2) \qquad (4.3)
\end{aligned}$$

Then as $\overline{\sigma}$ and $\nu(\pi_2)$ are uniquely determined, so is $\lambda(\pi_2)$.

EXAMPLE 4.1. We give here an illustration of the first steps of the injectivity proof. Let us suppose that we are given the parameters $N = 10, p = 3, q = 2$, the following partial permutation

and the bicolored ordere

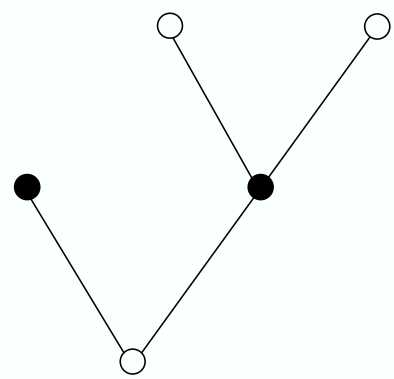

FIGURE 6. A Bicolored Tree

Consider the set $\lambda(S)$ in the first line of σ. Assuming a reverse labelling of the tree, the elements lacking in $\lambda(S)$ are

$$\lambda(m_1^1), \lambda(m_1^2), \lambda(\beta(m_2^1), \lambda(\beta(m_2^2)$$

The numbers lacking in $\lambda(S)$ to complete it up to $\lambda([N])$ are $1, 2, 7, 9$. According to the previous remarks, we can identify all these numbers in the following way:

$$\lambda(m_1^1) = 1, \lambda(m_1^2) = 2, \lambda(\beta(m_2^1)) = 7, \lambda(\beta(m_2^2)) = 9$$

Consider the set $(\nu \circ \beta)^{-1}(S)$ in the second line of σ. The elements lacking are

$$\nu(m_2^1), \nu(m_2^2), \nu(\beta)^{-1}(m_1^1), \nu(\beta)^{-1}(m_2^2)$$

We have

$$\nu(m_2^1) = 2, \nu(\beta)^{-1}(m_1^1) = 3, \nu(\beta)^{-1}(m_2^2) = 9, \nu(m_2^2) = 10$$

Now we can extend σ to the permutation $\overline{\sigma}$ on the set $[N]$:

$$\overline{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix}$$

Note, that as $\lambda(m_1^1) = 1, \lambda(m_1^2) = 2, \nu(m_2^1) = 2, \nu(m_2^2) = 10$, we also can identify the images of white blocks by λ and images of black blocks by ν:

$$\begin{aligned}
\lambda(\pi_1^1) &= \{1\}, \quad \lambda(\pi_1^2) = \{2\}, \lambda(\pi_1^3) = \{3, 4, 5, 6, 7, 8, 9, 10\}, \\
\nu(\pi_2^1) &= \{1, 2\}, \nu(\pi_2^2) = \{3, 4, 5, 6, 7, 8, 9, 10\}
\end{aligned}$$

Using (4.3) we obtain the relabelling of partition π_2:

$$\begin{aligned}
\lambda(\pi_2^1) &= \{6, 7\}, \qquad (4.4) \\
\lambda(\pi_2^2) &= \{1, 2, 3, 4, 5, 8, 9, 10\}
\end{aligned}$$

Now let us show that λ and ν are uniquely determined as well. As $1 \in \pi_1^p$ and λ is an increasing function on each block of π_1, $\lambda(1)$ is necessarily the least element of $\lambda(\pi_1^p)$. Let then $\lambda(\pi_2^k)$ be the block of $\lambda(\pi_2)$ such that $\lambda_1 \in \lambda(\pi_2^k)$. As ν is an increasing function on each block of π_2, necessarily $\nu(1)$ is the least element of $\nu(\pi_2^k)$.

Now assume that for a given i in $[N-1]$, $\lambda(1), \ldots, \lambda(i)$ and $\nu(1), \ldots, \nu(i)$ have been determined. As π_1 is stable by α, necessarily $\beta(i)$ and $i+1 = \gamma(i) = \alpha \circ \beta(i)$ belong to the same block of π_1. Hence $\lambda(i+1)$ and $\lambda(\beta(i))$ belong to the same block of $\lambda(\pi_1)$. But:

$$\lambda(\beta(i)) = \lambda \circ \beta \circ \nu^{-1}(\nu(i)) = \overline{\sigma}^{-1}(\nu(i)) \tag{4.5}$$

As a consequence, $\lambda(i+1)$ and $\overline{\sigma}^{-1}(\nu(i))$ belong to the same block of $\lambda(\pi_1)$. Finally, as λ is an increasing function on each block of π_1, $\lambda(i+1)$ is necessarily the least element of the block of $\lambda(\pi_1)$ containing $\overline{\sigma}^{-1}(\nu(i))$ that has not been used yet to identify $\lambda(1), \ldots, \lambda(i)$.

Let us denote by $\lambda(\pi_2^l)$ the block of $\lambda(\pi_2)$ containing $\lambda(i+1)$. Since ν is an increasing function on each block of π_2, $\nu(i+1)$ is uniquely determined as being the least element of the block $\nu(\pi_2^l)$ that has not already been used to identify $\nu(1), \ldots, \nu(i)$. By iterating the above procedure for all the integers in $[N-1]$ we see that λ and ν are uniquely determined.

To end this proof, we remark that :

$$\begin{aligned} \pi_1 &= \lambda^{-1}(\lambda(\pi_1)) \\ \pi_2 &= \nu^{-1}(\nu(\pi_2)) \\ \alpha &= \gamma \circ \beta^{-1} = \gamma \circ \nu^{-1} \circ \overline{\sigma} \circ \lambda \end{aligned}$$

As a result, at most one triple (π_1, π_2, α) can be associated by $\Theta_{N,p,q}$ to (t, σ). Moreover, if such a triple exists, it can be computed using the description of $\Theta_{N,p,q}^{-1}$ given by the above proof.

Example 4.2. We apply the iterative reconstruction of λ and ν to the previous example. A table of three lines and N columns will be used to sum up the available information on λ and ν on each step of the reconstruction: the first line is given by $[N]$ and represents the initial labelling γ of the edges of the partitioned map, the second and third lines represent the relabellings of the same edges by λ and ν. We initialize the procedure by putting $3 = \min(\lambda(\pi_1^3))$ at the first position of the line for λ:

$$\begin{array}{cccccccccccc} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & * & * & * & * & * & * & * & * & * \\ \nu & : & * & * & * & * & * & * & * & * & * & * \end{array}$$

Now, looking at equations (4.4) we establish, that the element 3 belongs to the second black block $\lambda(\pi_2^2)$. As the least element of $\nu(\pi_2^2)$ is 3, we put 3 in the first position of the third line of our table:

$$\begin{array}{cccccccccccc} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & * & * & * & * & * & * & * & * & * \\ \nu & : & 3 & * & * & * & * & * & * & * & * & * \end{array}$$

Let us establish now what is the next white block in our partitioned map. For this goal we take the image of the last discovered element $\nu(1)$ by $\overline{\sigma}^{-1}$:

$$\overline{\sigma}^{-1}(\nu(1)) = \overline{\sigma}^{-1}(3) = 5$$

Thus $\overline{\sigma}^{-1}(\nu(1))$ belongs to π_1^3. We then deduce that $\lambda(2)$ is the least element of $\lambda(\pi_1^3)$ which has not been met yet, i.e 4. We write 4 at the second position on the line for λ:

$$\begin{array}{cccccccccccc} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & 4 & * & * & * & * & * & * & * & * \\ \nu & : & 3 & * & * & * & * & * & * & * & * & * \end{array}$$

We iterate the process until λ and ν are fully reconstructed :

$$\begin{array}{cccccccccccc} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & 4 & 5 & 1 & 6 & 7 & 8 & 9 & 10 & 2 \\ \nu & : & 3 & 4 & 5 & 6 & 1 & 2 & 7 & 8 & 9 & 10 \end{array}$$

Once λ and ν are known, we have the partitioned map reconstructed:

$$\begin{aligned}\pi_1 &= \{\{4\}\{10\}\{1,2,3,5,6,7,8,9\}\}\\ \pi_2 &= \{\{5,6\}\{1,2,3,4,7,8,9,10\}\}\\ \alpha &= (13256798)(4)(10)\end{aligned}$$

The bicolored map represented by this permutation is the unicellular map of genus 2 on Figure 7.

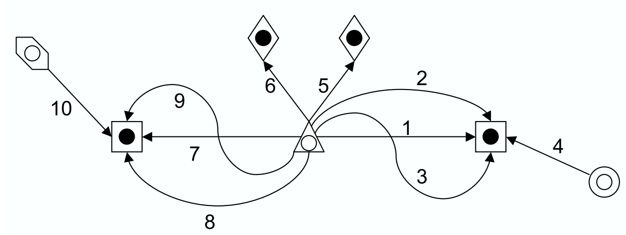

FIGURE 7. The Partitioned Bicolored Map Once Reconstructed

4.2. Surjectivity. Let us now proceed by showing that $\Theta_{N,p,q}$ is a surjection. Clearly, up to the reconstruction of λ and ν the first steps of the procedure described in the previous section can be applied to any couple (t,σ) belonging to $BT(p,q)\times PP(N,N-1,N-1-(p+q))$. Namely we can define the extension $\overline{\sigma}$ of σ to the whole set $[N]$ as well as the partitions $\lambda(\pi_1)$, $\lambda(\pi_2)$ and $\nu(\pi_2)$. Then we use lemma 4.3 to show that the reconstruction of λ and ν can also always be succesfully completed.

LEMMA 4.3. *Given any couple (t,σ) belonging to $BT(p,q)\times PP(N,N-1,N-1-(p+q))$, the iterative procedure for the reconstruction of λ and ν can always be performed and gives a valid output in any case.*

PROOF. First of all, we notice that only two reasons can prevent the procedure from being performed until its end. Either for a given i in $[N-1]$, $\overline{\sigma}^{-1}(\nu(i))$ belongs to a block of $\lambda(\pi_1)$ that has all its elements already used for the construction of $\lambda(1),\dots,\lambda(i)$ so that we cannot define $\lambda(i+1)$; or $\lambda(i+1)$ belongs to a block of $\lambda(\pi_2)$ such that the corresponding block of $\nu(\pi_2)$ has all its elements already been used for the construction of $\nu(1),\dots,\nu(i)$ and we are not able to define $\nu(i+1)$. We show by induction that this situation never occurs.

Assume that we have already successfully iterated the procedure up to $i\le N-1$. Also assume that we cannot define $\lambda(i+1)$ due to the reason stated above. We note $\lambda(\pi_1^k)$ the block containing $\overline{\sigma}^{-1}(\nu(i))$. Then:

(i) If $\lambda(\pi_1^k)$ does not contain $\lambda(1)$, the last assumption implies that $|\pi_1^k|+1$ different integers, including $\nu(i)$, used for the construction of ν have their image by $\overline{\sigma}^{-1}$ in $\lambda(\pi_1^k)$. This is of course a contradiction with the fact that $\overline{\sigma}^{-1}$ is a bijection.

(ii) If $\lambda(1)$ belongs to $\lambda(\pi_1^k)$ (thus $k=p$), we still have a contradiction. In this particular case, we only know that $|\pi_1^k|$ different integers have their image by $\overline{\sigma}^{-1}$ in $\lambda(\pi_1^p)$. However, according to our definition of $\overline{\sigma}$, $\lambda(\pi_1)$ and $\lambda(\pi_2)$, if the white vertex in t corresponding to a given block π_1^a is the direct descendant of the black vertex associated with π_2^b then :

$$\lambda(m_1^a)\in\lambda(\pi_2^b) \tag{4.6}$$

In other words we cannot have used all the elements of $\nu(\pi_2^b)$ for the construction of ν until the maximum element of $\lambda(\pi_1^a)$ (and henceforth all the elements of $\lambda(\pi_1^a)$) has been used for the construction of λ. In a similar fashion, if the black vertex associated to π_2^c is the direct descendant of the white one corresponding to π_1^d, we have:

$$\overline{\sigma}^{-1}(\nu(m_2^c))\in\lambda(\pi_1^d) \tag{4.7}$$

And all the elements of $\nu(\pi_2^c)$ must be used for the reconstruction of ν before all the elements of $\lambda(\pi_1^d)$ are used for the reconstruction of λ. To summarize, all the elements of a block associated to a vertex x (either black or white) are not used for the construction of λ and ν until all the elements of the blocks associated with vertices that are descendant of x are used for the same construction. As π_1^p is associated with the root of t, if all the elements of $\lambda(\pi_1^p)$ have already been used for the construction of λ and ν, it means that all the elements of all the other blocks of $\lambda(\pi_1)$ and $\nu(\pi_2)$ have been already used as well. The reconstruction is hence completed and $i = N$. That is a contradiction with our assumption $i \leq N-1$.

Once $\lambda(i+1)$ is found, we notice that $\nu(i+1)$ can always be defined. Indeed, if $\lambda(i+1)$ belonged to a block $\lambda(\pi_2^l)$ such that all the elements of $\nu(\pi_2^l)$ have been already used to construct ν, it would mean that $|\pi_2^l|+1$ different integers belong to $\lambda(\pi_2^l)$, which is a contradiction. Our induction is completed by an obvious remark that $\lambda(1)$ and $\nu(1)$ can always be defined. ∎

For the final step of this proof we need to show that once λ and ν are constructed the permutation α defined by

$$\alpha = \gamma \circ \nu^{-1} \circ \overline{\sigma} \circ \lambda \tag{4.8}$$

verifies the two following conditions:

$$\alpha(\pi_1) = \pi_1 \tag{4.9}$$

$$\alpha^{-1}\gamma(\pi_2) = \pi_2 \tag{4.10}$$

Condition (4.10) comes from the fact that we have defined:

$$\lambda(\pi_2) = \overline{\sigma}^{-1}(\pi_2) \tag{4.11}$$

Thus,

$$\pi_2 = \lambda^{-1} \circ \overline{\sigma}^{-1} \circ \nu \circ \gamma^{-1} \circ \gamma(\pi_2) \tag{4.12}$$

and by consequence

$$\pi_2 = \alpha^{-1} \circ \gamma(\pi_2) \tag{4.13}$$

Condition (4.9) can be shown using the fact that for all i in $[N]$, $\lambda(i)$ and $\overline{\sigma}^{-1} \circ \nu \circ \gamma^{-1}(i)$ belong to the same block of $\lambda(\pi_1)$. Hence, $\lambda^{-1} \circ \overline{\sigma}^{-1} \circ \nu \circ \gamma^{-1}(i)$ and i belong to the same block of π_1. Finally, the blocks of π_1 are stable by α^{-1} and henceforth by α.

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