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On the Combinatorics of Crystal Graphs, I

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Abstract. In this paper, we continue the development of a new combinatorial model for the irreducible characters of a complex semisimple Lie group. The main results of this paper are: (1) a combinatorial description of the crystal graphs corresponding to the irreducible representations (this result includes a transparent proof, based on the Yang-Baxter equation, of the fact that the mentioned description does not depend on the choice involved in our model); (2) a combinatorial realization (which is the first direct generalization of *Schützenberger's involution* on tableaux) of a certain fundamental involution on the canonical basis exhibiting the crystals as self-dual posets; (3) an analog for arbitrary root systems, based on the Yang-Baxter equation, of Schützenberger's sliding algorithm, which is also known as jeu de taquin (this algorithm has many applications to the representation theory of the Lie algebra of type A). Our approach is type-independent.

Résumé.
Dans cet article, nous continuons le développement d'un nouveau modèle combinatoire pour les caractères irréductibles d'un groupe de Lie complexe semisimple. Les résultats principaux de cet article sont : (1) une description combinatoire des graphes cristallins correspondant aux représentations irréductibles (ce résultat inclut une preuve transparente, basée sur l'équation de Yang-Baxter, du fait que la description mentionnée ne dépend pas du choix impliqué dans notre modèle) ; (2) une réalisation combinatoire (qui est la première généralisation directe de *l'involution de Schützenberger* sur les tableaux) d'une involution fondamentale sur la base canonique pour laquelle les cristaux sont des ensembles partiellement ordonnés auto-dual ; (3) un analogue de l'algorithme coulissant de Schützenberger, qui est également connu sous le nom "jeu de taquin", pour les systèmes de racine. Cet analogue est basé sur l'équation de Yang-Baxter. Notre approche est indépendante du choix du type du système de racine.

1. Introduction

We have recently given a simple combinatorial model for the irreducible characters of a complex semisimple Lie group G and, more generally, for the *Demazure characters* [**12**]. For reasons explained below, we call our model the *alcove path model*. This was extended to complex symmetrizable Kac-Moody algebras in [**13**] (that is, to infinite root systems).

The alcove path model leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux, for instance) to arbitrary type; our approach is type-independent. The present paper continues the study of the combinatorics of the new model, which was started in [**12, 13**].

The main results of this paper are:

(1) a combinatorial description of the crystal graphs corresponding to the irreducible representations (Corollary 4.4); this result includes a transparent proof, based on the Yang-Baxter equation, of the fact that the mentioned description does not depend on the choice involved in our model (Corollary 4.3);

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(2) a combinatorial realization of a certain fundamental involution on the canonical basis (Theorem 5.4, see also Example 5.6); this involution [**18**] exhibits the crystals as self-dual posets, and corresponds to the action of the longest Weyl group element on an irreducible representation; our combinatorial realization is the first direct generalization of *Schützenberger's involution* on tableaux (see e.g. [**6**]);

(3) an analog for arbitrary root systems, based on the Yang-Baxter equation, of Schützenberger's *sliding* algorithm, which is also known as *jeu de taquin* (Section 4); this algorithm has many applications to the representation theory of the Lie algebra of type A (see e.g. [**6**]).

Our model is based on the choice of an *alcove path*, which is a sequence of adjacent alcoves for the *affine Weyl group* W_{aff} of the Langland's dual group $G^\vee$. An alcove path is best represented as a λ-*chain*, that is, as a sequence of positive roots corresponding to the common walls of successive alcoves in the mentioned sequence of alcoves. These chains extend the notion of a *reflection ordering* [**5**]. Given a fixed λ-chain, the objects that generalize semistandard Young tableaux are all the subsequences of roots that give rise to saturated increasing chains in Bruhat order (on the Weyl group W) upon multiplying on the right by the corresponding reflections. We call these subsequences *admissible subsets*. In [**13**] we defined *root operators* on admissible subsets, which are certain partial operators associated with the simple roots; in type A, they correspond to the *coplactic operations* on tableaux [**17**]. The root operators produce a directed colored graph structure and a poset structure on admissible subsets. We showed in [**13**] that this graph is isomorphic to the crystal graph of the corresponding irreducible representation if the chosen λ-chain is a special one. All this background information on the alcove path model is explained in more detail in Section 3, following some general background material discussed in Section 2.

In Section 4, we study certain discrete moves which allow us to deform any λ-chain into any other λ-chain (for a fixed dominant weight λ), and to biject the corresponding admissible subsets. We call these moves *Yang-Baxter moves* since they express the fact that certain operators satisfy the *Yang-Baxter equation*. We will explain below the reason for which the Yang-Baxter moves can be considered an analog of jeu de taquin for arbitrary root systems. We show that the Yang-Baxter moves commute with the root operators; this means that the directed colored graph defined by the root operators is invariant under Yang-Baxter moves, and it is thus independent from the choice of a λ-chain. Based on the special case in [**13**] discussed above, this immediately implies that the mentioned graph is isomorphic to the corresponding crystal graph for any choice of a λ-chain.

In Section 5, we present a combinatorial description of a certain fundamental involution η_λ on the *canonical basis*. Such a description was given by Schützenberger in type A in terms of tableaux, and the corresponding procedure is known as *evacuation*. The importance of this involution stems from the fact that it exhibits the crystals as self-dual posets, and it corresponds to the action of the longest Weyl group element on an irreducible representation; it also appears in other contexts, such as the recent realization of the category of crystals as a *coboundary category* [**8**]. Our description of the mentioned involution is very similar to that of the evacuation map. The main ingredient in defining the latter map, namely Schützenberger's sliding algorithm (also known as jeu de taquin), is replaced by Yang-Baxter moves. There is another ingredient, which has to do with "reversing" a λ-chain and an associated admissible subset, by analogy with reversing the word of a tableau in the definition of the evacuation map. Our construction also leads to a purely combinatorial proof of the fact that the crystals (as defined by our root operators) are self-dual posets.

The relationship between the alcove path model and other models for the irreducible characters of semisimple Lie algebras, such as the Littelmann path model, LS paths [**14, 15, 16**], and LS-galleries [**7**], was discussed in [**12, 13**].

As far as analogs of jeu de taquin are concerned, let us mention that the only such analog known in the Littelmann path model is the one due to van Leeuwen [**11**]. The goal of the mentioned paper was to use this analog in order to express in a bijective manner the symmetry of the Littlewood-Richardson rule in the Littelmann path model.

Let us also mention that an explicit description of the involution η_λ discussed above is given in [**19**] in a different model for characters, which is based on *Lusztig's parametrization* and the *string parametrization* of the dual canonical basis [**2**]. Unlike the combinatorial approach in Schützenberger's evacuation procedure, the involution is now expressed as an affine map whose coefficients are entries of the corresponding Cartan matrix. No intrinsic explanation for the fact that this map is an involution is available.

We believe that the properties of our model that were investigated in [**12, 13**] as well as in this paper represent just a small fraction of a rich combinatorial structure yet to be explored, which would generalize most of the combinatorics of Young tableaux. A future publication will be concerned with the combinatorics of the product of crystals.

2. Preliminaries

We recall some background information on finite root systems, affine Weyl groups, and crystal graphs.

2.1. Root systems. Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. As usual, we denote by B^- be the opposite Borel subgroup, while N and N^- are the unipotent radicals of B and B^-, respectively. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ be the complex Lie algebras of G, T, N, and N^-, respectively. Let r be the rank of the Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, and let $\mathfrak{h}^*_\mathbb{R} \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- := -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*, which form a basis of $\mathfrak{h}^*_\mathbb{R}$. Let $\langle \cdot\,, \cdot \rangle$ denote the nondegenerate scalar product on $\mathfrak{h}^*_\mathbb{R}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha\rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.*

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*_\mathbb{R})$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha \in \Phi$, given by $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle\, \alpha$. In fact, the Weyl group W is generated by the *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$, subject to the *Coxeter relations*. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = \#\Phi^+$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = us_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.1}$$

The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

Let $\rho := \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $\langle \rho, \alpha^\vee \rangle = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. (In other words, $\theta^\vee$ is the highest root of the dual root system $\Phi^\vee$. It should not be confused with the coroot of the highest root of Φ.) We will also use the *Coxeter number*, that can be defined as $h := \langle \rho, \theta^\vee \rangle + 1$.

2.2. Affine Weyl groups. In this subsection, we remind a few basic facts about affine Weyl groups and alcoves, cf. Humphreys [**9**, Chaper 4] for more details.

Let W_{aff} be the *affine Weyl group* for the Langland's dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that reflect the space $\mathfrak{h}^*_\mathbb{R}$ with respect to the affine hyperplanes $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = k\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_\mathbb{R}$ into open regions, called *alcoves.*

The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

An important property of the affine Weyl group is that it acts simply transitively on the collection of all alcoves. This fact implies that, for any alcove A, there exists a unique element v_A of the affine Weyl group W_{aff} such that $v_A(A_\circ) = A$. Hence the map $A \mapsto v_A$ is a one-to-one correspondence between alcoves and elements of the affine Weyl group.

Recall that $\theta^\vee \in \Phi^\vee$ is the highest coroot. Let $\theta \in \Phi^+$ be the corresponding root, and let $\alpha_0 := -\theta$. The affine Weyl group is a Coxeter group generated by the set of reflections $s_0, s_1, \ldots, s_r$, where $s_0 := s_{\alpha_0,-1}$ and $s_1, \ldots, s_r \in W$ are the simple reflections $s_i = s_{\alpha_i,0}$.

We say that two alcoves A and B are *adjacent* if B is obtained by an affine reflection of A with respect to one of its walls. In other words, two alcoves are adjacent if they are distinct and have a common wall.

For a pair of adjacent alcoves, let us write $A \xrightarrow{\beta} B$ if the common wall of A and B is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

2.3. Crystal graphs and Schützenberger's involution. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$, which is generated by E_i, F_i, H_i, for $i = 1, \ldots, r$, subject to the Serre relations and some additional relations. Let $\mathcal{B}$ be the *canonical basis* of $U(\mathfrak{n}^-)$, and let $\mathcal{B}_\lambda := \mathcal{B} \cap V_\lambda$ be the canonical basis of the irreducible representation V_λ with highest weight λ. Let v_λ and v_λ^{low} be the highest and lowest weight vectors in $\mathcal{B}_\lambda$, respectively. Let $\widetilde{E}_i$, $\widetilde{F}_i$, for $i = 1, \ldots, r$, be Kashiwara's operators [**10, 18**]; these are also known as raising and lowering operators, respectively. The *crystal graph* of V_λ is the directed colored graph on $\mathcal{B}_\lambda$ defined by arrows $x \to y$ colored i for each $\widetilde{F}_i(x) = cy + \text{lower terms}$, or, equivalently, for each $\widetilde{E}_i(y) = cx + \text{lower terms}$, with c a constant. (In fact, Kashiwara defined the crystal graph of the q-deformation of $U(\mathfrak{g})$, also known as a *quantum group*; using the quantum deformation, one can associate a crystal graph to a $\mathfrak{g}$-representation.) One can also define partial orders $\preceq_i$ on $\mathcal{B}_\lambda$ by $x \preceq_i y$ if $x = \widetilde{F}_i^k(y)$ for some $k \geq 0$. We let $\preceq$ denote the partial order generated by all partial orders $\preceq_i$, for $i = 1, \ldots, r$. The poset $(\mathcal{B}_\lambda, \preceq)$ has maximum v_λ and minimum v_λ^{low}.

In order to proceed, we need the following general setup. Let V be a module over an associative algebra U and σ an automorphism of U. The twisted U-module V^σ is the same vector space V but with the new action $u * v := \sigma(u)v$ for $u \in U$ and $v \in V$. Clearly, $V^{\sigma\tau} = (V^\sigma)^\tau$ for every two automorphisms σ and τ of U. Furthermore, if V is a simple U-module, then so is V^σ. In particular, if $U = U(\mathfrak{g})$ and $V = V_\lambda$, then $(V_\lambda)^\sigma$ is isomorphic to $V_{\sigma(\lambda)}$ for some dominant weight $\sigma(\lambda)$. Thus there is an isomorphism of vector spaces $\sigma_\lambda : V_\lambda \to V_{\sigma(\lambda)}$ such that

$$\sigma_\lambda(uv) = \sigma(u)\sigma_\lambda(v)\,, \quad u \in U(\mathfrak{g})\,,\ v \in V_\lambda\,.$$

By Schur's lemma, σ_λ is unique up to a scalar multiple.

The longest Weyl group element $w_\circ$ defines an involution on the simple roots by $\alpha_i \mapsto \alpha_{i^*} := -w_\circ(\alpha_i)$. Consider the automorphisms of $U(\mathfrak{g})$ defined by

$$\phi(E_i) = F_i\,, \quad \phi(F_i) = E_i\,, \quad \phi(H_i) = -H_i\,, \tag{2.2}$$

$$\psi(E_i) = E_{i^*}\,, \quad \psi(F_i) = F_{i^*}\,, \quad \psi(H_i) = H_{i^*}\,, \tag{2.3}$$

and $\eta := \phi\psi$. Clearly, these three automorphisms together with the identity automorphism form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It also easily follows from (2.2)-(2.3) that

$$\phi(\lambda) = \psi(\lambda) = -w_\circ(\lambda)\,, \quad \eta(\lambda) = \lambda\,.$$

We can normalize each of the maps ϕ_λ, ψ_λ, and η_λ by the requirement that

$$\phi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}^{low}\,, \quad \psi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}\,, \quad \eta_\lambda(v_\lambda) = v_\lambda^{low}\,. \tag{2.4}$$

(Of course, we also set Id_λ to be the identity map on V_λ.) By [**18**, Proposition 21.1.2], cf. also [**1**, Proposition 7.1], we have the following result.

Proposition 2.1. [**1, 18**] (1) *Each of the maps ϕ_λ and ψ_λ sends $\mathcal{B}_\lambda$ to $\mathcal{B}_{-w_\circ(\lambda)}$, while η_λ sends $\mathcal{B}_\lambda$ to itself.*

(2) *For every two (not necessarily distinct) elements σ, τ of the group $\{\mathrm{Id}, \phi, \psi, \eta\}$, we have $(\sigma\tau)_\lambda = \sigma_{\tau(\lambda)}\tau_\lambda$. In particular, the map η_λ is an involution.*

(3) *For every $i = 1, \ldots, r$, we have*

$$\phi_\lambda\widetilde{F}_i = \widetilde{E}_i\phi_\lambda\,, \quad \psi_\lambda\widetilde{F}_i = \widetilde{F}_{i^*}\psi_\lambda\,, \quad \eta_\lambda\widetilde{F}_i = \widetilde{E}_{i^*}\eta_\lambda\,. \tag{2.5}$$

In particular, the poset $(\mathcal{B}_\lambda, \preceq)$ is self-dual, and η_λ is the corresponding antiautomorphism.

Berenstein and Zelevinsky [**1**] showed that, in type A_{n-1} (that is, in the case of the Lie algebra $\mathfrak{sl}_n$), the operator η_λ is given by Schützenberger's *evacuation* procedure for semistandard Young tableaux (see e.g. [**6**]).

3. The Alcove Path Model

In this section, we recall the model for the irreducible characters of semisimple Lie algebras that we introduced in [**12, 13**]. We refer to these papers for more details, including the proofs of the results mentioned below. Although some of these results hold for infinite root systems (cf. [**13**]), the setup in this paper is that of a finite irreducible root system, as discussed in Section 2.

Our model is conveniently phrased in terms of several sequences, so let us mention some related notation. Given a totally ordered index set $I = \{i_1 < i_2 < \ldots < i_n\}$, a sequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ is sometimes abbreviated to $\{a_j\}_{j\in I}$. We also let $[n] := \{1, 2, \ldots, n\}$.

3.1. λ-chains. The affine translations by weights preserve the set of affine hyperplanes $H_{\alpha,k}$. It follows that these affine translations map alcoves to alcoves. Let $A_\lambda = A_\circ + \lambda$ be the alcove obtained by the affine translation of the fundamental alcove $A_\circ$ by a weight $\lambda \in \Lambda$. Let v_λ be the corresponding element of W_{aff}, i.e,. v_λ is defined by $v_\lambda(A_\circ) = A_\lambda$. Note that the element v_λ may not be an affine translation itself.

Let us now fix a dominant weight λ. Let $v \mapsto \bar{v}$ be the homomorphism $W_{\text{aff}} \to W$ defined by ignoring the affine translation. In other words, $\bar{s}_{\alpha,k} = s_\alpha \in W$.

Definition 3.1. A *λ-chain of roots* is a sequence of positive roots $(\beta_1, \ldots, \beta_n)$ which is determined as indicated below by a reduced decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_n}$ of $v_{-\lambda}$ as a product of generators of W_{aff}:

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_n = \bar{s}_{i_1}\cdots\bar{s}_{i_{n-1}}(\alpha_{i_n})\,.$$

When the context allows, we will abbreviate "λ-chain of roots" to "λ-chain". The *λ-chain of reflections* associated with the above λ-chain of roots is the sequence $(\widehat{r}_1, \ldots, \widehat{r}_n)$ of affine reflections in W_{aff} given by

$$\widehat{r}_1 = s_{i_1},\ \widehat{r}_2 = s_{i_1}s_{i_2}s_{i_1},\ \widehat{r}_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ \widehat{r}_n = s_{i_1}\cdots s_{i_n}\cdots s_{i_1}.$$

We will present two equivalent definitions of a λ-chain of roots.

Definition 3.2. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_n)$ such that A_{i-1} and A_i are adjacent, for $i = 1, \ldots, n$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_n.

Given a finite sequence of roots $\Gamma = (\beta_1, \ldots, \beta_n)$, we define the sequence of integers $(l_1^\emptyset, \ldots, l_n^\emptyset)$ by $l_i^\emptyset := \#\{j < i \mid \beta_j = \beta_i\}$, for $i = 1, \ldots, n$. We also need the following two conditions on Γ.

(R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.

(R2) For each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Theorem 3.3. [**12**] *The following statements are equivalent.*

(a) *The sequence of roots $\Gamma = (\beta_1, \ldots, \beta_n)$ is a λ-chain, and $(\widehat{r}_1, \ldots, \widehat{r}_n)$ is the associated λ-chain of reflections.*

(b) *We have a reduced alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_n} A_n$ from $A_0 = A_\circ$ to $A_n = A_{-\lambda}$, and $\widehat{r}_i$ is the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \ldots, n$.*

(c) *The sequence Γ satisfies conditions* (R1) *and* (R2) *above, and $\widehat{r}_i = s_{\beta_i, -l_i^\emptyset}$, for $i = 1, \ldots, n$.*

A particular choice of a λ-chain was described in [**13**].

3.2. Admissible subsets. For the remainder of this section, we fix a λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$. Let $r_i := s_{\beta_i}$. We now define the centerpiece of our combinatorial model for characters, which is our generalization of semistandard Young tableaux in type A.

Definition 3.4. An *admissible subset* is a subset of $[n]$ (possibly empty), that is, $J = \{j_1 < j_2 < \ldots < j_s\}$, such that we have the following saturated chain in the Bruhat order on W:

$$1 \lessdot r_{j_1} \lessdot r_{j_1}r_{j_2} \lessdot \ldots \lessdot r_{j_1}r_{j_2}\ldots r_{j_s}\,.$$

We denote by $\mathcal{A}(\Gamma)$ the collection of all admissible subsets corresponding to our fixed λ-chain Γ. Given an admissible subset J, we use the notation

$$\mu(J) := -\widehat{r}_{j_1}\ldots\widehat{r}_{j_s}(-\lambda)\,, \qquad w(J) := r_{j_1}\ldots r_{j_s}\,.$$

We call $\mu(J)$ the *weight* of the admissible subset J.

THEOREM 3.5. [**12, 13**] *We have the following character formula:*

$$ch(V_\lambda) = \sum_{J \in \mathcal{A}(\Gamma)} e^{\mu(J)} \,.$$

A more general Demazure character formula is also given in [**12**]. In addition to these character formulas, a Littlewood-Richardson rule for decomposing tensor products of irreducible representations is presented in terms of our model in [**13**].

EXAMPLE 3.6. Consider the Lie algebra $\mathfrak{sl}_3$ of type A_2. The corresponding root system Φ can be realized inside the vector space $V := \mathbb{R}^3/\mathbb{R}(1,1,1)$ as $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \mid i \neq j,\, 1 \leq i,j \leq 3\}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V$ are the images of the coordinate vectors in $\mathbb{R}^3$. The reflection $s_{\alpha_{ij}}$ is denoted by s_{ij}. The simple roots are α_{12} and α_{23}, while $\alpha_{13} = \alpha_{12} + \alpha_{23}$ is the other positive root. Let $\lambda = \omega_1 = \varepsilon_1$ be the first fundamental weight. In this case, there is only one λ-chain $(\beta_1, \beta_2) = (\alpha_{12}, \alpha_{13})$. There are 3 admissible subsets: $\emptyset, \{1\}, \{1,2\}$. The subset $\{2\}$ is not admissible because the reflection s_{13} does not cover the identity element. We have $(l_1^\emptyset, l_2^\emptyset) = (0,0)$. Theorem 3.5 gives the following expression for the character of V_{ω_1}:

$$ch(V_{\omega_1}) = e^{\omega_1} + e^{s_{12}(\omega_1)} + e^{s_{12}s_{13}(\omega_1)}.$$

3.3. Root operators. We now define partial operators known as *root operators* on the collection $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to our fixed λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$. They are associated with a fixed simple root α_p, and are traditionally denoted by F_p (also called a lowering operator) and E_p (also called a raising operator). The notation is the one introduced above. Recall the sequence of integers $(l_1^\emptyset, \ldots, l_n^\emptyset)$ associated to Γ, and the corresponding affine reflections $\widehat{r}_i = s_{\beta_i, -l_i^\emptyset}$ for $i = 1, \ldots, n$. Let $J = \{j_1 < j_2 < \ldots < j_s\}$ be a fixed admissible subset. We associate with J the sequence of roots $(\gamma_1, \ldots, \gamma_n)$ and the sequence of integers $L(J) = (l_1, \ldots, l_n)$, as follows: given $i \in [n]$, we let $k := \max\{a \mid j_a < i\}$ and $\widehat{r}_{j_1} \ldots \widehat{r}_{j_k}(H_{\beta_i, -l_i^\emptyset}) = H_{\gamma_i, -l_i}$ for some positive root γ_i and some integer l_i. We also define $l_\infty^p := \langle \mu(J), \alpha_p^\vee \rangle$. Finally, we let

$$I(J,p) := \{i \in [n] \mid \gamma_i = \alpha_p\}\,, \quad L(J,p) := (\{l_i\}_{i \in I(J,p)},\, l_\infty^p)\,, \quad M(J,p) := \max L(J,p)\,. \tag{3.1}$$

It turns out that $M(J,p) \geq 0$.

We can define the root operator F_p on the admissible subset J whenever $M(J,p) > 0$. Let $m = m_F(J,p)$ be defined by

$$m_F(J,p) := \begin{cases} \min\{i \in I(J,p) \mid l_i = M(J,p)\} & \text{if this set is nonempty} \\ \infty & \text{otherwise}\,. \end{cases}$$

Let $k = k_F(J,p)$ be the predecessor of m in $I(J,p) \cup \{\infty\}$, which always exists. It turns out that $m \in J$ if $m \neq \infty$, but $k \notin J$ Finally, we set

$$F_p(J) := (J \setminus \{m\}) \cup \{k\}\,. \tag{3.2}$$

We showed in [**13**] that we have $\mu(F_p(J)) = \mu(J) - \alpha_p$.

Let us now define a partial inverse E_p to F_p. The operator E_p is defined on the admissible subset J whenever $M(J,p) > \langle \mu(J), \alpha_p^\vee \rangle$. Let $k = k_E(J,p)$ be defined by

$$k_E(J,p) := \max\{i \in I(J,p) \mid l_i = M(J,p)\}\,;$$

the above set turns out to be always nonempty. Let $m = m_E(J,p)$ be the successor of k in $I(J,p) \cup \{\infty\}$. It turns out that $k \in J$ but $m \notin J$. Finally, we set

$$E_p(J) := (J \setminus \{k\}) \cup (\{m\} \setminus \{\infty\})\,. \tag{3.3}$$

Similarly to Kashiwara's operators (see Subsection 2.3), the root operators above define a directed colored graph structure and a poset structure on the set $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to a fixed λ-chain Γ. According to [**13**, Proposition 6.9]), the admissible subset $J_{\max} = \emptyset$ is the maximum of the poset $\mathcal{A}(\Gamma)$.

4. Yang-Baxter Moves

In this section, we define the analog of Schützenberger's sliding algorithm in our model, which we call a *Yang-Baxter move*, for reasons explained below. We start with some results on dihedral subgroups of Weyl groups.

4.1. Dihedral reflection subgroups. Let $\overline{W}$ be a dihedral Weyl group of order $2q$, that is, a Weyl group of type $A_1 \times A_1$, A_2, B_2, or G_2 (with $q = 2, 3, 4, 6$, respectively). Let $\overline{\Phi}$ be the corresponding root system with simple roots α, β. The sequence

$$\beta_1 := \alpha,\ \ \beta_2 := s_\alpha(\beta),\ \ \beta_3 := s_\alpha s_\beta(\alpha),\ \ \ldots,\ \ \beta_{q-1} := s_\beta(\alpha),\ \ \beta_q := \beta \tag{4.1}$$

is a *reflection ordering* on the positive roots of $\overline{\Phi}$ (cf. [**5**]). We present the Bruhat order on the Weyl group of type G_2 in Figure 1. Here, as well as throughout this paper, we label a cover $v \lessdot vs_\gamma$ in Bruhat order by the corresponding root γ.

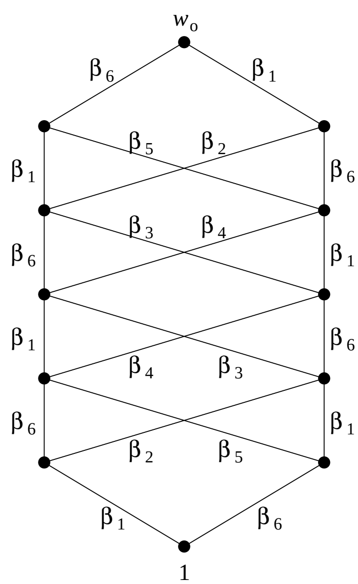

FIGURE 1. The Bruhat order on the Weyl group of type G_2.

With every pair of Weyl group elements $\overline{u} < \overline{w}$ in Bruhat order, we will associate a subset $J(\overline{u}, \overline{w})$ of $[q]$ as follows. Let $a := \ell(\overline{u})$ and $b := \ell(\overline{w})$. Given $\delta \in \{\alpha, \beta\}$, we will use the notation

$$\overline{W}_\delta := \{\overline{v} \in \overline{W} \mid \ell(\overline{v}s_\delta) > \ell(\overline{v})\}\,, \quad \overline{W}^\delta := \overline{W} \setminus \overline{W}_\delta = \{\overline{v} \in \overline{W} \mid \ell(\overline{v}s_\delta) < \ell(\overline{v})\}\,.$$

Case 0: $\overline{u} = \overline{w}$. We let $J(\overline{u}, \overline{u}) := \emptyset$.

Case 1: $b - a = 1$. We have the following disjoint subcases.

- **Case 1.1:** $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}^\alpha$, so $0 \le a \le q-1$. We let $J(\overline{u}, \overline{w}) := \{1\}$.
- **Case 1.2:** $\overline{u} \in \overline{W}^\beta$, $\overline{w} \in \overline{W}_\alpha$, so $0 < a < q-1$. We let $J(\overline{u}, \overline{w}) := \{q - a\}$.
- **Case 1.3:** $\overline{u} \in \overline{W}_\beta$, $\overline{w} \in \overline{W}^\beta$, so $0 \le a \le q-1$. We let $J(\overline{u}, \overline{w}) := \{q\}$.
- **Case 1.4:** $\overline{u} \in \overline{W}^\alpha$, $\overline{w} \in \overline{W}_\beta$, so $0 < a < q-1$. We let $J(\overline{u}, \overline{w}) := \{a + 1\}$.

Case 2: $1 < b - a < q$. We have the following disjoint subcases.

- **Case 2.1:** $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}_\beta$, so $0 \le a < a+2 \le b < q$.
  We let $J(\overline{u}, \overline{w}) := \{1, a+2, a+3, \ldots, b\}$.
- **Case 2.2:** $\overline{u} \in \overline{W}^\beta$, $\overline{w} \in \overline{W}^\beta$, so $0 < a < a+2 \le b \le q$.
  We let $J(\overline{u}, \overline{w}) := \{1, a+2, a+3, \ldots, b-1, q\}$.
- **Case 2.3:** $\overline{u} \in \overline{W}_\beta$, $\overline{w} \in \overline{W}_\alpha$, so $0 \le a < a+2 \le b < q$.
  We let $J(\overline{u}, \overline{w}) := \{a+1, a+2, \ldots, b-1, q\}$.

Case 2.4: $\overline{u} \in \overline{W}^{\alpha}$, $\overline{w} \in \overline{W}^{\alpha}$, so $0 < a < a+2 \le b \le q$.
We let $J(\overline{u}, \overline{w}) := \{a+1, a+2, \ldots, b\}$.

Case 3: $a = 0$ and $b = q$, that is, $\overline{u}$ is the identity and $\overline{w}$ is the longest Weyl group element $\overline{w}_\circ$. In this case, we let $J := [q]$.

In Case 2.2, if $b = a+2$ then the sequence $a+2, a+3, \ldots, b-1$ is considered empty.

Let $J(\overline{u}, \overline{w}) := \{j_1 < j_2 < \ldots < j_{b-a}\}$. We use the notation $r_i := s_{\beta_i}$, as above. It is easy to check that, in all cases above, we have a unique saturated increasing chain in Bruhat order from $\overline{u}$ to $\overline{w}$ whose labels form a subsequence of (4.1); this chain is

$$\overline{u} \lessdot \overline{u} r_{j_1} \lessdot \overline{u} r_{j_1} r_{j_2} \lessdot \ldots \lessdot \overline{u} r_{j_1} \ldots r_{j_{b-a}} = \overline{w}\,.$$

More generally, we have the result below for an arbitrary Weyl group W with a dihedral reflection subgroup $\overline{W}$ and corresponding root systems $\Phi \supseteq \overline{\Phi}$. The notation is the same as above. It is known that any element w of W can be written uniquely as $w = \lfloor w \rfloor \overline{w}$, where $\lfloor w \rfloor$ is the minimal representative of the left coset $w\overline{W}$, and $\overline{w} \in \overline{W}$. The following result can be easily deduced from the corresponding one for $W = \overline{W}$ via a standard fact about cosets modulo dihedral reflection subgroups, namely [**3**, Lemma 5.1].

Proposition 4.1. *For each pair of elements $u < w$ in the same (left) coset of W modulo $\overline{W}$, we have a unique saturated increasing chain in Bruhat order from u to w whose labels form a subsequence of* (4.1)*; this chain is*

$$u \lessdot u r_{j_1} \lessdot u r_{j_1} r_{j_2} \lessdot \ldots \lessdot u r_{j_1} \ldots r_{j_{b-a}} = w\,,$$

where $J(\overline{u}, \overline{w}) := \{j_1 < j_2 < \ldots < j_{b-a}\}$.

We obtain another reflection ordering by reversing the sequence (4.1). Let us denote the corresponding subset of $[q]$ by $J'(\overline{u}, \overline{w})$. We are interested in passing from the chain between u and w compatible with the ordering (4.1) to the chain compatible with the reverse ordering. If we fix $a := \ell(\overline{u})$ and $b := \ell(\overline{w})$, we can realize the passage from $J(\overline{u}, \overline{w})$ to $J'(\overline{u}, \overline{w})$ via the involution $Y_{q,a,b}$ described below in each of the cases mentioned above.

Case 0: $\emptyset \leftrightarrow \emptyset$ if $a = b$.

Case 1.1: $\{1\} \leftrightarrow \{q\}$ if $0 \le a = b-1 \le q-1$.

Case 1.2: $\{q-a\} \leftrightarrow \{a+1\}$ if $0 < a = b-1 < q-1$.

Case 2.1: $\{1, a+2, a+3, \ldots, b\} \leftrightarrow \{a+1, a+2, \ldots, b-1, q\}$ if $0 \le a < a+2 \le b < q$.

Case 2.2: $\{1, a+2, a+3, \ldots, b-1, q\} \leftrightarrow \{a+1, a+2, \ldots, b\}$ if $0 < a < a+2 \le b \le q$.

Case 3: $[q] \leftrightarrow [q]$ if $a = 0$ and $b = q$.

4.2. Yang-Baxter moves and their properties. Let us now consider an index set

$$I := \{\overline{1} < \ldots < \overline{t} < 1 < \ldots < q < \overline{t+1} < \ldots < \overline{n}\}\,, \tag{4.2}$$

and let $\overline{I} := \{\overline{1}, \ldots, \overline{n}\}$. Let $\Gamma = \{\beta_i\}_{i \in I}$ be a λ-chain, denote $r_i := s_{\beta_i}$ as before, and let $\Gamma' = \{\beta'_i\}_{i \in I}$ be the sequence of roots defined by

$$\beta'_i = \begin{cases} \beta_{q+1-i} & \text{if } i \in I \setminus \overline{I} \\ \beta_i & \text{otherwise}\,. \end{cases} \tag{4.3}$$

In other words, the sequence Γ' is obtained from the λ-chain Γ by reversing a certain segment. Now assume that $\{\beta_1, \ldots, \beta_q\}$ are the positive roots of a rank two root system $\overline{\Phi}$ (without repetition). Let $\overline{W}$ be the corresponding dihedral reflection subgroup of the Weyl group W. The following result is easily proved using the correspondence between λ-chains and reduced words for the affine Weyl group element $v_{-\lambda}$ mentioned in Definition 3.1; most importantly, we need to recall from the proof of [**12**, Lemma 9.3] that the moves $\Gamma \to \Gamma'$ correspond to Coxeter moves (on the mentioned reduced words) in this context.

Proposition 4.2. (1) *The sequence Γ' is also a λ-chain, and the sequence $\beta_1, \ldots, \beta_q$ is a reflection ordering.*

(2) *We can obtain any λ-chain for a fixed dominant weight λ from any other λ-chain by moves of the form $\Gamma \to \Gamma'$.*

Let us now map the admissible subsets in $\mathcal{A}(\Gamma)$ to those in $\mathcal{A}(\Gamma')$. Given $J \in \mathcal{A}(\Gamma)$, let

$$\overline{J} := J \cap \overline{I}\,, \quad u := w(J \cap \{\overline{1}, \dots, \overline{t}\})\,, \quad \text{and} \quad w := w(J \cap (\{\overline{1}, \dots, \overline{t}\} \cup [q]))\,. \tag{4.4}$$

Also let

$$u = \lfloor u \rfloor \overline{u}\,, \quad w = \lfloor w \rfloor \overline{w}\,, \quad a := \ell(\overline{u})\,, \quad \text{and} \quad b := \ell(\overline{w})\,, \tag{4.5}$$

as above. It is clear that we have a bijection $Y : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma')$ given by

$$Y(J) := \overline{J} \cup Y_{q,a,b}(J \setminus \overline{J})\,. \tag{4.6}$$

We call the moves $J \mapsto Y(J)$ *Yang-Baxter moves* (cf. the discussion following Theorem 4.1). Clearly, a Yang-Baxter move preserves the Weyl group element $w(\,\cdot\,)$ associated to an admissible subset, that is, $w(Y(J)) = w(J)$. In addition, Theorem 4.1 below holds.

THEOREM 4.1. *The map Y preserves the weight of an admissible subset. In other words, $\mu(Y(J)) = \mu(J)$ for all admissible subsets J.*

We now explain the way in which the Yang-Baxter moves are related to the *Yang-Baxter equation*, which justifies the terminology. In order to do this, we need to recall some information from [**12**]. Consider the ring $K := \mathbb{Z}[\Lambda/h] \otimes \mathbb{Z}[W]$, where $\mathbb{Z}[W]$ is the group algebra of the Weyl group W, and $\mathbb{Z}[\Lambda/h]$ is the group algebra of $\Lambda/h := \{\lambda/h \mid \lambda \in \Lambda\}$ (i.e., of the weight lattice shrunk h times, h being the Coxeter number defined in Subsection 2.1). We define $\mathbb{Z}[\Lambda/h]$-linear operators B_α and X^λ on K, where α is a positive root and λ is a weight:

$$B_\alpha : w \longmapsto \begin{cases} ws_\alpha & \text{if } \ell(ws_\alpha) = \ell(w) + 1 \\ 0 & \text{otherwise,} \end{cases} \qquad X^\lambda : w \mapsto e^{w(\lambda/h)} w.$$

Let us now consider the operators $R_\alpha := X^\rho (X^\alpha + B_\alpha) X^{-\rho}$ for $\alpha \in \Phi^+$; if $\alpha \in \Phi^-$, we define R_α by setting $B_\alpha := -B_{-\alpha}$. It was proved in [**12**, Theorem 10.1] that the operators $\{R_\alpha \mid \alpha \in \Phi\}$ satisfy the Yang-Baxter equation in the sense of Cherednik [**4**]. The main application of the operators R_α was to show that, given a λ-chain $\Gamma = (\beta_1, \dots, \beta_n)$, we have

$$R_{\beta_n} \dots R_{\beta_1}(1) = \sum_{J \in \mathcal{A}(\Gamma)} e^{\mu(J)} w(J)\,. \tag{4.7}$$

Due to the Yang-Baxter property, the right-hand side of the above formula does not change when we replace the λ-chain Γ by Γ', as defined above. The Yang-Baxter moves described above implement the passage from Γ to Γ' at the level of the individual terms in (4.7).

We now present the main result related to Yang-Baxter moves.

THEOREM 4.2. *The root operators commute with the Yang-Baxter moves, that is, a root operator F_p is defined on an admissible subset J if and only if it is defined on $Y(J)$ and we have*

$$Y(F_p(J)) = F_p(Y(J))\,.$$

Theorem 4.2 asserts that the map Y above is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Given two arbitrary λ-chains Γ and Γ', we know from Proposition 4.2 (2) that they can be related by a sequence of λ-chains $\Gamma = \Gamma_0, \Gamma_1, \dots, \Gamma_m = \Gamma'$ to which correspond Yang-Baxter moves $Y_1, \dots, Y_m$. Hence the composition $Y_m \dots Y_1$ is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Since every directed graph $\mathcal{A}(\Gamma)$ has a unique source (cf. [**13**, Proposition 6.9]), its automorphism group as a directed colored graph consists only of the identity. Thus, we have the following corollary of Theorem 4.2.

COROLLARY 4.3. *Given two arbitrary λ-chains Γ and Γ', the directed colored graph structures on $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ are isomorphic. This isomorphism is unique and, therefore, is given by the composition of Yang-Baxter moves corresponding to* any *sequence of λ-chains relating Γ and Γ'.*

We have thus given a combinatorial explanation for the independence of the directed colored graph defined by our root operators from the chosen λ-chain.

Corollary 4.4. *Given* any *λ-chain Γ, the directed colored graph on the set $\mathcal{A}(\Gamma)$ defined by the root operators is isomorphic to the crystal graph of the irreducible representation V_λ with highest weight λ. Under this isomorphism, the weight of an admissible subset gives the weight space in which the corresponding element of the canonical basis lies.*

The above result follows, based on Corollary 4.3, from its special case corresponding to the particular choice of a λ-chain Γ that was described in [**13**] and was mentioned in Subsection 3.1. Based on Corollary 4.4, we will now identify the elements of the canonical basis with the corresponding admissible subsets.

5. Schützenberger's Involution

In this section, we present an explicit description of the involution η_λ in Subsection 2.3 in the spirit of Schützenberger's evacuation. We will show that the role of jeu de taquin in the definition of the evacuation map is played by the Yang-Baxter moves.

Throughout the remainder of this paper, we fix an index set $I := \{\overline{1} < \ldots < \overline{q} < 1 < \ldots < n\}$ and a λ-chain $\Gamma = \{\beta_i\}_{i\in I}$ such that $l_i^\emptyset = 0$ if and only if $i \in \overline{I} := \{\overline{1} < \ldots < \overline{q}\}$. In other words, the second occurence of a root can never be before the first occurence of another root. We will also write $\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, \beta_1, \ldots, \beta_n)$. Let us recall the notation $r_i := s_{\beta_i}$ for $i \in I$. It is easy to see that the set $J_{\min} := \overline{I}$ is the minimum of the poset $\mathcal{A}(\Gamma)$.

Given a Weyl group element w, we denote by $\lfloor w \rfloor$ and $\lceil w \rceil$ the minimal and the maximal representatives of the coset wW_λ, respectively (where W_λ is the stabilizer of the weight λ). Let $w_\circ^\lambda$ be the longest element of W_λ.

Definition 5.1. Let J be an admissible subset. Let $J \cap \overline{I} = \{\overline{j}_1 < \ldots < \overline{j}_a\}$ and $J \setminus \overline{I} = \{j_1 < \ldots < j_s\}$. The *initial key* $\kappa_0(J)$ and the *final key* $\kappa_1(J)$ of J are the Weyl group elements defined by

$$\kappa_0(J) := r_{\overline{j}_1} \ldots r_{\overline{j}_a}\,, \qquad \kappa_1(J) := w(J) = \kappa_0(J) r_{j_1} \ldots r_{j_s}\,.$$

Remark 5.2. The keys $\kappa_0(J)$ and $\kappa_1(J)$ are the analogs of the left and right keys of a semistandard Young tableau, as well as of the final and the initial directions of an LS path, respectively.

We associate with our fixed λ-chain Γ another sequence $\Gamma^{\mathrm{rev}} := \{\beta_i'\}_{i\in I}$ by

$$\beta_i' := \begin{cases} \beta_i & \text{if } i \in \overline{I} \\ w_\circ^\lambda(\beta_{n+1-i}) & \text{otherwise}\,. \end{cases}$$

In other words, we have

$$\Gamma^{\mathrm{rev}} = (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, w_\circ^\lambda(\beta_n), w_\circ^\lambda(\beta_{n-1}), \ldots, w_\circ^\lambda(\beta_1))\,. \tag{5.1}$$

Let $r_i' := s_{\beta_i'}$ for $i \in I$. Fix an admissible subset

$$J = \{\overline{j}_1 < \ldots < \overline{j}_a < j_1 < \ldots < j_s\} \tag{5.2}$$

in $\mathcal{A}(\Gamma)$, where $\{\overline{j}_1 < \ldots < \overline{j}_a\} \subseteq \overline{I}$ and $\{j_1 < \ldots < j_s\} \subseteq I \setminus \overline{I}$. Let $u := \kappa_0(J)$ and $w := \kappa_1(J)$. We have the increasing saturated chain

$$1 \lessdot r_{\overline{j}_1} \lessdot r_{\overline{j}_1} r_{\overline{j}_2} \lessdot \ldots \lessdot r_{\overline{j}_1} \ldots r_{\overline{j}_a} = u \lessdot u r_{j_1} \lessdot u r_{j_1} r_{j_2} \lessdot \ldots \lessdot u r_{j_1} \ldots r_{j_s} = w\,. \tag{5.3}$$

According to [**5**], there is a unique saturated increasing chain in Bruhat order of the form

$$1 \lessdot r_{\overline{k}_1}' \lessdot r_{\overline{k}_1}' r_{\overline{k}_2}' \lessdot \ldots \lessdot r_{\overline{k}_1}' \ldots r_{\overline{k}_b}' = \lfloor w_\circ w \rfloor = w_\circ w w_\circ^\lambda\,,$$

where $\{\overline{k}_1 < \overline{k}_2 < \ldots < \overline{k}_b\} \subseteq \overline{I}$. Define

$$J^{\mathrm{rev}} := \{\overline{k}_1 < \ldots < \overline{k}_b < k_1 < \ldots < k_s\}\,, \tag{5.4}$$

where $k_i := n + 1 - j_{s+1-i}$ for $i = 1, \ldots, s$. Note that $\beta_{k_i}' = w_\circ^\lambda(\beta_{j_{s+1-i}})$ for $i = 1, \ldots, s$.

Proposition 5.1. *Γ^{rev} is a λ-chain, and J^{rev} is an admissible subset in $\mathcal{A}(\Gamma^{\mathrm{rev}})$. We have*

$$\kappa_0(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_1(J) \rfloor\,, \quad \kappa_1(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_0(J) \rfloor\,, \quad \mu(J^{\mathrm{rev}}) = w_\circ(\mu(J))\,, \tag{5.5}$$

as well as $(J^{\mathrm{rev}})^{\mathrm{rev}} = J$.

We will now present the main result related to the map $J \mapsto J^{\mathrm{rev}}$, which involves its commutation with the root operators.

THEOREM 5.3. *A root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on J^{rev}, and we have*

$$(F_p(J))^{\mathrm{rev}} = E_{p^*}(J^{\mathrm{rev}})\,.$$

We can summarize the construction so far as follows: given the λ-chain Γ (for a fixed dominant weight λ), we defined the λ-chain Γ^{rev}, and given $J \in \mathcal{A}(\Gamma)$, we defined $J^{\mathrm{rev}} \in \mathcal{A}(\Gamma^{\mathrm{rev}})$. Hence we can map J^{rev} to an admissible subset $J^* \in \mathcal{A}(\Gamma)$ using Yang-Baxter moves, as it is described in Section 4 and it is recalled below. To be more precise, let $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\mathrm{rev}})$ denote the bijection $J \mapsto J^{\mathrm{rev}}$. On the other hand, we know from Proposition 4.2 (2) that the λ-chains Γ^{rev} and Γ can be related by a sequence of λ-chains $\Gamma^{\mathrm{rev}} = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ to which correspond Yang-Baxter moves $Y_1, \ldots, Y_m$. By Corollary 4.3, the composition $Y := Y_m \ldots Y_1$ does not depend on the sequence of intermediate λ-chains, and it defines a bijection from $\mathcal{A}(\Gamma^{\mathrm{rev}})$ to $\mathcal{A}(\Gamma)$. We let $J^* := YR(J)$ and conclude that it is a bijection on $\mathcal{A}(\Gamma)$. The main result of this section, namely Theorem 5.4 below, now follows directly from Theorems 4.2 and 5.3.

THEOREM 5.4. *The bijection $J \mapsto J^*$ constructed above coincides with the fundamental involution η_λ on the canonical basis. In other words, a root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on J^*, and we have*

$$(J_{\min})^* = J_{\max}\,, \quad (J_{\max})^* = J_{\min}\,, \quad \textit{and} \quad (F_p(J))^* = E_{p^*}(J^*)\,, \quad \textit{for } p = 1, \ldots, r\,. \tag{5.6}$$

In particular, the map $J \mapsto J^*$ expresses combinatorially the self-duality of the poset $\mathcal{A}(\Gamma)$.

REMARK 5.5. The above construction is analogous to the definition of Schützenberger's evacuation map (see, for instance, [**6**]). Below, we recall the three-step procedure defining this map and we discuss the analogy with our construction in the case of each step.

(1) REVERSE: We rotate a given semistandard Young tableau by 180°. This corresponds to reversing its word, which is similar to the procedure used to construct Γ^{rev} from Γ.
(2) COMPLEMENT: We complement each entry via the map $i \mapsto w_\circ(i)$, where $w_\circ$ is the longest element in the corresponding symmetric group. This corresponds to using $w_\circ$ for the arbitrary Weyl group in the definition (5.4) of J^{rev}.
(3) SLIDE: We apply jeu de taquin on the obtained skew tableau. This corresponds to the Yang-Baxter moves $Y_1, \ldots, Y_m$ discussed above.

EXAMPLE 5.6. Consider the Lie algebra $\mathfrak{sl}_3$ of type A_2, cf. Example 3.6. Consider the dominant weight $\lambda = 4\varepsilon_1 + 2\varepsilon_2$ and the following λ-chain:

$$\begin{array}{cccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma = (& \alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13} &). \end{array}$$

Here we indicated the index corresponding to each root, using the notation above; more precisely, we have $I = \{\overline{1} < \overline{2} < \overline{3} < 1 < 2 < 3 < 4 < 5\}$ and $\overline{I} = \{\overline{1} < \overline{2} < \overline{3}\}$. By the defining relation (5.1), we have

$$\begin{array}{cccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma^{\mathrm{rev}} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13} &). \end{array}$$

Consider the admissible subset $J = \{2, 4\}$. This is indicated above by the underlined roots in Γ. In order to define J^{rev}, cf. (5.4), we need to compute

$$\kappa_0(J^{\mathrm{rev}}) = w_\circ w(J) = (s_{12}s_{23}s_{12})(s_{12}s_{23}) = s_{12}\,.$$

Hence we have $J^{\mathrm{rev}} = \{\overline{1}, 2, 4\}$. This is indicated above by the underlined positions in Γ^{rev}.

In order to transform the λ-chain Γ^{rev} into Γ, we need to perform a single Yang-Baxter move; this consists of reversing the order of the bracketed roots below:

$$\begin{array}{ccccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & & \\ \Gamma^{\mathrm{rev}} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13} &) & \longrightarrow \\ & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & & \\ \Gamma \quad = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13} &). & \end{array}$$

The underlined roots indicate the way in which the Yang-Baxter move $J^{\text{rev}} \mapsto Y(J^{\text{rev}}) = J^*$ works. All we need to know is that there are two saturated chains in Bruhat order between the permutations u and w, cf. the notation in (4.4):

$$u = s_{12} \lessdot s_{12}s_{23} \lessdot s_{12}s_{23}s_{12} = w\,, \quad u = s_{12} \lessdot s_{12}s_{13} \lessdot s_{12}s_{13}s_{23} = w\,.$$

The first chain is retrieved as a subchain of Γ^{rev} and corresponds to J^{rev}, while the second one is retrieved as a subchain of Γ and corresponds to J^*. Hence we have $J^* = \{\overline{1}, 3, 4\}$.

We can give an intrinsic explanation for the fact that the map $J \mapsto J^*$ is an involution on $\mathcal{A}(\Gamma)$; this explanation is only based on the results in Sections 4 and 5, so it does not rely on Proposition 2.1 (2). Let us first recall the bijections $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\text{rev}})$ and $Y : \mathcal{A}(\Gamma^{\text{rev}}) \to \mathcal{A}(\Gamma)$ defined above. We claim that $YR = R^{-1}Y^{-1}$, which would prove that the composition YR is an involution. In the same way as we proved Theorem 5.4 (that is, as a direct consequence of Theorems 4.2 and 5.3), we can verify that the composition $R^{-1}Y^{-1}$ satisfies the conditions in (5.6). Since these conditions uniquely determine the corresponding map from $\mathcal{A}(\Gamma)$ to itself, our claim follows.

REMARK 5.7. According to the above discussion, we have a second way of realizing the fundamental involution η_λ on the canonical basis, namely as $R^{-1}Y^{-1}$. In some sense, this is the analog of the construction of the evacuation map based on the *promotion* operation (see, for instance, [**6**, p. 184]).

We have the following corollary of Proposition 5.1.

COROLLARY 5.8. *For any $J \in \mathcal{A}(\Gamma)$, we have*

$$\mu(J^*) = w_\circ(\mu(J))\,, \quad \kappa_0(J^*) = \lfloor w_\circ \kappa_1(J) \rfloor\,, \quad \kappa_1(J^*) = \lfloor w_\circ \kappa_0(J) \rfloor\,. \tag{5.7}$$

References

[1] A. Berenstein and A. Zelevinsky. Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics. *Duke Math. J.*, 82:473–502, 1996.

[2] A. Berenstein and A. Zelevinsky. Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.*, 143:77–128, 2001.

[3] F. Brenti, S. Fomin, and A. Postnikov. Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Internat. Math. Res. Notices*, 8:419–441, 1999.

[4] I. Cherednik. Quantum Knizhnik-Zamolodchikov equations and affine root systems. *Comm. Math. Phys.*, 150:109–136, 1992.

[5] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89(1):91–115, 1993.

[6] W. Fulton. *Young Tableaux*, volume 35 of *London Math. Soc. Student Texts*. Cambridge Univ. Press, Cambridge and New York, 1997.

[7] S. Gaussent and P. Littelmann. LS-galleries, the path model and MV-cycles. *Duke Math. J.*, 127:35–88, 2005.

[8] A. Henriques and J. Kamnitzer. Crystals and coboundary categories. `arXiv:math.QA/0406478`.

[9] J. E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[10] M. Kashiwara. Crystalizing the q-analogue of universal enveloping algebras. *Commun. Math. Phys.*, 133:249–260, 1990.

[11] M. A. A. van Leeuwen. An analogue of jeu de taquin for Littelmann's crystal paths. *Sém. Loth. Comb.*, 41, Art. B41b, 23 pp., 1998.

[12] C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. `arXiv:math.RT/0309207`.

[13] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. `arXiv:math.RT/0502147`.

[14] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116:329–346, 1994.

[15] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math. (2)*, 142:499–525, 1995.

[16] P. Littelmann. Characters of representations and paths in $\mathfrak{h}^*_{\mathbb{R}}$. In *Representation Theory and Automorphic Forms (Edinburgh, 1996)*, volume 61 of *Proc. Sympos. Pure Math.*, pages 29–49. Amer. Math. Soc., Providence, RI, 1997.

[17] M. Lothaire. The plactic monoid (by A. Lascoux, B. Leclerc, and J-Y. Thibon). In *Algebraic Combinatorics on Words*, pages 144–172. Cambridge University Press, Cambridge, 2002.

[18] G. Lusztig. *Introduction to Quantum Groups*, volume 110 of *Progress in Mathematics*. Birkhäuser, Boston, 1993.

[19] S. Morier-Genoud. Relèvement géométrique de la base canonique et involution de Schützenberger. (French) [Geometrical lifting of the canonical base and Schützenberger involution]. *C. R. Math. Acad. Sci. Paris*, 337:371–374, 2003.

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