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A Rook Theory Model for the Generalized p, q-Stirling Numbers of the First and Second Kind

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Abstract. In (EJC 11 (2004), #R84), Remmel and Wachs presented two natural ways to define p, q-analogues of the generalized Stirling numbers of the first and second kind, $S^1(\alpha, \beta, r)$ and $S^2(\alpha, \beta, r)$ as introduced by Hsu and Shiue (Adv. App. Math 20 (1998), 366-384). In this paper, we present a rook theoretic model for each type of p, q-analogue based on a pair of boards parametrized by the nonnegative integers α, β, and r, so that rooks attack cells on its own board as well as on its companion board. For each model, we provide an analogue of Goldman, Joichi and White's product formula (Proc. Amer. Math. Soc. 52 (1975), 485-492) and demonstrate how each type of the generalized p, q-Stirling numbers of the first and second kind arises as a special case of these p, q-rook numbers.

Résumé. Remmel et Wachs, dans (EJC 11 (2004), #R84), ont présenté deux façons naturelles pour définir les p, q-analogues des nombres de Stirling généralisés, des première et deuxième sortes, $S^1(\alpha, \beta, r)$ et $S^2(\alpha, \beta, r)$, introduits par Hsu et Shiue (Adv. App. Math 20 (1998), 366-384). Dans cet article, nous présentons un model théorique des mouvements de la tour pour chaque type des p, q-analogues basé sur une paire de jeux paramétrisés par les entiers non-négatifs α, β, et r. Ainsi, la tour attaque les cases sur son propre jeu et celles de l'autre jeu. Pour chacun des modèles, nous donnons une formule analogue à celle du produit de Goldman, Joichi et White (Proc. Amer. Math. Soc. 52 (1975), 485-492) et démontrons comment chaque type de p, q-analogues des nombres de Stirling généralisés des première et deuxième sortes forment un cas spécial de nombres p, q-analogues pour les mouvements de la tour.

1. Introduction

In [**11**], Remmel and Wachs presented two natural ways to give p, q-analogues of Hsu and Shiue's generalized Stirling numbers of the first and second kind [**7**], respectively denoted $\overline{S}^1_{n,k}(\alpha, \beta, r)$ and $\overline{S}^2_{n,k}(\alpha, \beta, r)$ for $0 \leq k \leq n$, and defined by

$$x(x-\alpha)\cdots(x-(n-1)\alpha) = \sum_{k=0}^{n} \overline{S}^1_{n,k}(\alpha, \beta, r)(x-r)(x-r-\beta)\cdots(x-r-(k-1)\beta), \tag{1.1}$$

and

$$x(x-\beta)\cdots(x-(n-1)\beta) = \sum_{k=0}^{n} \overline{S}^2_{n,k}(\alpha, \beta, r)(x+r)(x+r-\alpha)\cdots(x+r-(k-1)\alpha). \tag{1.2}$$

From these definitions, one can clearly see that $\overline{S}^1_{n,k}(\alpha, \beta, r) = \overline{S}^2_{n,k}(\beta, \alpha, -r)$. Moreover, we find that $\overline{S}^1_{n,k}(1, 0, 0) = s_{n,k}$ and $\overline{S}^2_{n,k}(1, 0, 0) = S_{n,k}$ where $s_{n,k}$ and $S_{n,k}$ respectively denote the classical Stirling numbers of the first and second kind.

By setting

$$S^1_{n,k}(\alpha, \beta, r) = \overline{S}^1_{n,k}(\alpha, \beta, -r) \text{ and } S^2_{n,k}(\alpha, \beta, r) = \overline{S}^2_{n,k}(\alpha, \beta, -r)$$

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and replacing x by $t-r$ in equation (1.1) and x by t in equation (1.2), Remmel and Wachs obtained the following pair of equations:

$$(t-r)(t-r-\alpha)\cdots(t-r-(n-1)\alpha)=\sum_{k=0}^{n}S^1_{n,k}(\alpha,\beta,r)t(t-\beta)\cdots(t-(k-1)\beta), \tag{1.3}$$

and

$$t(t-\beta)\cdots(t-(n-1)\beta)=\sum_{k=0}^{n}S^2_{n,k}(\alpha,\beta,r)(t-r)(t-r-\alpha)\cdots(t-r-(k-1)\alpha). \tag{1.4}$$

Replacing $(t-\gamma)$ by two distinctly natural p,q-analogues, Remmel and Wachs then defined their two types of p,q-analogues of $S^1_{n,k}(\alpha,\beta,r)$ and $S^2_{n,k}(\alpha,\beta,r)$. The p,q-analogue of any real number γ is defined by

$$[\gamma]_{p,q}=\frac{p^\gamma-q^\gamma}{p-q},$$

so that when $\gamma=n$ is a nonnegative integer, $[n]_{p,q}=q^{n-1}+pq^{n-2}+\cdots p^{n-2}q+p^{n-1}$. Then, the p,q-analogues of $n!$ and $\binom{n}{k}$ are naturally defined by $[n]_{p,q}!=[n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$ and

$$\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

For their type-I (p,q)-analogues of $S^1_{n,k}(\alpha,\beta,r)$ and $S^2_{n,k}(\alpha,\beta,r)$, Remmel and Wachs replaced $(t-\gamma)$ by $([t]_{p,q}-[\gamma]_{p,q})$ in (1.3) and (1.4). That is, they defined $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ for $0\le k\le n$ respectively by the following equations:

$$\begin{aligned}&([t]_{p,q}-[r]_{p,q})([t]_{p,q}-[r+\alpha]_{p,q})\cdots([t]_{p,q}-[r+(n-1)\alpha]_{p,q})\\ &\quad=\sum_{k=0}^{n}S^{1,p,q}_{n,k}(\alpha,\beta,r)([t]_{p,q})([t]_{p,q}-[\beta]_{p,q})\cdots([t]_{p,q}-[(k-1)\beta]_{p,q})\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}&([t]_{p,q}-[\beta]_{p,q})\cdots([t]_{p,q}-[(n-1)\beta]_{p,q})\\ &\quad=\sum_{k=0}^{n}S^{2,p,q}_{n,k}(\alpha,\beta,r)([t]_{p,q}-[r]_{p,q})([t]_{p,q}-[r+\alpha]_{p,q})\cdots([t]_{p,q}-[r+(k-1)\alpha]_{p,q}).\end{aligned} \tag{1.6}$$

Moreover, they proved that when $0\le k\le n$, the $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ defined according to equations (1.5) and (1.6) satisfy the following recursions:

$$S^{1,p,q}_{0,0}(\alpha,\beta,r)=1 \text{ and } S^{1,p,q}_{n,k}(\alpha,\beta,r)=0 \text{ if } k<0 \text{ or } k>n \tag{1.7}$$

and

$$S^{1,p,q}_{n+1,k}(\alpha,\beta,r)=S^{1,p,q}_{n,k-1}(\alpha,\beta,r)+([k\beta]_{p,q}-[n\alpha+r]_{p,q})S^{1,p,q}_{n,k}(\alpha,\beta,r), \tag{1.8}$$

$$S^{2,p,q}_{0,0}(\alpha,\beta,r)=1 \text{ and } S^{2,p,q}_{n,k}(\alpha,\beta,r)=0 \text{ if } k<0 \text{ or } k>n \tag{1.9}$$

and

$$S^{2,p,q}_{n+1,k}(\alpha,\beta,r)=S^{2,p,q}_{n,k-1}(\alpha,\beta,r)+([k\alpha+r]_{p,q}-[n\beta]_{p,q})S^{2,p,q}_{n,k}(\alpha,\beta,r). \tag{1.10}$$

For their type-II (p,q)-analogues of $S^1_{n,k}(\alpha,\beta,r)$ and $S^2_{n,k}(\alpha,\beta,r)$, Remmel and Wachs replaced $(t-\gamma)$ by $[t-\gamma]_{p,q}$ in (1.3) and (1.4). That is, they defined $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)$ for $0\le k\le n$ by the following equations:

$$[t-r]_{p,q}[t-r-\alpha]_{p,q}\cdots[t-r-(n-1)\alpha]_{p,q}=\sum_{k=0}^{n}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t]_{p,q}[t-\beta]_{p,q}\cdots[t-(k-1)\beta]_{p,q} \tag{1.11}$$

and

$$[t]_{p,q}[t-\beta]_{p,q}\cdots[t-(k-1)\beta]_{p,q}=\sum_{k=0}^{n}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)[t-r]_{p,q}[t-r-\alpha]_{p,q}\cdots[t-r-(k-1)\alpha]_{p,q}. \tag{1.12}$$

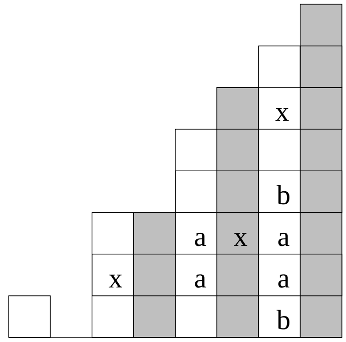

FIGURE 1. A placement in $(\mathcal{N}|\mathcal{F})_3^2(B_{\mathrm{BIP}}(1,0,3,3,5,6,7,8))$.

They further proved that when $0 \leq k \leq n$, the $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)$ defined according to equations (1.11) and (1.12) satisfy the following recursions:

$$\tilde{S}^{1,p,q}_{n+1,k}(\alpha,\beta,r) = q^{(k-1)\beta-n\alpha-r}\ \tilde{S}^{1,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r), \tag{1.13}$$

with initial conditions $\tilde{S}^{1,p,q}_{0,0}(\alpha,\beta,r) = 1$ and $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$, and

$$\tilde{S}^{2,p,q}_{n+1,k}(\alpha,\beta,r) = q^{r+(k-1)\alpha-n\beta}\ \tilde{S}^{2,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-r-k\alpha}[k\alpha+r-n\beta]_{p,q}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r), \tag{1.14}$$

with initial conditions $\tilde{S}^{2,p,q}_{0,0}(\alpha,\beta,r) = 1$ and $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$.

Remmel and Wachs gave rook theory interpretations to $c^{i,j}_{n,k}(p,q) = (-1)^{n-k}S^{1,p,q}_{n,k}(j,0,i)$ and $S^{i,j}_{n,k}(p,q) = S^{2,p,q}_{n,k}(j,0,i)$ as well as $\tilde{c}^{i,j}_{n,k}(p,q) = (-1)^{n-k}\tilde{S}^{1,p,q}_{n,k}(j,0,i)$ and $\tilde{S}^{i,j}_{n,k}(p,q) = \tilde{S}^{1,p,q}_{n,k}(j,0,i)$ where i,j are nonnegative integers. Moreover, they were able to give combinatorial proofs of certain product formulas involving these polynomials. In this paper, we provide a generalization of their results by giving combinatorial interpretations to $S^{i,p,q}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{i,p,q}_{n,k}(\alpha,\beta,r)$ when α, β and r are integers and $i \in \{1,2\}$, and we give combinatorial proofs to the product formulas that Remmel and Wachs did not provide.

2. A Rook Theoretic Model for $\mathbf{S^{1,p,q}_{n,k}}(\alpha,\beta,\mathbf{r})$ and $\mathbf{S^{2,p,q}_{n,k}}(\alpha,\beta,\mathbf{r})$

In this section, we give a rook theoretic model to interpret the type-I generalized p,q-Stirling numbers $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$. The boards in our model are constructed as follows. Given any two finite sequences of nonnegative integers $\{a_1,a_2,\ldots,a_n\}$ and $\{b_1,b_2,\ldots,b_n\}$, we construct the *bipartite board* $B_{\mathrm{BIP}}(a_1,b_1,a_2,b_2,\ldots,a_n,b_n)$ whose column heights from left to right are $a_1,b_1,a_2,b_2,\ldots,a_n,b_n$. We will call the collection of columns whose heights are $a_1,a_2,\ldots,a_n$, the *Premier-columns* (P-columns), and the collection of columns whose heights are $b_1,b_2,\ldots,b_n$ the *Secondary-columns* (S-columns). For example, from the sequences $\{1,3,5,7\}$ and $\{0,3,6,8\}$, we obtain the board $B_{\mathrm{BIP}}(1,0,3,3,5,6,7,8)$ which is illustrated in Figure 1 with the P-columns given in white and the S-columns shaded in gray.

For any bipartite board $B = B_{\mathrm{BIP}}(a_1,b_1,a_2,b_2,\ldots,a_n,b_n)$, a rook r placed in a P-column (resp. S-column) of B is said to *j-attack* the cells in the P-columns of B that are strictly to the right of r in the first j rows that are weakly above r (resp. in the first j rows beginning with row 1) that are not j-attacked by any other rook that lies in a column to the left of r. Then, a placement $\mathbb{P}$ of rooks in $B_{\mathrm{BIP}}(a_1,b_1,a_2,b_2,\ldots,a_n,b_n)$ is called *j-nonattacking* if no rook in $\mathbb{P}$ is j-attacked by any rook in $\mathbb{P}$ to its left and there is at most one j-attacking rook per column pair $\{a_i,b_i\}$ for each $1 \leq i \leq n$. We let $(\mathcal{N}|\mathcal{F})^j_k(B)$ denote the set of all placements of k j-nonattacking rooks in B.

A placement in $(\mathcal{N}|\mathcal{F})_3^2(B_{\mathrm{BIP}}(1,0,3,3,5,6,7,8))$ is illustrated in Figure 1. As usual, rooks are denoted in the figure by an "x". In this example, the leftmost rook in B is placed in row 2 of column a_1 and 2-attacks the cells in rows 2 and 3 of columns a_3 and a_4. These cells 2-attacked by the leftmost rook contain an "a" in Figure 1. The second rook from the left in row 3 of column b_3 2-attacks the cells in rows 1 and 4 of column a_4. These cells 2-attacked by this second rook contain a "b" in Figure 1. The final rook of the placement is in row 6 of column a_4. Since there are no P-columns to the right of a_4, this rook does not j-attack any cells in the board.

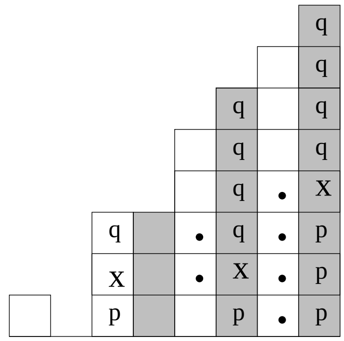

FIGURE 2. The p,q-weight of $\mathbb{P} \in (\mathcal{N}|\mathcal{F})_3^2(B_{\text{BIP}}(1,0,3,3,5,6,7,8))$ as contributed to $r_k^j(B,p,q)$.

For a nonnegative integer j, we say that a board $B = B_{\text{BIP}}(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ is a *j-attacking bipartite board* if $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$, and for all placements of rooks in B, there are a sufficient number of cells in the P-columns of B for each rook to j-attack. By this definition, note that in the case when $b_1 = \cdots = b_n = 0$, the board $B_{\text{BIP}}(a_1, 0, a_2, 0, \cdots, a_n, 0)$ is a j-attacking bipartite board provided that for all $1 \leq i < n$, $a_i \neq 0$ implies that $a_{i+1} \geq a_i + j - 1$. However, in the case when $b_i \neq 0$ for some $1 \leq i < n$, B is a j-attacking bipartite board provided that $a_{j+1} \geq a_j + j$ for all $j > i \geq 1$. In Figure 1, the board $B_{\text{BIP}}(1,0,3,3,5,6,7,8)$ is a 2-attacking bipartite board.

Suppose that $\mathbb{P} \in (\mathcal{N}|\mathcal{F})_k^j(B)$ and set

$$\begin{aligned} n_S(\mathbb{P}) &= \text{the number of rooks of } \mathbb{P} \text{ placed in an } S\text{-column}, \\ \mathcal{A}_B &= \text{the number of non-attacked cells in } B \text{ directly above some rook in } \mathbb{P}, \\ \mathcal{B}_B &= \text{the number of non-attacked cells in } B \text{ directly below some rook in } \mathbb{P}, \\ w_{p,q,B}^j(\mathbb{P}) &= (-1)^{n_S(\mathbb{P})} q^{\mathcal{A}_B} p^{\mathcal{B}_B}. \end{aligned}$$

The *type-I p,q-rook numbers*, denoted $r_k^j(B,p,q)$, are defined by

$$r_k^j(B,p,q) = \sum_{\mathbb{P} \in (\mathcal{N}|\mathcal{F})_k^j(B)} w_{p,q,B}^j(\mathbb{P}). \tag{2.1}$$

Here and in what follows, we will place a "$\bullet$" in the cells j-attacked by rooks in a given placement $\mathbb{P}$, a q in the cells that contribute a factor of q to $w_{p,q,B}^j(\mathbb{P})$, and a p in the cells that contribute a factor of p to $w_{p,q,B}^j(\mathbb{P})$. As illustrated in Figure 2 for $\mathbb{P} \in (\mathcal{N}|\mathcal{F})_3^2(B_{\text{BIP}}(1,0,3,3,5,6,7,8))$, $w_{p,q,B}^j(\mathbb{P}) = (-1)^2 q^9 p^5$.

Our first result is a p,q-analogue of Goldman, Joichi, and White's product formula [**5**].

THEOREM 2.1. *Let $B = B_{BIP}(s, b_1, s+j, b_2, \ldots, s+(n-1)j, b_n)$. Then*

$$\begin{aligned} \sum_{k=0}^{n} r_{n-k}^j(B,p,q)([x]_{p,q} - [s]_{p,q})([x]_{p,q} - [s+j]_{p,q}) \cdots ([x]_{p,q} - [s+(k-1)j]_{p,q}) \\ = \quad ([x]_{p,q} - [b_1]_{p,q})([x]_{p,q} - [b_2]_{p,q}) \cdots ([x]_{p,q} - [b_n]_{p,q}). \end{aligned} \tag{2.2}$$

PROOF. Given $B = B_{\text{BIP}}(s, b_1, s+j, b_2, \ldots, s+(n-1)j, b_n)$, we let $B_{(x,j)}$ be the board obtained from B by adjoining a single column of height $x + s + (i-1)j$ beneath the column pair $\{a_i, b_i\}$ for each $1 \leq i \leq n$. Here we call the line separating B from the adjoined rows the *bar*, the first x rows below the bar in $B_{(x,j)}$ the *x-adjoined rows* and the last $s + (n-1)j$ rows in $B_{(x,j)}$ below the bar the *j-adjoined rows*. Further, we will call the collection of cells in the column pair $\{a_i, b_i\}$ together with the $x + s + (i-1)j$ adjoined cells below it the *ith joined column*. The augmented board $B_{(x,j)}$ is illustrated in Figure 3.

For a given board B, placements of j-attacking rooks placed above the bar in $B_{(x,j)}$ will j-attack the same cells above the bar as described above. Additionally, any j-attacking rook r placed above the bar will attack all of the cells below it in its joined column as well as the first j rows in the j-adjoined rows strictly to the right of r not attacked by any rook to the left. A rook that is placed in one of the x-adjoined rows will attack all of the cells directly above it in the board B as well as the cells directly below it in the j-adjoined rows. A rook that is placed in a j-adjoined row will attack the cells directly above it in the x-adjoined

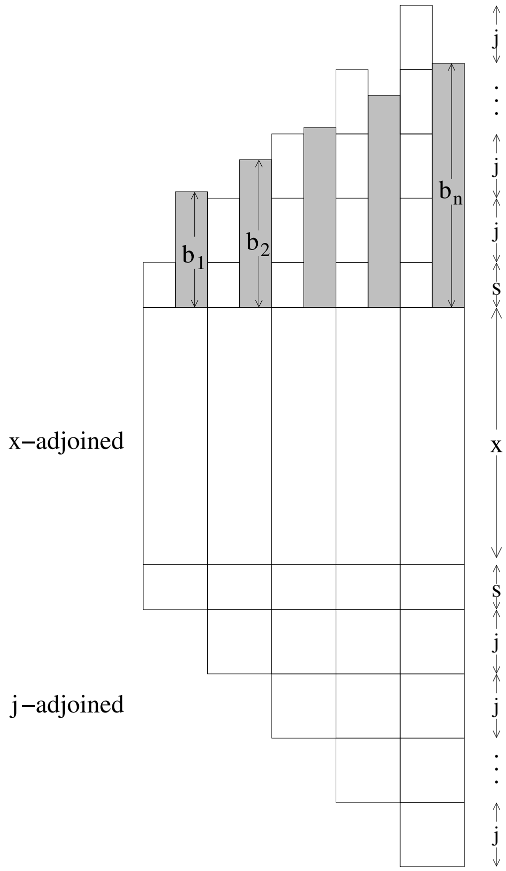

FIGURE 3. The board $B_{(x,j)}(s, b_1, s+j, b_2, \ldots, s+(n-1)j, b_n)$

rows and in the board B. The cells attacked by rooks of a placement in $B_{(x,j)}(1,0,3,3,5,6,7,8)$ have been illustrated in Figure 4.

Let $(\mathcal{N}|\mathcal{F})^j_n(B_{(x,j)})$ be the set of all placements of n rooks in $B_{(x,j)}$ such that no two rooks lie in the same joined column, no rook in $B_{(x,j)}$ is j-attacked by any rook in $B_{(x,j)}$ to its left. Thus, the placement in Figure 4 is in $(\mathcal{N}|\mathcal{F})^2_4(B_{\mathrm{BIP}}(1,0,3,3,5,6,7,8)_{(x,2)})$.

Then for positive integers x, the identity in (2.2) arises from two ways of counting

$$N = \sum_{\mathbb{P} \in (\mathcal{N}|\mathcal{F})^j_n(B_{(x,j)})} w^j_{p,q,B_{(x,j)}}(\mathbb{P}), \tag{2.3}$$

where $w^j_{p,q,B_{(x,j)}}(\mathbb{P})$ is defined as

$$w^j_{p,q,B_{(x,j)}}(\mathbb{P}) = (-1)^{n_S+n_j} q^{\mathcal{A}_{B_{(x,j)}}} p^{\mathcal{B}_{B_{(x,j)}}},$$

with

$$\begin{aligned} n_S(\mathbb{P}) &= \text{the number of rooks of } \mathbb{P} \text{ placed in an } S\text{-column},\\ n_j(\mathbb{P}) &= \text{the number of rooks of } \mathbb{P} \text{ placed in a } j\text{-adjoined row},\\ \mathcal{A}_{B_{(x,j)}} &= \text{the number of non-attacked cells in } B_{(x,j)} \text{ directly above some rook in } \mathbb{P},\\ \mathcal{B}_{B_{(x,j)}} &= \text{the number of non-attacked cells in } B_{(x,j)} \text{ directly below some rook in } \mathbb{P}. \end{aligned}$$

First we note that each placement $\mathbb{P} \in (\mathcal{N}|\mathcal{F})^j_n(B_{(x,j)})$ can be obtained by placing exactly one rook in each of the joined columns of $B_{(x,j)}$ proceeding from left to right. In the first joined column, the rook can be placed in either the first P-column, the first S-column, below the bar in the x-adjoined rows, or in the j-adjoined rows. It is easy to see that the contribution of the first joined column to N by placing the rook in the ith row from the top in the first P-column is $q^{i-1}p^{s-i}$ for a total contribution of $[s]_{p,q}$ to N. Likewise, the contribution of the first joined column to N by placing the rook in the ith row from the top in the first

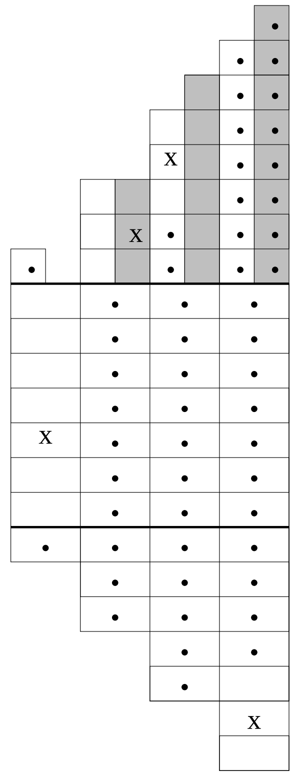

Figure 4. A placement in the board $B_{(x,2)}(B(1,0,3,3,5,6,7,8))$.

S-column is $-q^{i-1}p^{b_1-i}$ for a total contribution of $-[b_1]_{p,q}$ to N. Using the same analysis, we find that when the rook is placed below the bar in the x-adjoined rows, the contribution of the first joined column to N is $[x]_{p,q}$ while the total contribution is $-[s]_{p,q}$ from the placements in the j-adjoined rows. Therefore, the total contribution of the first joined column to N is $[x]_{p,q} - [b_1]_{p,q}$.

We now argue that regardless of the placement of the rook in the first joined column, the contributions from the P-column and the j-adjoined of the second adjoined column will cancel. To see this, first consider the case when a rook in the first joined column had been placed in B. Such a rook would attack exactly j cells in the P-columns as well as j cells in each row of the j-adjoined columns weakly to the right of the rook. In this case, the contribution from the second P-column is $[s]_{p,q}$ while the contribution from the second j-adjoined column is $-[s]_{p,q}$. On the other hand, if the rook in the first joined column had been placed below the bar, then the contribution from the second P-column is $[s+j]_{p,q}$ while the contribution from the second j-adjoined column is $-[s+j]_{p,q}$. To this end, we can argue as above, that the contribution of the second adjoined column to N is $[x]_{p,q} - [b_2]_{p,q}$.

Continuing in this way, we find that the total contribution of all n adjoined columns to N is

$$([x]_{p,q} - [b_1]_{p,q})([x]_{p,q} - [b_2]_{p,q}) \cdots ([x]_{p,q} - [b_n]_{p,q}).$$

Now suppose that a placement $\mathbb{Q}$ of $n-k$ rooks is fixed in B. Then a placement $\mathbb{P} \in (\mathcal{N}|\mathcal{F})_n^j(B_{(x,j)})$ can be obtained from $\mathbb{Q}$ by placing the remaining k rooks below the bar. As prescribed, each of the $n-k$ rooks in B will attack j cells in the j-adjoined rows in the columns weakly to the right of each rook. As such, there will be x places in the x-adjoined rows and $s+(i-1)j$ places in the j-adjoined rows in which to place the ith rook below the bar from the left, for $1 \leq i \leq k$. Therefore, the placement of the k rooks below the bar will contribute a factor of $([x]_{p,q} - [s]_{p,q})([x]_{p,q} - [s+j]_{p,q}) \cdots ([x]_{p,q} - [s+(k-1)j]_{p,q})$ to N. Furthermore, each rook placed below the bar will attack the cells above it in B implying that $w^j_{p,q,B}(\mathbb{Q}) = w^j_{p,q,B}(\mathbb{P} \cap B)$.

Thus,

$$\begin{aligned} N &= \sum_{k=0}^{n} \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^j_{n-k}(B)} \sum_{\substack{\mathbb{P}\in(\mathcal{N}|\mathcal{F})^j_n(B_{(x,j)})\\ \mathbb{P}\cap B=\mathbb{Q}}} w^j_{p,q,B_{(x,j)}}(\mathbb{P}) \\ &= \sum_{k=0}^{n} \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^j_{n-k}(B)} w^j_{p,q,B}(\mathbb{P}\cap B)([x]_{p,q}-[s]_{p,q})([x]_{p,q}-[s+j]_{p,q})\cdots([x]_{p,q}-[s+(k-1)j]_{p,q}) \\ &= \sum_{k=0}^{n} r^j_{n-k}(B,p,q)([x]_{p,q}-[s]_{p,q})([x]_{p,q}-[s+j]_{p,q})\cdots([x]_{p,q}-[s+(k-1)j]_{p,q}). \end{aligned}$$

□

We are now in a position to give combinatorial interpretations to $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ defined by (1.5) and (1.6). To begin, let x, y, μ and ν be nonnegative integers and let $B^{x,y}_{\mu,\nu,n}$ denote the bipartite board $B_{\mathrm{BIP}}(x,y,x+\mu,y+\nu,x+2\mu,y+2\nu,\ldots,x+(n-1)\mu,y+(n-1)\nu)$. Then,

THEOREM 2.2. *If n and k are nonnegative integers for which $0<k<n$, then*

$$S^{1,p,q}_{n,k}(\alpha,\beta,r) = r^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n},p,q) \quad \text{and} \tag{2.4}$$

$$S^{2,p,q}_{n,k}(\alpha,\beta,r) = r^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n},p,q). \tag{2.5}$$

PROOF. We begin by noting that the identities in (2.4) and (2.5) can be proved by showing that the p,q-rook numbers $r^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n},p,q)$ satisfy the same recursion as $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ given in (1.7) and (1.8) and that $r^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n},p,q)$ satisfy the same recursion as $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ given in (1.9) and (1.10).

For $n=0$, $B^{x,y}_{\mu,\nu,0}=\emptyset$. So, it immediately follows from our definition that

$$r^{\beta}_0(B^{0,r}_{\beta,\alpha,0},p,q)=1 \quad \text{and} \quad r^{\alpha}_0(B^{r,0}_{\alpha,\beta,0},p,q)=1.$$

Clearly, $r^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n},p,q)=0$ and $r^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n},p,q)=0$ if $k>n$ or $k<0$ since both $(\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})$ and $(\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})$ are empty if $k>n$ or $k<0$. Therefore, to verify the equalities in (2.4) and (2.5), it remains to show that for all $n\geq 1$ and $0\leq k\leq n$,

$$r^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1},p,q) = r^{\beta}_{n-(k-1)}(B^{0,r}_{\beta,\alpha,n},p,q) + ([k\beta]_{p,q}-[n\alpha+r]_{p,q})\, r^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n},p,q) \tag{2.6}$$

and

$$r^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1},p,q) = r^{\alpha}_{n-(k-1)}(B^{r,0}_{\alpha,\beta,n},p,q) + ([k\alpha+r]_{p,q}-[n\beta]_{p,q})\, r^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n},p,q). \tag{2.7}$$

To prove (2.6), we note that the set of elements in $(\mathcal{N}|\mathcal{F})^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1})$ can be partitioned into the sets No, $P-Last$, and $S-Last$ where No consists of the placements of $(\mathcal{N}|\mathcal{F})^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1})$ with no rook in the column pair $\{a_{n+1},b_{n+1}\}$, $P-Last$ consists of the placements of $(\mathcal{N}|\mathcal{F})^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1})$ with a rook in the P-column a_{n+1}, and $S-Last$ consists of the placements of $(\mathcal{N}|\mathcal{F})^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1})$ with a rook in the S-column b_{n+1}. Then,

$$\begin{aligned} r^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1},p,q) &= \sum_{\mathbb{P}\in(\mathcal{N}|\mathcal{F})^{\beta}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1})} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) \\ &= \sum_{\mathbb{P}\in No} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) + \sum_{\mathbb{P}\in P-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) + \sum_{\mathbb{P}\in S-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}). \end{aligned}$$

It is easy to see that a placement in $\mathbb{P}\in No$ has $n-(k-1)$ rooks to the left of the column pair $\{a_{n+1},b_{n+1}\}$. Thus, $w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) = w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q})$ where $\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\beta}_{n-(k-1)}(B^{0,r}_{\beta,\alpha,n})$ is the placement that would result in eliminating the last pair of columns $\{a_{n+1},b_{n+1}\}$ from $B^{0,r}_{\beta,\alpha,n+1}$. Therefore,

$$\sum_{\mathbb{P}\in No} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) = \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\beta}_{n-(k-1)}(B^{0,r}_{\beta,\alpha,n})} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q}) = r^{\beta}_{n-(k-1)}(B^{0,r}_{\beta,\alpha,n},p,q).$$

To compute $\sum_{\mathbb{P}\in P-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P})$, we first observe that each $\mathbb{Q} \in (\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})$ can be extended to $k\beta$ placements in $P - Last$ by placing an additional rook in a non-attacked cell of column a_{n+1}. This follows since each of the $n-k$ rooks of a fixed $\mathbb{Q} \in (\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})$ attacks β cells in column a_{n+1} leaving $n\beta - (n-k)\beta = k\beta$ non-attacked cells in column a_{n+1} in which to place the additional rook. Next, we note that $n_S(\mathbb{P}) = n_S(\mathbb{Q})$. So, if the additional rook is placed in the ith non-attacked cell from the top, then the weight of the corresponding placement $\mathbb{P}^i$ is $q^{i-1}p^{k\beta-i}w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q})$. Therefore,

$$\begin{aligned}\sum_{\mathbb{P}\in P-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) &= \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})} (p^{k\beta-1} + qp^{k\beta-2} + \cdots + q^{k\beta-2}p + q^{k\beta-1})\ w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q}) \\ &= [k\beta]_{p,q} r^{\beta}_{n,k}(B^{0,r}_{\beta,\alpha,n}, p, q).\end{aligned}$$

Finally, we observe that each rook of $\mathbb{P} \in S - Last$ attacks β cells of column a_{n+1} but no cells of column b_{n+1}. Accordingly, $\sum_{\mathbb{P}\in S-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P})$ could be computed by extending each $\mathbb{Q} \in (\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})$ to $n\alpha + r$ distinct placements in $(\mathcal{N}|\mathcal{F})^{\beta}_{n-k+1}(B^{0,r}_{\beta,\alpha,n+1})$ by placing an additional rook in any of the $n\alpha + r$ non-attacked cells of column b_{n+1}. For such a placement $\mathbb{P}^i$ obtained by placing the additional rook in the ith non-attacked cell from the top, we note that $n_S(\mathbb{P}^i) = 1 + n_S(\mathbb{Q})$ and consequently $w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}^i) = -q^{i-1}p^{n\alpha+r+i}w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q})$. Therefore, it follows that

$$\begin{aligned}&\sum_{\mathbb{P}\in S-Last} w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}) \\ &= \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})} -(p^{n\alpha+r-1} + qp^{n\alpha+r-2} + \cdots + q^{n\alpha+r-2}p + q^{n\alpha+r-1})\ w^{\beta}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q}) \\ &= -[n\alpha + r]_{p,q}\, r^{\beta}_{n,k}(B^{0,r}_{\beta,\alpha,n}, p, q).\end{aligned}$$

In the same way, we prove (2.7) by partitioning $(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1})$ into the sets No, $P - Last$, and $S - Last$ where No consists of the placements of $(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1})$ with no rook in the column pair $\{a_{n+1}, b_{n+1}\}$, $P - Last$ consists of the placements of $(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1})$ with a rook in the P-column a_{n+1}, and $S - Last$ consists of the placements of $(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1})$ with a rook in the S-column b_{n+1}. The recursion in (2.7) will follow by showing that

$$\begin{aligned}r^{\alpha}_{n+1-k}(B^{0,r}_{\beta,\alpha,n+1}, p, q) &= \sum_{\mathbb{P}\in(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1-k}(B^{r,0}_{\alpha,\beta,n+1})} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) \\ &= \sum_{\mathbb{P}\in No} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) + \sum_{\mathbb{P}\in P-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) + \sum_{\mathbb{P}\in S-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}).\end{aligned}$$

Again, it is easy to see that

$$\sum_{\mathbb{P}\in No} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) = r^{\alpha}_{n-(k-1)}(B^{r,0}_{\alpha,\beta,n}, p, q).$$

To compute $\sum_{\mathbb{P}\in P-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P})$, we observe that each fixed placement $\mathbb{Q} \in (\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})$ can be extended to $k\alpha + r$ distinct placements in $(\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})$ by placing an additional rook in one of the $n\alpha + r - \alpha(n-k) = k\alpha + r$ non-attacked cells of column a_{n+1}. If the additional rook is placed in the ith non-attacked cells from the top of column a_{n+1}, then the weight of the corresponding placement $\mathbb{P}^i$ is $q^{i-1}p^{k\alpha+r-i}w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n}}(\mathbb{Q})$. It follows that

$$\begin{aligned}\sum_{\mathbb{P}\in P-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) &= \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})} (p^{k\alpha+r-1} + qp^{k\alpha+r-2} + \cdots + q^{k\alpha+r-2}p + q^{k\alpha+r-1})\ w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n}}(\mathbb{Q}) \\ &= [k\alpha + r]_{p,q} r^{\alpha}_{n,k}(B^{r,0}_{\alpha,\beta,n}, p, q).\end{aligned}$$

As above, we observe that each rook of $\mathbb{P} \in S-Last$ attacks α cells of column a_{n+1} but no cells of column b_{n+1}. Therefore, $\sum_{\mathbb{P}\in S-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P})$ could be computed by extending each $\mathbb{Q} \in (\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})$ to $n\beta$ distinct placements in $(\mathcal{N}|\mathcal{F})^{\alpha}_{n-k+1}(B^{r,0}_{\alpha,\beta,n+1})$ by placing an additional rook in any of the $n\beta$ non-attacked cells of column b_{n+1}. For such a placement $\mathbb{P}^i$ obtained by placing the additional rook in the ith non-attacked cell from the top, we note that $n_S(\mathbb{P}^i) = 1 + n_S(\mathbb{Q})$ and thus $w^{\alpha}_{p,q,B^{0,r}_{\beta,\alpha,n+1}}(\mathbb{P}^s) = -q^{i-1}p^{n\beta+i}w^{\alpha}_{p,q,B^{0,r}_{\beta,\alpha,n}}(\mathbb{Q})$. To this end,

$$\begin{aligned}\sum_{\mathbb{P}\in S-Last} w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n+1}}(\mathbb{P}) &= \sum_{\mathbb{Q}\in(\mathcal{N}|\mathcal{F})^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})} -(p^{n\beta-1} + qp^{n\beta-2} + \cdots + q^{n\beta-2}p + q^{n\beta-1})\; w^{\alpha}_{p,q,B^{r,0}_{\alpha,\beta,n}}(\mathbb{Q}) \\ &= -[n\beta]_{p,q} r^{\alpha}_{n,k}(B^{r,0}_{\alpha,\beta,n}, p, q).\end{aligned}$$

□

We end this section by noting that as a consequence of Theorems 2.1 and 2.2, our single model described above yields a combinatorial interpretation to both (1.5) and (1.6). In particular, (1.5) is obtained from (2.2) by setting $s = 0$, $j = \beta$, and $b_i = r + (i-1)\alpha$. Likewise, setting $s = r$, $j = \alpha$, and $b_i = (i-1)\beta$ in (2.2) produces (1.5).

3. A Rook Theoretic Model for $\tilde{\mathbf{S}}^{\mathbf{1},\mathbf{p},\mathbf{q}}_{\mathbf{n},\mathbf{k}}(\alpha,\beta,\mathbf{r})$ and $\tilde{\mathbf{S}}^{\mathbf{2},\mathbf{p},\mathbf{q}}_{\mathbf{n},\mathbf{k}}(\alpha,\beta,\mathbf{r})$

To define the second type of p,q-rook numbers, let B be a j-attacking bipartite board and suppose $\mathbb{P} \in (\mathcal{N}|\mathcal{F})^{j}_{k}(B)$. Assume additionally that the k rooks are in the column pairs $\{a_i, b_i\}$ with labels $1 \le c_1 < \cdots < c_k \le n$ and that there are j_i non-attacked cells in the column containing the rook among the pair $\{a_{c_i}, b_{c_i}\}$ for $1 \le i \le k$. Setting

$$\begin{aligned} a_B &= \text{the number of non-attacked cells in } B \text{ directly above some rook in } \mathbb{P},\\ b_B &= \text{the number of non-attacked cells in } B \text{ directly below some rook in } \mathbb{P},\\ \varepsilon_B &= \text{the number of non-attacked cells in a } P\text{-column of an } \{a_i,b_i\} \text{ pair containing no rook,}\end{aligned}$$

we define the *type-II p,q-rook numbers*, denoted $\tilde{r}^{j}_{k}(B,p,q)$, by

$$\tilde{r}^{j}_{k}(B,p,q) = q^{-(b_1+\cdots+b_n)} \sum_{\mathbb{P}\in(\mathcal{N}|\mathcal{F})^{j}_{k}(B)} (-1)^{n_S} q^{\varepsilon_B(\mathbb{P})+a_B(\mathbb{P})} p^{b_B(\mathbb{P})+kt-(j_1+j_2+\cdots+j_k)}. \tag{3.1}$$

The following result gives the generalized product formula for the type-II p,q-rook numbers.

THEOREM 3.1. *Let $B = B_{BIP}(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ be a j-attacking bipartite board. Then for each nonnegative integer n,*

$$\begin{aligned}&\sum_{k=0}^{n} \tilde{r}^{j}_{n-k}(B,p,q)[t]_{p,q}[t-j]_{p,q}\cdots[t-(k-1)j]_{p,q} \\ &\qquad = \prod_{i=1}^{n} q^{-b_i}\left([t+a_i-(i-1)j]_{p,q} - p^{t+a_i-(i-1)j-b_i}[b_i]_{p,q}\right).\end{aligned} \tag{3.2}$$

The proof of Theorem 3.1 is similar to that of Theorem 2.1. The idea is to consider all placements of n j-attacking rooks in the board $B^{j}_{t}(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ which is obtained from B by adjoining t rows below the n P-columns, labeled from bottom to top by $1, 2, \ldots, t$. The board $B^{j}_{t}(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ is illustrated in Figure 5. Here, rooks placed in B will j-attack in the P-columns as usual, while a rook that is placed in row i below a P-column will j-attack the cells to the right in the first j rows weakly above it in the list of rows $i, i+1, \ldots, t, 1, \ldots, i-1$ that have not been j-attacked by a rook from the left. Such a placement of n rooks is illustrated in Figure 5 with $j = 2$.

It can also be shown that the type-II p,q-rook numbers on specific j-attacking bipartite boards satisfy the same recursions as the type-II Stirling numbers of the first and second kind. To see this, we first note that

$$[k\beta - n\alpha - r]_{p,q} = q^{-n\alpha-r}\left([k\beta]_{p,q} - p^{k\beta-n\alpha-r}[n\alpha+r]\right) \tag{3.3}$$

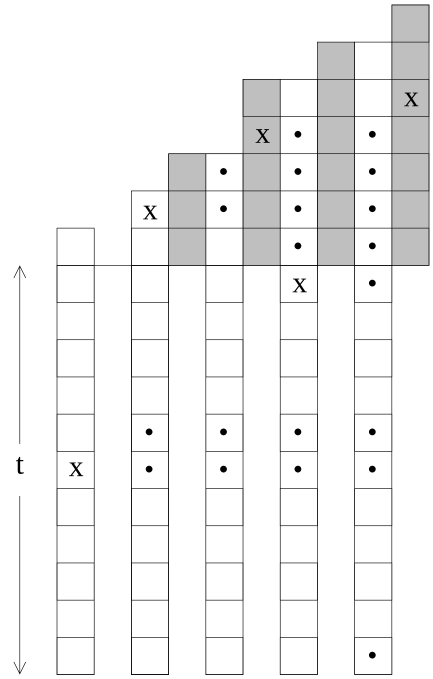

Figure 5. The board $B_t^j(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$

and

$$[k\alpha + r - n\beta]_{p,q} = q^{-n\beta}\left([k\alpha + r]_{p,q} - p^{k\alpha+r-n\beta}[n\beta]\right). \tag{3.4}$$

Then, substituting the identity (3.3) into the recursion (1.13) and (3.4) into (1.14) yields the following three term recursions:

$$\begin{aligned} \tilde{S}^{1,p,q}_{n+1,k}(\alpha,\beta,r) &= q^{(k-1)\beta-n\alpha-r}\ \tilde{S}^{1,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-k\beta}q^{-n\alpha-r}[k\beta]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r) \\ & -p^{t-n\alpha-r}q^{-n\alpha-r}[n\alpha+r]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r). \end{aligned} \tag{3.5}$$

$$\begin{aligned} \tilde{S}^{2,p,q}_{n+1,k}(\alpha,\beta,r) &= q^{r+(k-1)\alpha-n\beta}\ \tilde{S}^{2,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-r-k\alpha}q^{-n\beta}[k\alpha+r]_{p,q}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r) \\ & -p^{t-n\beta}q^{-n\beta}[n\beta]_{p,q}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r). \end{aligned} \tag{3.6}$$

As in the proof of Theorem 2.2, we can show that the rook numbers $\tilde{r}^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n})$ and $\tilde{r}^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n})$ satisfy the respective recursions in (3.5) and (3.5) by again partitioning the set of placements $(\mathcal{N}|\mathcal{F})^{\beta}_{n+1}(B^{0,r}_{\beta,\alpha,n+1})$ and $(\mathcal{N}|\mathcal{F})^{\alpha}_{n+1}(B^{r,0}_{\alpha,\beta,n+1})$ into *No*, $P - $*Last*, and $S - $*Last*. We summarize these results in the following:

Theorem 3.2. *If n and k are nonnegative integers for which $0 < k < n$, then*

$$\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r) = \tilde{r}^{\beta}_{n-k}(B^{0,r}_{\beta,\alpha,n}) \quad \text{and} \tag{3.7}$$

$$\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r) = \tilde{r}^{\alpha}_{n-k}(B^{r,0}_{\alpha,\beta,n}). \tag{3.8}$$

As a consequence of Theorems 3.1 and 3.2, this single rook theoretic model yields a combinatorial interpretation for the identities given in (1.11) and (1.12). To see this, we observe that

$$q^{-b_i}\left([t + a_i - (i-1)j]_{p,q} - p^{t+a_i-(i-1)j-b_i}[b_i]_{p,q}\right) = [t + a_i - (i-1)j - b_i]_{p,q}.$$

Then from (3.2), (1.11) is obtained by setting $j = \beta$, $a_i = (i-1)\beta$, and $b_i = r + (i-1)\alpha$ as is (1.12) by setting $j = \alpha$, $a_i = r + (i-1)\alpha$, and $b_i = (i-1)\beta$, and replacing t with $t - r$.

4. Directions

While our models have provided a rook theoretic interpretation for both types of p, q-analogues of the generalized Stirling numbers of the first and second kind, their product formulas and recursions, we have yet to produce analogues of the orthogonality relations given by Hsu and Shiue [**7**] for arbitrary parameters α, β, and r:

$$\sum_{k=i}^{n} \overline{S}^{1}_{n,k}(\alpha,\beta,r)\overline{S}^{2}_{k,i}(\alpha,\beta,r) = \sum_{k=i}^{n} \overline{S}^{2}_{n,k}(\alpha,\beta,r)\overline{S}^{1}_{k,i}(\alpha,\beta,r) = \chi(i=n).$$

Although, Remmel and Wachs gave direct combinatorial interpretations of the following p, q-analogues of the orthogonality relations

$$\sum_{k=r}^{n} S^{2,p,q}_{n,k}(j,0,i) S^{1,p,q}_{k,r}(j,0,i) = \chi(r=n)$$

and

$$\sum_{k=r}^{n} p^{\binom{n-k+1}{2}} \tilde{S}^{2,p,q}_{n,k}(j,0,i)(pq)^{\binom{k}{2}j} p^{-ir} q^{-ik} \tilde{S}^{1,p,q}_{k,r}(j,0,i) = \chi(r=n),$$

they did not provide the p, q-orthogonality relations for arbitrary parameters α, β, and r. We will pursue this problem in a subsequent paper.

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