Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Some Expansions of the Dual Basis of Z_λ

Amanda Riehl

Abstract.
A zigzag or ribbon is a connected skew diagram that contains no 2×2 boxes. Given a composition $\beta = (\beta_1, \ldots \beta_k)$, we let Z_β denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \ldots \beta_k$ reading from top to bottom. For each n, the set $\{Z_\lambda\}_{\lambda \vdash n}$ is a basis for Λ_n, the space of homogeneous symmetric functions of degree n. In this paper, we investigate some characteristics of the dual basis of $\{Z_\lambda\}_{\lambda \vdash n}$ relative to the Hall inner product which we denote by $\{DZ_\lambda\}_{\lambda \vdash n}$. We give a combinatorial interpretation for the coefficients in the expansion of DZ_λ in terms of the monomial symmetric functions $\{m_\mu\}_{\mu \vdash n}$ as a certain signed sum of paths in the partition lattice under refinement. We shall show that in many cases, we can give an explicit formulas for the coefficients $a_{\mu,\lambda} = DZ_\lambda \mid_{m_\mu}$. In addition, we give explicit formulas for the coefficients that arise in the expansion of DZ_λ in terms of Schur functions for several special cases. As an application, we obtain combinatorial interpretations for the coefficients in the expansion of Schur functions and general ribbon Schur functions in terms of ribbon Schur functions indexed by partitions.

Résumé. Un zigzag ou un ruban est un diagramme connexe oblique qui ne contient aucune boite 2×2. Soit une composition $\beta = (\beta_1, \ldots \beta_k)$, notons Z_β la fonction oblique de Schur correspondant la forme de zigzag dont les longueurs des lignes sont $\beta_1, \ldots \beta_k$ lu de haut en bas. Pour chaque n, l'ensemble $\{Z_\lambda\}_{\lambda \vdash n}$ est une base Λ_n, de l'éspace des fonctions symétriques homogène de degré n. Dans cet article, nous étudions certaines caractéristiques de la base duale de $\{Z_\lambda\}_{\lambda \vdash n}$ relativement au produit intérieur de Hall que nous dénotons par $\{DZ_\lambda\}_{\lambda \vdash n}$. Nous donnons une interprétation combinatoire des coefficients dans l'expansion de DZ_λ en termes des fonctions symétriques monômiales $\{m_\mu\}_{\mu \vdash n}$ comme une somme signée de chemin dans le treillis des partages (l'ordre est le raffinement). Nous montrerons que, dans beaucoup de cas, nous pouvons donner des formules explicites pour les coefficients $a_{\mu,\lambda} = DZ_\lambda \mid_{m_\mu}$. De plus, nous donnons dans plusieurs cas des formules explicites pour les coefficients dans l'expansion de DZ_λ en termes de fonctions de Schur. Comme application, nous obtenons des interprétations combinatoires pour les coefficients dans l'expansion des fonctions de Schur et des fonctions Schur de ruban en termes de fonctions de Schur ruban indexées par les partages.

1. Introduction

Zigzag (or ribbon) Schur functions are the skew Schur functions with a ribbon shape and indexed by compositions. A composition $\beta = (\beta_1, \ldots, \beta_k)$ of n, denoted $\beta \models n$, is a sequence of positive integers such that $\beta_1 + \beta_2 + \ldots + \beta_k = n$. We define a zigzag shape to be a connected skew shape that contains no 2 x 2 array of boxes. Given a composition $\beta = (\beta_1, \ldots \beta_k)$, we let Z_β denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \ldots \beta_k$ reading from top to bottom. For example Figure 1 shows the zigzag shape corresponding to the composition $(2, 3, 1, 4)$. As pointed out in [**2**], zigzag Schur functions arise in many contexts. For example, the scalar product of any two zigzags gives the number of permutations σ such that σ and σ^{-1} have the associated pair of descent sets [**9**]. Zigzags can also be used to compute the number of permutations with a given descent set and cycle structure [**5**]. MacMahon [**8**] showed their coefficients in terms of the monomial symmetric functions count descents in permutations with repeated

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 05A17.
Key words and phrases. symmetric functions, zigzag Schur functions, dual basis.
This research was done under the supervision of Jeffrey Remmel as part of the author's thesis, and the author would like to thank him for his support and numerous helpful conversations.

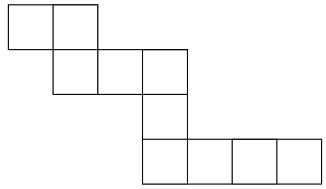

FIGURE 1. The ribbon shape corresponding to the composition $(2,3,1,4)$, so that $s_{(7,4,4,2)/(3,3,1)} = Z_{(2,3,1,4)}$.

elements. They also show up as sl_n-characters of the irreducible components of the Yangian representation in level 1 modules of $\hat{sl}_n$[**6**].

The zigzag Schur functions corresponding to partitions of n form a basis of Λ_n, the space of homogeneous symmetric functions of degree n, and therefore they have a dual basis relative to the Hall inner product which we denote by $\{DZ_\lambda\}_{\lambda\vdash n}$. We shall call DZ_λ the dual zigzag symmetric function corresponding to λ. The basis $\{DZ_\lambda\}_{\lambda\vdash n}$ has not been extensively studied. Let $\{m_\lambda\}_{\lambda\vdash n}$ denote the set of monomial symmetric functions, $\{h_\lambda\}_{\lambda\vdash n}$ denote the set of homogeneous symmetric functions, and $\{s_\lambda\}_{\lambda\vdash n}$ denote the set of Schur functions. The main result of this paper is to give a combinatorial interpretation to coefficients that arise in the expansion of DZ_λ in terms of the monomial symmetric functions. That is, we shall give a combinatorial interpretation to $a_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu a_{\mu,\lambda} m_\mu. \tag{1.1}$$

Our main result will show that $a_{\mu,\lambda}$ is a signed sum over the weights of certain paths in the lattice of partitions under refinement. In general such a signed sum is complicated, but we will show that in many special cases, we can explicitly evaluate this sum. For example, we will show that $a_{\mu,(n)} = 1$ for all μ so that

$$DZ_{(n)} = \sum_\mu m_\mu = s_{(n)}$$

where $s_{(n)}$ is the Schur function associated to the partition with only one part.

Once we have found our combinatorial interpretation for $a_{\mu,\lambda}$, we can obtain combinatorial interpretations for the expansion of DZ_λ in terms of any other basis by using the combinatorial interpretations of the transition matrices between bases of symmetric functions found in [**1**]. In particular, we shall use this method to find explicit values for $b_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu b_{\mu,\lambda} s_\mu \tag{1.2}$$

for certain special cases.

We now give brief explanations of the concepts to state our main result. There is a natural correspondence between a composition β of n and subsets of $[n-1]$. That is, given a composition $\beta = (\beta_1, \ldots, \beta_k)$ of n, we define a subset of $[n-1]$ by

$$Set(\beta) = \{\beta_1, \beta_1+\beta_2, \beta_1+\beta_2+\beta_3, \ldots, \beta_1+\beta_2+\ldots+\beta_{k-1}\}. \tag{1.3}$$

We also let $\lambda(\beta)$ denote the partition that arises from β by arranging its parts in decreasing order and $\ell(\beta)$ denote the number of parts of β. For example, if $\beta = (2,3,1,2)$, then $Set(\beta) = \{2,5,6\}$ and $\lambda(\beta) = (3,2,2,1)$. Given a subset $S = \{a_1 < a_2 < \cdots < a_r\} \subseteq [n-1]$, we define a composition of n by

$$\beta_n(S) = (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, n - a_r). \tag{1.4}$$

For example, if $S = \{2,4,8\}$, then $\beta_{10}(S) = (2,2,4,2)$. We also define $shape_n(S) = \lambda(\beta_n(S))$. Given two compositions β and γ, we say that β is a refinement of γ, denoted $\beta \leq_r \gamma$, if by adding together adjacent components of β, we can obtain γ. For two partitions μ and λ with $\mu \leq_r \lambda$, we define $Path(\mu,\lambda)$ to be the set of all $P = (\mu_0, \mu_1, \ldots, \mu_k)$, such that $\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda$. If $P = (\mu_0, \mu_1, \ldots, \mu_k)$ is such a path, we let $\ell(P) = k$ denote the length of P. Finally, μ and λ are partitions of n, then we define

$$[\mu \to \lambda] = |\{S \subseteq Set(\mu) : shape_n(S) = \lambda\}|$$

For example, if $\mu = (2,2,2,1)$ and $\lambda = (4,2,1)$, then $[\mu \to \lambda] = 2$, since $Set(\mu) = \{2,4,6\}$ and $\lambda(\beta_7(\{2,6\})) = \lambda(\beta_7(\{4,6\})) = (4,2,1)$.

This given, our main result is to give a combinatorial interpretation of for the coefficients $a_{\mu,\lambda}$ that arise in (1.1).

THEOREM 1.1. *If λ and μ are partitions of n, then*

$$a_{\mu,\lambda} = (-1)^{l(\mu)-l(\lambda)} \sum_{P \in Path(\mu,\lambda)} [P](-1)^{l(P)}$$

where $P = (\mu_0, \mu_1, \ldots, \mu_k)$, $\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda$ and $[P] = [\mu_0 \to \mu_1][\mu_1 \to \mu_2]\ldots[\mu_{k-1} \to \mu_k]$.

As one application of our main result, we can give a combinatorial interpretation of the expansion of Z_α in terms of Z_λ's, where α is a composition of n, and λ is a partition of n. It is known, see [**4**], that

$$Z_\alpha = \sum_{T \subseteq Set(\alpha)} (-1)^{|Set(\alpha)-T|} h_{\lambda(\beta(T))}.$$

Thus if $Z_\alpha = \sum_{\mu \vdash n} f_{\mu,\alpha} Z_\mu$, then

$$f_{\mu,\alpha} = \langle Z_\alpha, DZ_\mu \rangle = \sum_{T \subseteq Set(\alpha)} (-1)^{|Set(\alpha)-T|} a_{\lambda(\beta(T)),\mu}. \tag{1.5}$$

In principle, (1.5) gives rise to a combinatorial algorithm to compute the coefficients $f_{\mu,\alpha}$. However, such an algorithm is not necessarily the most efficient way to compute these coefficients.

The outline of this paper is as follows. In Section 2, we shall review the necessary background for symmetric functions and the combinatorial interpretation of the entries of the transition matrices between various bases of symmetric functions that we shall need. In particular, we shall use the Jacobi-Trudi identity to give a combinatorial interpretation of the coefficients $Z_\lambda \mid_{h_\mu}$. In Section 3, we outline the proof of our main theorem and give some examples of the computations involved in computing the coefficients $a_{\mu,\lambda}$. In Section 4, we give closed forms for several of the coefficients, independent of the size of the composition. In Section 5, we give the expansion of several dual zigzags in terms of Schur functions which are independent of the size of the partition. In Section 6, we give a brief explanation of two applications of our main result.

2. Background Information

We say that $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ is a partition of n, written $\lambda \vdash n$ if $\lambda_1 + \ldots + \lambda_k = n = |\lambda|$. We $\ell(\lambda)$ denote the number of parts of λ. We let F_λ denote the Ferrers diagram of λ. If $\mu = (\mu_1, \ldots, \mu_m)$ is a partition where $m \leq k$ and $\lambda_i \geq \mu_i$ for all $i \leq m$, we let $F_{\lambda/\mu}$ denote the skew shape that results by removing the cells of F_μ from F_λ.

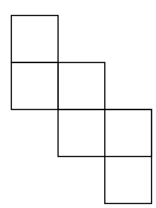

FIGURE 2. The skew Ferrers diagram of $(3,3,2,1)/(2,1)$.

A *column-strict tableau* T of shape λ is any filling of F_λ with natural numbers such that entries in each row are weakly increasing from left to right, and entries in each column are strictly increasing from bottom to top. We define the content of T to be $c(T) = (\alpha_1, \alpha_2, \ldots,)$ where α_i is the number of times that i occurs in T. If λ is a partition denoted by $\lambda = (\lambda_1, \ldots, \lambda_l) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, where m_i is the number of parts of λ equal to i, then we define $z_\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \ldots m_n!$.

There are six standard bases of the space of homogeneous symmetric functions of degree n, $\Lambda_n(x)$, which are generally notated as: $\{m_\lambda\}_{\lambda \vdash n}$ (the monomial symmetric functions), $\{h_\lambda\}_{\lambda \vdash n}$ (the complete homogeneous symmetric function), $\{e_\lambda\}_{\lambda \vdash n}$ (the elementary symmetric functions), $\{p_\lambda\}_{\lambda \dashv n}$ (the power sum

5		
4	4	
2	3	
1	1	2

FIGURE 3. A column strict tableau of shape $(3, 2, 2, 1)$ and content $(2, 2, 1, 2, 1)$.

symmetric functions), $\{f_\lambda\}_{\lambda \vdash n}$ (the forgotten symmetric functions) and $\{s_\lambda\}_{\lambda \vdash n}$ (the Schur functions), where λ is a partition of n.

The Hall inner product is a standard scalar product on the space of homogeneous symmetric functions $\Lambda_n(x)$, which is defined by:

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

where

$$\delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Under this scalar product, $\{s_\lambda\}_{\lambda \vdash n}$ and $\{p_\lambda/\sqrt{z_\lambda}\}_{\lambda \vdash n}$ are known to be self-dual, and $\{e_\lambda\}_{\lambda \vdash n}$ and $\{f_\lambda\}_{\lambda \vdash n}$ are dual [**1**].

When given two bases of $\Lambda_n(x)$, $\{a_\lambda\}_{\lambda \vdash n}$ and $\{b_\lambda\}_{\lambda \vdash n}$, we first fix some ordering of the partitions of n, e.g. the lexicographic order, and then we may think of the bases as row vectors, $\langle a_\lambda \rangle_{\lambda \vdash n}$ and $\langle b_\lambda \rangle_{\lambda \vdash n}$. We can define the transition matrix $M(a, b)$ that transforms the basis $\langle a_\lambda \rangle_{\lambda \vdash n}$ into the basis $\langle b_\lambda \rangle_{\lambda \vdash n}$ by

$$\langle b_\lambda \rangle_{\lambda \vdash n} = \langle a_\lambda \rangle_{\lambda \vdash n} M(a, b).$$

The (λ, μ) entry of $M(a, b)$ is given by the equation

$$b_\lambda = \sum_{\mu \vdash n} a_\mu M(a, b)_{\mu,\lambda}.$$

The main goal of this paper is to find a combinatorial interpretation of the entries of $M(m, DZ)$. That is, we want find a combinatorial interpretation for the $a_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu a_{\mu,\lambda} m_\mu.$$

In addition, we shall also be interested in finding a combinatorial interpretation for the entries of $M(s, DZ)$. That is, we want to find a combinatorial interpretation for $b_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu b_{\mu,\lambda} s_\mu.$$

We now give examples of the expansion of $\{DZ_\lambda\}_{\lambda\vdash n}$ when $n=6$. We first give the expansion of DZ_λ in terms of the monomial symmetric functions, when $\lambda\vdash 6$.

$$\begin{aligned}
DZ_{(6)} &= m_6+m_{5,1}+m_{4,2}+m_{4,1,1}+m_{3,3}+m_{3,2,1}+m_{3,1,1,1}\\
&\quad +m_{2,2,2}+m_{2,2,1,1}+m_{2,1,1,1,1}+m_{1,1,1,1,1,1}\\
DZ_{(5,1)} &= m_{5,1}+m_{4,1,1}+m_{3,2,1}+2m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}-2m_{1,1,1,1,1,1}\\
DZ_{(4,2)} &= m_{4,2}+m_{4,1,1}+2m_{2,2,2}+m_{2,2,1,1}+2m_{2,1,1,1,1}+7m_{1,1,1,1,1,1}\\
DZ_{(4,1,1)} &= m_{4,1,1}+m_{3,1,1,1}+m_{2,2,1,1}+3m_{2,1,1,1,1}+8m_{1,1,1,1,1,1}\\
DZ_{(3,3)} &= m_{3,3}+m_{3,2,1}+m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}\\
DZ_{(3,2,1)} &= m_{3,2,1}+2m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}-3m_{1,1,1,1,1,1}\\
DZ_{(3,1,1,1)} &= m_{3,1,1,1}+m_{2,1,1,1,1}+m_{1,1,1,1,1,1}\\
DZ_{(2,2,2)} &= m_{2,2,2}+m_{2,2,1,1}+2m_{2,1,1,1,1}+5m_{1,1,1,1,1,1}\\
DZ_{(2,2,1,1)} &= m_{2,2,1,1}+3m_{2,1,1,1,1}+9m_{1,1,1,1,1,1}\\
DZ_{(2,1,1,1,1)} &= m_{2,1,1,1,1}+5m_{1,1,1,1,1,1}\\
DZ_{(1,1,1,1,1,1)} &= m_{1,1,1,1,1,1}.
\end{aligned}$$

We note that we can get an indirect combinatorial interpretation of the coefficients $b_{\mu,\gamma}$ by using the combinatorial interpretation of the entries of the transition matrix $M(s,m)$ given in [**3**]. That is,

$$M(s,m)_{\lambda\mu}=K^{-1}_{\mu,\lambda},$$

where $||K^{-1}_{\mu,\lambda}||$ is the inverse Kostka matrix which will be described below. Thus

$$DZ_\lambda=\sum_{\mu\le_r\lambda}a_{\mu,\lambda}\sum_\gamma s_\gamma K^{-1}_{\mu,\gamma}=\sum_\gamma s_\gamma\sum_{\mu\le_r\lambda}a_{\mu,\lambda}K^{-1}_{\mu,\gamma}. \tag{2.1}$$

Hence

$$b_{\mu,\gamma}=\sum_{\mu\le_r\lambda}a_{\mu,\lambda}K^{-1}_{\mu,\gamma}. \tag{2.2}$$

The expansion of DZ_λ in terms of the Schur functions, when $\lambda\vdash 6$, is given below.

$$\begin{aligned}
DZ_{(6)} &= s_6\\
DZ_{(5,1)} &= s_{5,1}-s_{4,2}+s_{3,2,1}-s_{2,2,2}-s_{2,2,1,1}\\
DZ_{(4,2)} &= s_{4,2}-s_{3,3}-s_{3,2,1}+2s_{2,2,2}+s_{2,2,1,1}\\
DZ_{(4,1,1)} &= s_{4,1,1}-s_{3,2,1}+s_{2,2,2}+s_{2,2,1,1}\\
DZ_{(3,3)} &= s_{3,3}-s_{2,2,2}\\
DZ_{(3,2,1)} &= s_{3,2,1}-2s_{2,2,2}-s_{2,2,1,1}.
\end{aligned}
\qquad
\begin{aligned}
DZ_{(3,1,1,1)} &= s_{3,1,1,1}-s_{2,2,1,1}\\
DZ_{(2,2,2)} &= s_{2,2,2}\\
DZ_{(2,2,1,1)} &= s_{2,2,1,1}\\
DZ_{(2,1,1,1,1)} &= s_{2,1,1,1,1}\\
DZ_{(1,1,1,1,1,1)} &= s_{1,1,1,1,1,1}
\end{aligned}$$

Next we shall describe the combinatorial interpretation of the coefficients that arise in expanding a skew Schur function in terms of the homogeneous symmetric functions. In particular, we will need to use the expansion of skew-Schur functions in terms of h_λ. To do so, we introduce rim hooks, special rim hooks and special rim hook tabloids. More detail is given in [**3**] where they are used to give a combinatorial interpretation of the inverse Kostka matrix.

For a partition λ, consider the Ferrers diagram F_λ. A rim hook of λ is a sequence of cells, h, along the northeast boundary of F_λ such that any two consecutive cells in h share an edge and if we remove h from F_λ, we are left with the Ferrers diagram of another partition. More generally, h is a rim hook of a skew shape λ/μ if h is a rim hook of λ which does not intersect μ.

A rim hook tableau of shape λ/ν and type μ, T, is a sequence of partitions

$$T=(\nu=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\lambda^{(k)}=\lambda),$$

such that for each $1\le i\le k, \lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ of size μ_i. We define the sign of a rim hook $h_i=\lambda^{(i)}/\lambda^{(i-1)}$ to be

$$sgn(h_i)=(-1)^{r(h_i)-1},$$

where $r(h_i)$ is the number of rows that h_i occupies. The sign of a rim hook tableau T is

$$sgn(T) = \Pi_{i=1}^{k} sgn(h_i).$$

Given two partitions $\lambda^{(i-1)} \subset \lambda^{(i)}$, we say that $\lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook if $\lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ and $\lambda^{(i)}/\lambda^{(i-1)}$ contains a cell from the first column of λ. A special rim hook tabloid (SRHT) T of shape λ/μ is a sequence of partitions

$$T = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda),$$

such that for each $1 \leq i \leq k, \lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook of $\lambda^{(i)}$. We have a partition determined by the integers $|\lambda^{(i)}/\lambda^{(i-1)}|$ which is the type of the special rim hook tabloid T. Notice that we have used the word *tabloid* instead of *tableau* in order to highlight there is no implicit order in the size of each successive special rim hook, unlike rim hook tableau.

The sign of a special rim hook, $h_i = \lambda^{(i)}/\lambda^{(i-1)}$, and the sign of a special rim hook tabloid T, are defined as we did for rim hooks and rim hook tableaux. We show an example of a special rim hook tabloid of type $(6, 5, 4, 2)$ and shape $(5, 4, 4, 3, 1)$ in Fig 4. For $|\lambda/\nu| = |\mu|$, Eğecioğlu and Remmel [**3**] show that

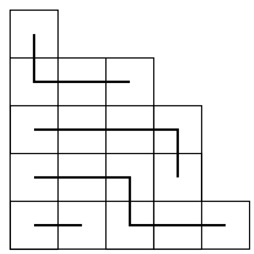

FIGURE 4. A special rim hook tabloid of shape (5,4,4,3,1) and type (6,5,4,2).

$$s_{\lambda/\nu} = \sum_{\mu} K^{-1}_{\mu,\lambda/\nu} h_\mu \tag{2.3}$$

where

$$K^{-1}_{\mu,\lambda/\nu} = \sum_{T \text{ is a SRHT of shape } \lambda/\nu \text{ and type } \mu} sgn(T).$$

Hence we obtain a combinatorial description of

$$M(s,m)_{\lambda,\mu} = K^{-1}_{\mu,\lambda}.$$

Recall that we defined a composition β of n, denoted $\beta \models n$, as a list of positive integers $(\beta_1, \beta_2, \ldots, \beta_k)$ such that $\beta_1 + \beta_2 + \ldots + \beta_k = n$. We call β_i a *component* of β, and we say that β has length $l(\beta) = k$ and size $|\beta| = n$. From this definition, we can see that β is a partition if each of its components are weakly decreasing. For any composition β, we define the partition determined by β, $\lambda(\beta)$, which we obtain by reordering the components of β in weakly decreasing order, e.g. $\lambda(2, 8, 9, 4) = (9, 8, 4, 2)$. Notice that two compositions β, γ can determine the same partition, e.g. if $\beta = (2, 8, 9, 4)$ and $\gamma = (2, 9, 8, 4)$, then $\lambda(2, 8, 9, 4) = (9, 8, 4, 2) = \lambda(2, 9, 8, 4)$.

There is a natural correspondence between a composition $\beta \models n$ and a subset $Set(\beta) \subseteq [n-1] = \{1, 2, \ldots, n-1\}$ where

$$Set(\beta) = \{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \ldots + \beta_{k-1}\}.$$

We can also reverse this process so that for any subset $S = \{j_1, j_2, \ldots, j_{k-1}\} \subseteq [n-1]$, we can find the composition $\beta_n(S) \models n$ where

$$\beta_n(S) = (j_1, j_2 - j_1, \ldots, n - j_{k-1}).$$

For example, the composition $\beta = (2, 9, 8, 4)$ has $Set(\beta) = \{2, 11, 19\} \subseteq [22]$. We also define $shape_n(S) = \lambda(\beta_n(S))$. For example if $S = \{2, 5, 6, 10\}$ and $n = 11$, then $\beta_{11}(S) = (2, 3, 1, 4, 1)$, and $shape_{11}(S) = (4, 3, 2, 1, 1)$.

Given two partitions λ and μ of n, we say that λ is a refinement of μ, written $\lambda \leq_r \mu$, if λ can be created from μ by splitting some of the parts of μ into pieces. For example, $(4, 2, 1, 1, 1, 1) \leq_r (5, 3, 2)$ since we can split 5 into $4+1$ and 3 into $1+1+1$ to obtain λ. The cover relations in the lattice of partitions of n under refinement arise by starting with a partition λ and combining two of the parts of λ to get μ. Similarly, given two compositions β and γ, we say that β is a refinement of γ, denoted $\beta \leq_r \gamma$, if by adding together adjacent components of β, we can obtain γ. For example, $421131 \leq_r 4314$, meaning $\gamma = 421131$ is a refinement of $\beta = 4314$. If we only add together a single pair of adjacent components of a partition β to get γ, then we will say that γ covers β.

The refinement ordering restricted to the set of partitions forms a lattice which we call the lattice of partitions under refinement, or more briefly, the refinement lattice. For two partitions μ and λ, with $\mu \leq_r \lambda$ we define $Path(\mu, \lambda)$ to be the set of all $P = (\mu_0, \mu_1, \ldots, \mu_k)$, such that $\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda$. We define the length of P, $l(P) = k$.

Given two partitions of λ and μ of n such that $\mu \leq_r \lambda$, we define

$$[\mu \to \lambda] = |\{S \subseteq Set(\mu) : shape_n(S) = \lambda\}|.$$

As an example, let's calculate $[(2,1^4) \to (4,2)]$. Note that $Set(2,1^4) = \{2,3,4,5\}$. We want to find $|\{S \subseteq \{2,3,4,5\} : shape_6(S) = (4,2)|$. The only two subsets of $\{2,3,4,5\}$ that have the appropriate shape are $\{2\}$ and $\{4\}$, so $[(2,1^4) \to (4,2)] = 2$.

3. A sketch of the proof of Theorem 1.1

Before proceeding with the proof of Theorem 1.1, we shall demonstrate how it can be used to calculate $a_{\mu,\lambda}$ in the case where $\mu = (1^6)$ and $\lambda = (3,2,1)$. Since our theorem says we sum over all paths in the refinement lattice, we give the relevant portion of the refinement lattice in Fig. 5. First we give several

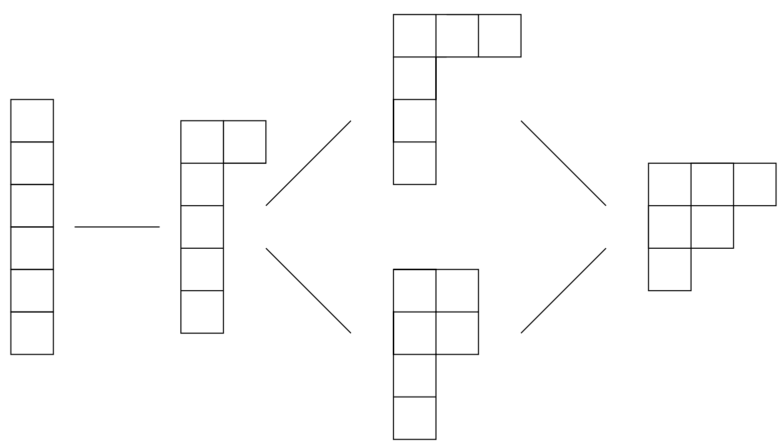

FIGURE 5. The refinement lattice from (1,1,1,1,1,1) to (3,2,1).

examples of how to calculate $[\alpha \to \beta]$. Recall that $Set(\lambda) = \{\lambda_1, \lambda_1+\lambda_2, \ldots, \lambda_1+\cdots+\lambda_{k-1}\}$. We first calculate $[(1^6) \to (2,1^4)]$, which is equal to $|\{S \subset Set(1^6) : shape_6(S) = (2,1^4)\}|$. $Set(1^6) = \{1,2,3,4,5\}$, and the subsets $\{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}$, and $\{1,2,3,4\}$ all have shape equal to $(2,1^4)$. Therefore $[(1^6) \to (2,1^4)] = 5$. Similarly $[(1^6) \to (3,2,1)] = 6$ since $\{3,4\}, \{3,5\}, \{2,5\}, \{2,3\}, \{1,3\}, \{1,4\}$ are the only subsets T of $Set(1^6) = \{1,2,3,4,5\}$ such that $shape_6(T) = (3,2,1)$. Finally we calculate $[(2,1^4) \to (3,1^3)]$. In this case, $Set(2,1^4) = \{2,3,4,5\}$ and the only subset T of $Set(2,1^4)$ such that $shape_6(T) = (3,1^3)$ is $\{3,4,5\}$. Thus $[(2,1^4) \to (3,1^3)] = 1$.

From these three examples we see that a considerable amount of work goes into calculating $[\alpha \to \beta]$ for every possibility in our refinement lattice. In Table 1, we give the values needed to calculate $[\alpha \to \beta]$ for all pairs in the refinement lattice from (1^6) to $(3,2,1)$.

Once we have calculated those values, we can easily calculate the weights of each possible path in our refinement lattice. These paths and weights are listed in Table 2. The length of the path will be used in our calculation of $a_{\mu,\lambda}$.

$[1^6 \to 2, 1^4] = 5$	$[2, 1^4 \to 3, 1^3] = 1$	$[3, 1^3 \to 3, 2, 1] = 2$
$[1^6 \to 3, 1^3] = 4$	$[2, 1^4 \to 2^2, 1^1] = 3$	$[2^2, 1^2 \to 3, 2, 1] = 1$
$[1^6 \to 2^2, 1^2] = 6$	$[2, 1^4 \to 3, 2, 1] = 4$	
$[1^6 \to 3, 2, 1] = 6$		

Table 1. Values for $[\alpha \to \beta]$ for pairs in the refinement lattice from (1^6) to $(3, 2, 1)$.

Possible Paths	Length of Path	Weight of Path
$[(1^6) \to (3, 2, 1)]$	1	6
$[(1^6) \to (3, 1^3)][(3, 1^3) \to (3, 2, 1)]$	2	8
$[(1^6) \to (2^2, 1^2)][(2^2, 1^2) \to (3, 2, 1)]$	2	6
$[(1^6) \to (2, 1^4)][(2, 1^4) \to (3, 2, 1)]$	2	20
$[(1^6) \to (2, 1^4)][(2, 1^4) \to (3, 1^3)][(3, 1^3 \to (3, 2, 1)]$	3	10
$[(1^6) \to (2, 1^4)][(2, 1^4) \to (2^2, 1^2)][(2^2, 1^2 \to (3, 2, 1)]$	3	15

Table 2. The weight of each possible path in the refinement lattice from (1^6) to $(3, 2, 1)$.

Finally, we combine this information:

$$\begin{aligned} a_{(1^6),(3,2,1)} &= (-1)^{6-3} \sum_{P \in Path((1^6),(3,2,1))} -1^{l(P)}[P] \\ &= -1^3(-1^1(6) + -1^2(8+6+20) + -1^3(10+15)) \\ &= -(-6+34-25) \\ &= -3. \end{aligned}$$

We should note that although this first example required many calculations, we have now done almost all of the work for several other coefficients for $n = 6$ since our the set of paths that we considered also arise in the computation of $a_{\alpha,\beta}$ for other pairs of partitions. In addition, we will see later that the same calculations allow us to evaluate an infinite number of coefficients $a_{\alpha,\beta}$ where α and β are partitions of $n > 6$.

Outline of proof of Theorem 1.1:

We start by expanding the zigzag Schur functions in terms of the homogeneous symmetric functions $\{h_\lambda\}_{\lambda \vdash n}$ derived from the Jacobi-Trudi by Egecioglu and Remmel [**3**],

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j}) = \sum_\nu K^{-1}_{\nu,\lambda/\mu} h_\mu$$

where $h_0 = 1$ and $h_k = 0$ if $k < 0$. Applying it specifically to zigzag Schur functions and using compositions as subscripts, we can show that for any $\alpha \models n$,

$$Z_\alpha = (-1)^{l(\alpha)} \sum_{\beta \leq_r \alpha} (-1)^{l(\beta)} h_{\lambda(\beta)}.$$

Alternatively,

$$Z_\alpha = h_{\lambda(\beta(\alpha))} + \sum_{T \subset Set(\alpha)} (-1)^{|Set(\alpha) - T|} h_{\lambda(\beta(\alpha))}. \tag{3.1}$$

The result in 3.1 is well-known and can be proved by inclusion-exclusion [**4**]. Recall that $[\mu \to \lambda] = |\{S \subseteq Set(\mu) : shape_n(S) = \lambda\}|$. So

$$Z_\lambda = h_\lambda + \sum_{\lambda \leq_r \alpha} (-1)^{l(\lambda) - l(\alpha)} [\lambda \to \alpha] h_\alpha.$$

Since $\{Z_\lambda\}_{\lambda \vdash n}$ and $\{DZ_\lambda\}_{\lambda \vdash n}$ are dual bases, it follows that

$$\sum_\gamma Z_\gamma(x) DZ_\gamma(y) = \sum_\gamma h_\gamma(x) m_\gamma(y)$$

or, equivalently,

$$\sum_{\gamma} Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x) m_\mu(y)} = \delta_{\lambda,\mu}.$$

Given our expansion of $Z_\lambda(x)$ in terms of $h_\lambda(x)$'s and the fact that $\langle h_\lambda(x), m_\mu(x)\rangle = \delta_{\lambda,\mu}$, we can then show that

$$\sum_{\gamma} Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x)} = \sum_{\alpha \leq_r \lambda} (-1)^{l(\alpha)-l(\lambda)} [\alpha \to \lambda] m_\alpha(y)$$

and

$$\begin{aligned} \sum_{\gamma} Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x) m_\mu(y)} &= \sum_{\mu \leq_r \alpha \leq_r \lambda} (-1)^{l(\alpha)-l(\lambda)} [\alpha \to \lambda] a_{\mu,\alpha} \\ &= \sum_{\mu \leq_r \alpha \leq_r \lambda} \sum_{P \in Path(\mu,\alpha)} [P][\alpha \to \lambda] \\ &= \sum_{Q \in Paths(\mu,\lambda)} sgn(Q)[Q] \end{aligned}$$

Thus we need only show that $\sum_{Q\in Path(\mu,\lambda)} sgn(Q)[Q] = \delta_{\lambda,\mu}$. This can be done by defining a weight preserving involution on the set of paths in the lattice of partitions under refinement but we do not have the space to give the argument in this paper.

4. Special Cases of the $a_{\mu,\lambda}$'s

We saw in our example calculating $a_{(1^6),(3,2,1)}$ how difficult and time-consuming it can be to find these coefficients. However, in a number of special cases, we can actually compute a closed form for the sum $a_{\mu,\lambda} = (-1)^{l(\mu)-l(\lambda)} \sum_{P\in Path(\mu,\lambda)} [P](-1)^{l(P)}$. For example, if $\mu <_r \lambda$ is a cover relation in the refinement lattice, then there is only one path and the formula for the coefficient $a_{\mu,\lambda}$ consists of a single term. In fact, we can prove the following.

1. If λ and μ are a cover relation in the refinement lattice, then $a_{\mu,\lambda} = [\mu \to \lambda]$.
2. Similarly, we can show that $a_{\mu,\mu} = 1$ for all μ.
3. For any μ such that $\mu \vdash n$, $a_{\mu,(n)} = 1$, so that we find $DZ_{(n)} = \sum_\mu m_\mu = s_{(n)}$.

We outline a proof of 3 by induction on the length of the refinement.

$$\begin{aligned} a_{\mu,(n)} &= (-1)^{l(\mu)-1} \sum_{P\in Path(\mu,(n))} (-1)^{l(P)}[P] \\ &= (-1)^{l(\mu)-1} \sum_{\mu <_r \alpha <_r (n)} (-1)[\mu \to \alpha] \sum_{P \in Path(\alpha,(n))} (-1)^{l(P)}[P] \\ &\quad +(-1)^{l(\mu)-1}(-1)[\mu \to (n)] \end{aligned}$$

Our inductive assumption that $a_{\alpha,(n)} = 1$ gives that $\sum_{P\in Path(\alpha,(n))} (-1)^{l(P)}[P] = (-1)^{l(\alpha)-1}$. Thus Note that

$$a_{\mu,(n)} = (-1)^{l(\mu)-1}\Big(\sum_{\mu <_r \alpha <_r (n)} (-1)[\mu \to \alpha](-1)^{l(\alpha)-1}\Big) + (-1)^{l(\mu)-1}(-1)[\mu \to (n)].$$

But if we think about the definition of $[\mu \to \alpha]$, now we are summing over all possibilities of ways to remove at least one element from $Set(\mu)$ so

$$\begin{aligned} a_{\mu,(n)} &= (-1)^{l(\mu)-1} \sum_{\emptyset \subsetneq S \subseteq Set(\mu)} (-1)^{|Set(\mu)|-|S|} \\ &= (-1)^{l(\mu)-1}\Big(\Big(\sum_{\emptyset \subseteq S \subseteq Set(\mu)} (-1)^{|Set(\mu)|-|S|}\Big) - (-1)^{|Set(\mu)|}\Big) \end{aligned}$$

But $\sum_{S\subseteq Set(\mu)} (-1)^{|S|} = 0$. So

$$a_{\mu,(n)} = (-1)^{l(\mu)}(0 - (-1)^{|Set(\mu)|}) = (-1)^{l(\mu)}((-1)^{|Set(\mu)|+1})$$

But $|Set(\mu)| + 1 = l(\mu)$, so $a_{\mu,(n)} = 1$.

Other results can be found using careful examination of the lattice of refinement. The proofs of some of the below items are very straightforward. For example, the proof of item 4 is plain because the relevant portion of the refinement lattice contains only two shapes. Moreover, $Set(1^k) = \{1, 2, \ldots, k-1\}$ and when we remove any element from $Set(1^k)$, one ends up with a set that has shape $(2, 1^{k-2})$. Since there are $k-1$ ways to remove one element from $Set(1^k)$, it follows that $a_{(1^k),(2,1^{k-2})} = k-1$. The proofs of other items are more involved.

Results with $\mu = (1^k)$ and $\lambda = (b, 1^{k-b})$ for $b = 1, 2, \ldots, 7$:

4. $a_{(1^k),(2,1^{k-2})} = k-1$
5. $a_{(1^k),(3,1^{k-3})} = 1$
6. $a_{(1^k),(4,1^{k-4})} = \binom{k-1}{2} - 2$
7. $a_{(1^k),(5,1^{k-5})} = -\frac{1}{2}(k-1)(k-4) + 3$
8. $a_{(1^k),(6,1^{k-6})} = \frac{1}{6}(k^3 - 3k^2 - 16k - 6)$
9. $a_{(1^k),(7,1^{k-7})} = -\frac{1}{3}(k)(k+1)(k-7) + 1$

Here are some other results which are useful for the computation of the coefficients $b_{\mu,\lambda}$ of (**??**):

10. $a_{(1^k),(3^2,1^{k-6})} = 0$
11. $a_{(1^k),(3,2,1^{k-5})} = -\frac{1}{2}k(k-5)$
12. $a_{(2,1^{k-2}),(4,1^{k-4})} = k-3$
13. $a_{(2,1^{k-2}),(3,2,1^{k-5})} = 1$

Theorem 4.1. *If $d \neq 1$,*

$$a_{(2^c,1^b),(2^{c+d},1^{b-2d})} = \frac{b(b-1)\cdots(b-d+2)}{d!}(b-2d+1)$$

Note that if $d = 1$, the product on the right is not defined, so that Theorem 4.1 would not make sense. However the case where $d = 1$ and $c = 0$ is a special case of one our previous formulas.

Finding the value of one coefficient also tells us the value of an infinite number of other coefficients. Let $\mu = (\mu_1, \ldots, \mu_j)$. That is, define $k\mu$ to be the partition obtained when each part of μ is multiplied by k so that $k\mu = (k\mu_1, \ldots, k\mu_j)$. Then we can prove the following result.

Theorem 4.2. *For all $k \in \mathbb{N}$,*

$$a_{\mu,\lambda} = a_{k\mu,k\lambda}.$$

In particular, if we apply Theorem 4.2 to Theorem 4.1, we obtain infinite number of cases where we have explicit formulas for $a_{\mu,\lambda}$. The proof of Theorem 4.2 follows from an obvious bijection between paths in the refinement lattice of (μ, λ) to paths in the refinement lattice of $(k\mu, k\lambda)$.

Here is another result of the same sort.

Theorem 4.3. *Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\lambda = (\lambda_1, \ldots, \lambda_t)$. Then for any j such that $1 \leq j < min(\mu_s, \lambda_t)$,*

$$a_{\mu,\lambda} = a_{(\mu_1,\ldots,\mu_s,j),(\lambda_1,\ldots,\lambda_t,j)}.$$

The proof of Theorem 4.3 follows from examining the compositions and noticing that we must always have the last element of the composition in our subsets S in order for $shape_n(S)$ to match $(\lambda_1, \ldots, \lambda_t, k)$. This theorem works in "both directions", so to speak. Knowledge of the coefficients $a_{\mu,\lambda}$ where $\mu \vdash n$ and $\lambda \vdash n$ both with smallest part larger than 1 allows us to compute values of $a_{\alpha,\beta}$ for certain partitions α and β of size larger than n. Conversely, knowledge of coefficients $a_{\mu,\lambda}$ where μ and λ have identical unique smallest part allows us to compute values of $a_{\alpha,\beta}$ where α and β are partitions of size smaller than n by removing that smallest part from both μ and λ.

Thus the combination of Theorem 4.2 and Theorem 4.3 enables us to calculate the value $a_{\alpha,\beta}$ for infinitely many α and β starting with a single value of $a_{\mu,\lambda}$. That is, starting with $a_{\mu,\lambda}$, we can first multiply each part by k, then add smaller parts on the end, and so on.

5. Special Cases of the $b_{\mu,\lambda}$'s

Our method of expansion in terms of Schur functions in section 2 is useful not only in calculating particular expansions, but can also be used to make general statements independent of the size of λ.

T	$(-1)^{\lvert Set(\alpha)-T\rvert}$	$\lambda(\beta(T))$
$\emptyset$	-1	(10)
$\{2\}$	1	(8,2)
$\{4\}$	1	(6,4)
$\{8\}$	1	(8,2)
$\{2,4\}$	-1	(6,2,2)
$\{2,8\}$	-1	(6,2,2)
$\{4,8\}$	-1	(4,4,2)
$\{2,4,8\}$	1	(4,2,2,2)

TABLE 3. Values for $(-1)^{\lvert Set(\alpha)-T\rvert}$ and $\lambda(\beta(T))$ for each possible $T \subseteq Set(2,2,4,2)$.

μ	$(4,2,2,2)$	$(4,4,2)$	$(6,2,2)$	$(6,4)$	$(8,2)$	(10)
$a_{(4,2,2,2),\mu}$	1	2	1	1	2	1
$-a_{(4,4,2),\mu}$	0	-1	0	-1	-1	-1
$-2a_{(6,2,2),\mu}$	0	0	-2	-2	-2	-2
$a_{(6,4),\mu}$	0	0	0	1	0	1
$2a_{(8,2),\mu}$	0	0	0	0	2	2
$-a_{(10),\mu}$	0	0	0	0	0	-1
Sum for each μ	1	1	-1	-1	1	0

TABLE 4. Values for $a_{\gamma,\mu}$ used to compute $Z_{(2,2,4,2)} = \sum_{\mu \vdash n} f_{\mu,(2,2,4,2)} Z_\mu$.

We can use the fact that $b_{\mu,\lambda}$ can be expressed as $a_{\mu,\lambda}$ to prove further results, in particular that

1. $DZ_{(n)} = s_{(n)}$
2. $DZ_{(1^n)} = s_{1^n}$
3. $DZ_{(2^k,1^{n-2k})} = s_{(2^k,1^{n-2k})}\ \forall\ k$
4. $DZ_{(3^k,1^{n-3k})} = s_{(3,1^{n-3})} - s_{(2^2,1^{n-4})}\ \forall\ k$
5. $DZ_{(3,2,1^{n-5})} = s_{(3,2,1^{n-5})} - 2s_{(2^3,1^{n-6})} - s_{(2^2,1^{n-4})}$
6. $DZ_{(4,1^{n-4})} = s_{(4,1^{n-4})} - s_{(3,2,1^{n-5})} + s_{(2^2,1^{n-4})} + s_{(2^3,1^{n-6})}$

The proof of 1 was given above. The proofs of the others involve using the combinatorial interpretation of the coefficients that arise in (2.1) and defining some appropriate involutions to simplify the sum.

6. Applications of Our Main Result

As noted in the introduction, one application of our main result is to give a combinatorial interpretation of the expansion of Z_α in terms of Z_λ's, where α is a composition of n and λ is a partition of n. We noted that if $Z_\alpha = \sum_{\mu \vdash n} f_{\mu,\alpha} Z_\mu$, then

$$f_{\mu,\alpha} = \langle Z_\alpha, DZ_\mu \rangle = \sum_{T \subseteq Set(\alpha)} (-1)^{\lvert Set(\alpha)-T\rvert} a_{\lambda(\beta(T)),\mu}.$$

We now present an example of this fact; we will expand $Z_{(2,2,4,2)}$ as a sum of Z_λ's indexed by partitions of 10.

Table 3 tells us that

$$f_{\mu,(2,2,4,2)} = a_{(4,2,2,2),\mu} - a_{(4,4,2),\mu} - 2a_{(6,2,2),\mu} + a_{(6,4),\mu} + 2a_{(8,2),\mu} - a_{(10),\mu}.$$

Then Table 4 gives that $Z_{(2,2,4,2)} = Z_{(4,2,2,2)} + Z_{(4,4,2)} - Z_{(6,2,2)} - Z_{(6,4)} + Z_{(8,2)}$.

As another application of our results is that we can give a combinatorial interpretation of the coefficients that arise in the expansion of a Schur function s_γ in terms of the Z_λ's where $, \gamma, \lambda \vdash n$. That is, we can give a combinatorial interpretation of $e_{\mu,\gamma}$ where $s_\gamma = \sum_{\mu \vdash n} e_{\mu,\gamma} Z_\mu$.

Note that by 2.3, $s_\gamma = \sum_\mu K^{-1}_{\mu,\gamma} h_\mu$, so that

λ	$(3,2,1)$	$(4,1,1)$	$(3,3)$	$(4,2)$	$(5,1)$	(6)
$a_{(3,2,1),\lambda}$	1	0	1	0	1	1
$-a_{(4,1,1),\lambda}$	0	-1	0	-1	-1	-1
$-a_{(3,3),\lambda}$	0	0	-1	0	0	-1
$a_{(5,1),\lambda}$	0	0	0	0	1	1
Sum for each λ	1	-1	0	-1	1	0

Table 5. Values for $a_{\gamma,\lambda}$ used to compute $s_{(3,2,1)} = \sum_{\mu \vdash n} e_{\mu,(3,2,1)} Z_\mu$.

$$\begin{aligned} e_{\lambda,\gamma} &= \langle s_\gamma, DZ_\lambda \rangle \\ &= \langle \sum_\mu K^{-1}_{\mu,\gamma} h_\mu, \sum_{\beta \leq_r \lambda} a_{\beta,\lambda} m_\beta \rangle \\ &= \sum_{\beta \leq_r \lambda} K^{-1}_{\beta,\gamma} a_{\beta,\lambda}. \end{aligned}$$

We now present an example by expanding $s_{(3,2,1)}$ as a sum of ribbon Schur functions indexed by partitions. We can easily see that $s_{(3,2,1)} = h_1h_2h_3 - h_1h_1h_4 - h_3h_3 + h_1h_5$ by writing down all the special rim hook tabloids of shape $(3,2,1)$. Then

$$\langle s_{(3,2,1)}, DZ_\lambda \rangle = a_{(3,2,1),\lambda} - a_{(4,1,1),\lambda} - a_{(3,3),\lambda} + a_{(5,1),\lambda)}.$$

In Table 5, we present the relevant values of $a_{\mu,\lambda}$.

Thus

$$s_{(3,2,1)} = Z_{(3,2,1)} - Z_{(4,1,1)} - Z_{(4,2)} + Z_{(5,1)}.$$

This may not be the most efficient algorithm in all cases, for example another approach is to use a result of Lascoux and Pragacz [**7**] which gives the expansion of a Schur function as a product of ribbon Schur functions using a determinantal formula. Any product ribbon Schur functions can be simplified to a sum of ribbon Schur functions. However the ribbon Schur functions that result from such an expansion are just arbitrary Z_α where α is a composition. Thus one would need to expand $Z_\alpha = \sum_{\lambda \vdash n} f_{\lambda,\alpha} Z_\lambda$, where α is a composition of n and λ is a partition of n, as we did above. In special cases, such as when γ is a double hook, this method may be more efficient. However this method does not give a combinatorial interpretation of the coefficients of the Z_λ's that arise in the expansion.

7. Conclusions and Further Research

In this paper we have given combinatorial interpretations of the coefficients in the expansion of DZ_λ in terms of the monomial symmetric functions. We also found more indirect combinatorial interpretations of the expansion DZ_λ in terms of the Schur functions by using the inverse Kostka matrix. Moreover, we have given explicit formulas for such coefficients in many special cases.

There are many unanswered questions in this area. Of particular interest is what happens when we apply the ω transformation to DZ_λ. That is, recall the $\omega : \Lambda_n \to \Lambda_n$ is defined by the fact for all $\lambda \vdash n$, $\omega(h_\lambda) = e_\lambda$. Then the question is: can we give a combinatorial interpretation of $\omega(DZ_\lambda)$ in terms of $\{Z_\lambda\}_{\lambda \vdash n}$ or $\{DZ_\lambda\}_{\lambda \vdash n}$? We can clearly give a combinatorial interpretations of $\omega(DZ_\lambda)$ in terms of $\{f_\lambda\}_{\lambda \vdash n}$, since we can already expand DZ in terms of $\{m_\lambda\}_{\lambda \vdash n}$ and $\omega(m_\lambda) = f_\lambda$.

We also examined the coefficients in the expansion in terms of the power and elementary symmetric functions. Again the coefficients that arise in such expansions are not all positive. Thus another unanswered question is to find good combinatorial interpretations for the coefficients in the expansion of DZ_λ in terms of the other standard bases for the space of symmetric functions.

References

[1] D. Beck, J. Remmel, and T. Whitehead, The combinatorics of transition matrices between the bases of the symmetric functions and the B_n analogues, *Discrete Mathematics* **153** 1996, 3–27.

[2] L. Billera, H. Thomas, and S. van Willigenburg, Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions, preprint 2005, to appear in *Adv. Math.*.

[3] Ö. Eğecioğlu and J. Remmel, A combinatorial interpretation of the inverse Kostka matrix, *Linear Multilinear Algebra* **26** (1990), 59–84.

[4] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, Noncommutative symmetric functions, *Adv. in Math.*, **112** (1995), 218–348.

[5] I. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A* **64** (1993), 189–215.

[6] A.N. Kirillov, A. Kuniba and T. Nakanishi, Skew diagram method in spectral decomposition of integrable lattice models, *Comm. Math. Phys.* **185** (1997), 441–465.

[7] A. Lascoux and P. Pragacz, Ribbon Schur functions, *European J. Combin.* **9** (1988), no. 6, 561–574.

[8] P.A. MacMahon, *Combinatory Analysis*, Cambridge University Press, 1917 (Vol. I), 1918 (Vol. II), reprint, Chelsea, New York, USA, 1960.

[9] R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge Studies in Advanced Mathematics, Vol. 62, Cambridge University Press, Cambridge, UK, 1999.

Department of Mathematics, University of California San Diego, La Jolla CA 92093-0112. USA
E-mail address: ariehl@math.ucsd.edu