Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Shellable complexes and topology of diagonal arrangements

Sangwook Kim

Abstract. We prove that if a simplicial complex Δ is shellable, then the intersection lattice L_Δ for the corresponding diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres. Furthermore, we describe precisely the spheres in the wedge, based on the data of shelling.

Résumé. Nous prouvons que si un complexe simplicial Δ est shellable, alors le treillis d'intersection L_Δ pour le correspondre l'arrangement diagonal $\mathcal{A}_\Delta$ est l'équivalent de homotopy à un bouquet de sphères. De plus, nous décrivons précisément les sphères dans le bouquet, basé sur les données d'écaler.

1. Introduction

Consider $\mathbb{R}^n$ with coordinates $u_1, \ldots, u_n$. A *diagonal subspace* $U_{i_1 \cdots i_r}$ is a linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$ with $r \geq 2$. A *diagonal arrangement* (or a *hypergraph arrangement*) $\mathcal{A}$ is a finite set of diagonal subspaces of $\mathbb{R}^n$.

For a simplicial complex Δ on $[n] = \{1, \ldots, n\}$ such that $\dim \Delta \leq n-3$, one can associate a diagonal arrangement $\mathcal{A}_\Delta$ as follows. For a facet F of Δ, let $U_{\overline{F}}$ be the diagonal subspace $u_{i_1} = \cdots = u_{i_r}$ where $\overline{F} = [n] - F = \{i_1, \ldots, i_r\}$. Define

$$\mathcal{A}_\Delta = \{U_{\overline{F}} | F \text{ is a facet of } \Delta\}.$$

For each diagonal arrangement $\mathcal{A}$, one can find a simplicial complex Δ such that $\mathcal{A} = \mathcal{A}_\Delta$.

Two important spaces associated with an arrangement $\mathcal{A}$ of linear subspaces in $\mathbb{R}^n$ are

$$\mathcal{M}_\mathcal{A} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{V}^\circ_\mathcal{A} = \mathbb{S}^{n-1} \cap \bigcup_{H \in \mathcal{A}} H,$$

called the *complement* and the *singularity link* of $\mathcal{A}$.

We are interested in the topology of $\mathcal{M}_\mathcal{A}$ and $\mathcal{V}^\circ_\mathcal{A}$ for a diagonal arrangement $\mathcal{A}$. We mention here some applications. In computer science, Björner, Lovász and Yao [**3**] find lower bounds on complexity of k-equal problems using the topology of diagonal arrangements (see also [**2**]). In group cohomology, it is well-known that $\mathcal{M}_{\mathcal{B}_n}$ for the braid arrangement $\mathcal{B}_n$ in $\mathbb{C}^n$ is a $K(\pi, 1)$ space with the fundamental group isomorphic to the pure braid group ([**6**]). Khovanov [**9**] shows that $\mathcal{M}_{\mathcal{A}_{n,3}}$ for the 3-equal arrangement $\mathcal{A}_{n,3}$ in $\mathbb{R}^n$ is also a $K(\pi, 1)$ space.

Note that $\mathcal{M}_\mathcal{A}$ and $\mathcal{V}^\circ_\mathcal{A}$ are related by Alexander duality as follows:

$$H^i(\mathcal{M}_\mathcal{A}; \mathbb{F}) = H_{n-2-i}(\mathcal{V}^\circ_\mathcal{A}; \mathbb{F}) \qquad (\mathbb{F} \text{ is any field}) \tag{1.1}$$

In the mid 1980's Goresky and MacPherson [**7**] found a formula for the Betti numbers of $\mathcal{M}_\mathcal{A}$, while the homotopy type of $\mathcal{V}^\circ_\mathcal{A}$ was computed by Ziegler and Živaljević [**14**] (see Section 4). The answers are phrased in terms of the lower intervals in the *intersection lattice* $L_\mathcal{A}$ of the subspace arrangement $\mathcal{A}$, that is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion. For general subspace arrangements, these lower intervals in $L_\mathcal{A}$ can have arbitrary homotopy type (see [**14**, Corollary 3.1]).

2000 *Mathematics Subject Classification.* Primary 06A07; Secondary 05E99.

Key words and phrases. shellable, hypergraph complex, diagonal subspace arrangement, Golod ring.

This research forms part of the author's doctoral dissertation at the University of Minnesota, under the supervision of Victor Reiner, and partially supported by NSF grant DMS-0245379.

Our goal is to find a general sufficient condition for the intersection lattice $L_{\mathcal{A}}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved. Björner and Welker [**4**] show that $L_{\mathcal{A}_{n,k}}$ is shellable, and hence has the homotopy type of a wedge of spheres, where $\mathcal{A}_{n,k}$ is the k-equal arrangement consisting of all $U_{i_1 \cdots i_k}$ for all $1 \leq i_1 < \cdots < i_k \leq n$ (see Section 2). Kozlov [**11**] shows that $L_{\mathcal{A}}$ is shellable if $\mathcal{A}$ satisfies some conditions (see Section 2). Suggested by a homological calculation (Theorem 4.4 below), we will prove the following main result, capturing the homotopy type assertion from [**11**] (see Section 3).

THEOREM 1.1. *Let Δ be a shellable simplicial complex. Then the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.*

Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. Let Δ have vertex set $[n]$ with a shelling order $F_1, \ldots, F_q$ on its facets. Let σ be the intersection of all facets, and $\bar{\sigma}$ its complement. For each i, let G_i be the face of F_i obtained by intersecting the walls of F_i that lie in the subcomplex generated by $F_1, \ldots, F_{i-1}$, where a *wall of F_i* is a codimension 1 face of F_i. An (*unordered*) *shelling-trapped decomposition* (*of $\bar{\sigma}$ over* Δ) is defined to be a family $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ such that $\{\bar{\sigma}_1, \ldots, \bar{\sigma}_p\}$ is a decomposition of $\bar{\sigma}$ as a disjoint union

$$\bar{\sigma} = \bigsqcup_{j=1}^{p} \bar{\sigma}_j$$

and $F_{i_1} < \cdots < F_{i_p}$ are facets of Δ such that $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all j. Then the wedge of spheres in Theorem 1.1 consists of $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition D of $\bar{\sigma}$ and a permutation ω of $[p-1]$, there exists a saturated chain $\overline{C}_{D,\omega}$ (see Section 3) such that removing the simplices corresponding to these chains in $\overline{L}_\Delta$ leaves a contractible simplicial complex.

The following example shows that the intersection lattice in Theorem 1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres.

EXAMPLE 1.2. Let Δ be a simplicial complex on $\{1,2,3,4,5,6,7,8\}$ with the shelling $123456, 127, 237, 137, 458, 568, 468$. Then $\Delta(U_{78}, \hat{1})$ is a disjoint union of two circles, hence is not shellable. Therefore, the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is also not shellable. The intersection lattice L_Δ is shown in Figure 1 (thick lines represent the open interval $(U_{78}, \hat{1})$).

The next example shows that there is a nonshellable simplicial complex whose intersection lattice is shellable.

EXAMPLE 1.3. Let Δ be a simplicial complex on $\{1,2,3,4\}$ whose facets are 12 and 34. Then Δ is not shellable. But the order complex of $\overline{L}_\Delta$ consists of two vertices, hence is shellable.

2. Some known special cases

In this section, we give Kozlov's theorem and show how its consequence for homotopy type follows from Theorem 1.1. Also, we give Björner and Welker's theorem about the intersection lattice of the k-equal arrangements which can be recovered using Theorem 1.1.

Kozlov [**11**] shows that $\mathcal{A}_\Delta$ has shellable intersection lattice if Δ satisfies some conditions. This class includes k-equal arrangements and all other diagonal arrangements for which the intersection lattice was proved shellable up to now.

THEOREM 2.1. ([**11**, Corollary 3.2]) *Consider a partition of*

$$[n] = E_1 \cup \cdots \cup E_r$$

such that $\max E_i < \min E_{i+1}$ *for* $i = 1, \ldots, r-1$. *Let*

$$f : \{1, 2, \ldots, r\} \to \{2, 3, \ldots\}$$

be a nondecreasing map. Let Δ be a simplicial complex on $[n]$ such that F is a facet of Δ if and only if

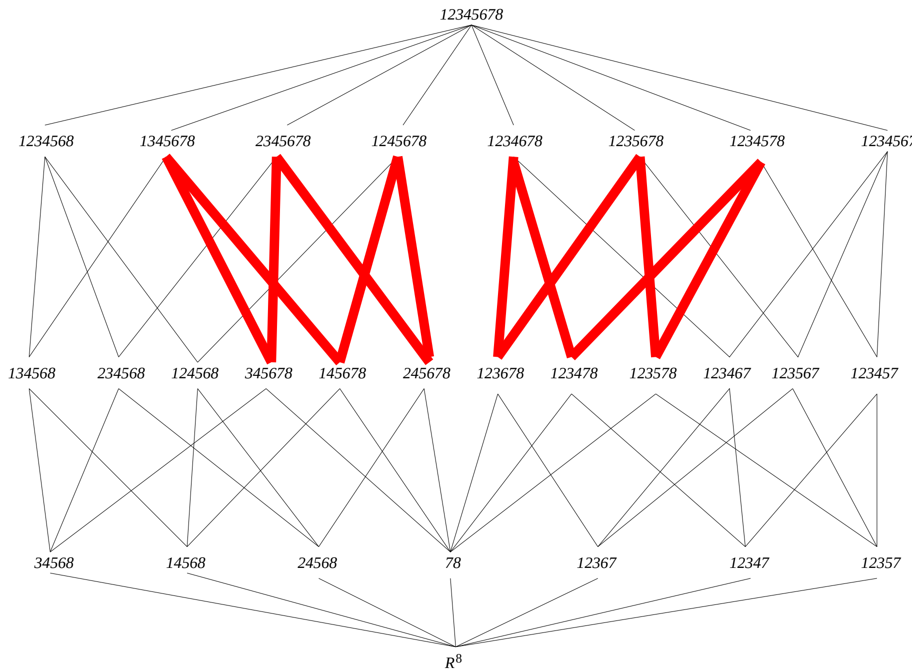

FIGURE 1. The intersection lattice for $\mathcal{A}_\Delta$

$\min \overline{F}$	F	w	$\min \overline{F}$	F	w	$\min \overline{F}$	F	w
1	23456	17	2	1356	247	3	1256	347
	23457	16		1357	246		1257	346
	23467	15		1367	245		1267	345
	23567	14						
	24567	13						
	34567	12						

TABLE 1. Table for Example 2.2

(1) $|E_i - F| \leq 1$ *for* $i = 1, \ldots, r$*;*
(2) *if* $\min \overline{F} \in E_i$ *then* $|F| = n - f(i)$.

Then the intersection lattice for $\mathcal{A}_\Delta$ *is shellable.*

In particular, this intersection lattice has the homotopy type of a wedge of spheres.

PROPOSITION 2.1. Δ *in Theorem 2.1 is shellable.*

PROOF SKETCH. One checks that a shelling order is $F_1, F_2, \ldots, F_q$ such that the words $w_1, w_2, \ldots, w_q$ are in lexicographic order, where w_i is the increasing array of elements in $\overline{F}_i$. $\square$

EXAMPLE 2.2. Consider the partition of

$$[7] = \{1\} \cup \{2,3\} \cup \{4\} \cup \{5,6,7\}$$

and the function f given by $f(1) = 2, f(2) = 3, f(3) = 4$, and $f(4) = 5$. Then the facets of the simplicial complex that satisfy the conditions of Theorem 2.1 and the corresponding words can be found in Table 1. Thus the ordering 34567, 24567, 23567, 23467, 23457, 23456, 1367, 1357, 1356, 1267, 1257 and 1256 is a shelling for this simplicial complex.

One can also use Theorem 1.1 to recover the following theorem of Björner and Welker [**4**].

Theorem 2.3. *The order complex of the intersection lattice $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres consisting of*

$$(p-1)! \sum_{0=i_0\le i_1\le\cdots\le i_p=n-pk} \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}$$

copies of $(n-3-p(k-2))$-dimensional spheres for $1\le p\le \lfloor \frac{n}{k} \rfloor$.

3. Proof of main theorem

Theorem 1.1 will be deduced from a more general statement about homotopy types of lower intervals $\Delta(\hat{0}, H)$ in $L_{\mathcal{A}}$, Theorem 3.1 below.

Theorem 3.1. *Let Δ be a shellable simplicial complex on $[n]$ with a shelling $F_1,\ldots,F_q$ and $\dim\Delta\le n-3$. Let $U_{\bar\sigma}$ be a subspace in L_Δ for some subset $\bar\sigma$ of $[n]$. Then $\Delta(\hat 0, U_{\bar\sigma})$ is homotopy equivalent to a wedge of spheres, consisting of $(p-1)!$ copies of spheres of dimension*

$$\delta(D) := p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar\sigma| - 3$$

for each shelling-trapped decomposition $D=\{(\bar\sigma_1, F_{i_1}),\ldots,(\bar\sigma_p, F_{i_p})\}$ of $\bar\sigma$.

Moreover, for each such shelling-trapped decomposition D and each permutation ω of $[p-1]$, one can construct a saturated chain $\overline{C}_{D,\omega}$ (see Section 3.1 below), such that if one removes the corresponding $\delta(D)$-dimensional simplices for all pairs (D,ω), the remaining simplicial complex $\widehat{\Delta}(\hat 0, U_{\bar\sigma})$ is contractible.

To prove this result, we begin with some preparatory lemmas.

First of all, one can characterize exactly which subspaces lie in L_Δ when Δ is shellable. Recall that for $\bar\sigma=\{i_1,\ldots,i_r\}\subseteq[n]$, we denote by $U_{\bar\sigma}$ the linear subspace of the form $u_{i_1}=\cdots=u_{i_r}$.

Lemma 3.2. *Let Δ be a simplicial complex on $[n]$ with $\dim\Delta\le n-3$.*

(1) Every subspace H in L_Δ has the form

$$H = U_{\bar\sigma_1}\cap\cdots\cap U_{\bar\sigma_p}$$

for pairwise disjoint subsets $\bar\sigma_1,\ldots,\bar\sigma_p$ of $[n]$ such that σ_i can be expressed as an intersection of facets of Δ for $i=1,2,\ldots,p$.

(2) Conversely, when Δ is shellable, every subspace H of $\mathbb{R}^n$ that has the above form lies in L_Δ.

The next example shows that Lemma 3.2(2) can fail when Δ is not assumed to be shellable.

Example 3.3. Let Δ be a simplicial complex with two facets 123 and 345. Then Δ is not shellable. Since L_Δ has only three subspaces U_{12}, U_{45} and $U_{12}\cap U_{45}$, it does not have the subspace U_{1245}, even though $\overline{1245}=3$ is an intersection of facets 123 and 345 of Δ. Thus Lemma 3.2(2) fails for Δ.

In fact, Lemma 3.2(2) is true for a more general class of simplicial complexes. A simplicial complex is called *locally gallery-connected* if any pair F, F' of facets are connected by a path

$$F=F_0, F_1,\ldots,F_{r-1}, F_r=F'$$

of facets in which $F_i\cap F_{i-1}$ share a $(\min\{\dim F_i, \dim F_{i-1}\}-1)$-dimensional face for each i. It is not hard to show that sequentially Cohen-Macaulay simplicial complexes (and hence shellable simplicial complexes) are locally gallery-connected. One can show that Lemma 3.2(2) is true when Δ is locally gallery-connected. Although Lemma 3.2(2) is true for locally gallery-connected simplicial complexes, Theorem 3.1 can fail when Δ is locally gallery-connected. E.g., any triangulation of $\mathbb{RP}^2$ gives a counterexample.

The following lemma shows that every lower interval $[\hat 0, H]$ can be written as a product of lower intervals of the form $[\hat 0, U_{\bar\sigma}]$.

Lemma 3.4. *Let Δ be a simplicial complex on $[n]$ with $\dim\Delta\le n-3$ and let $H\in L_\Delta$ be a subspace of the form*

$$H = U_{\bar\sigma_1}\cap\cdots\cap U_{\bar\sigma_p}$$

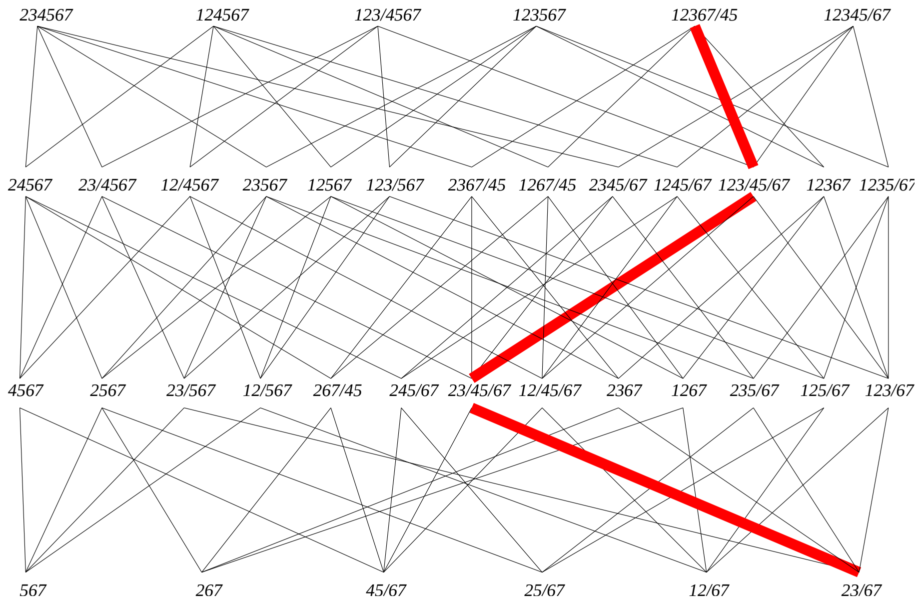

FIGURE 2. The upper interval $(U_{67}, \hat{1})$ in L_Δ

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ *of* $[n]$*. Then*

$$[\hat{0}, H] = [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

In particular,

$$\Delta(\hat{0}, H) = \Delta(\hat{0}, U_{\bar{\sigma}_1}) * \cdots * \Delta(\hat{0}, U_{\bar{\sigma}_p}) * \mathbb{S}^{p-2},$$

where $*$ *denotes the join of topological spaces.*

PROOF. The first assertion is straightforward, and the second then follows from [**13**, Theorem 4.3]. $\square$

The next lemma, whose proof is completely straightforward and omitted, shows that the lower interval $[\hat{0}, U_{\bar{\sigma}}]$ is *isomorphic* to the intersection lattice for the diagonal arrangement corresponding to $\mathrm{link}_\Delta \sigma$.

LEMMA 3.5. *Let* Δ *be a simplicial complex on* $[n]$ *with* $\dim \Delta \leq n-3$ *and let* $U_{\bar{\sigma}}$ *be a subspace in* L_Δ *for some face* σ *of* Δ*. Then the lower interval* $[\hat{0}, U_{\bar{\sigma}}]$ *is isomorphic to the intersection lattice of the diagonal arrangement* $\mathcal{A}_{link_\Delta(\sigma)}$ *corresponding to* $link_\Delta(\sigma)$ *on the vertex set* $\bar{\sigma}$*.*

The following lemma shows that upper intervals in L_Δ are at least still *homotopy equivalent* to the intersection lattice of a diagonal arrangement.

LEMMA 3.6. *Let* Δ *be a simplicial complex on* $[n]$ *with* $\dim \Delta \leq n-3$ *and let* $U_{\bar{\sigma}}$ *be a subspace in* L_Δ *for some face* $\sigma = \{v_1, \ldots, v_t\}$ *of* Δ*. Then the upper interval* $[U_{\bar{\sigma}}, \hat{1}]$ *is homotopy equivalent to the intersection lattice of the diagonal arrangement* $\mathcal{A}_{\Delta_\sigma}$ *corresponding to the simplicial complex* Δ_σ *on the vertex set* $\{v_1, \ldots, v_t, v\}$ *whose facets are obtained in the following way:*

(A) If $F \cap \sigma$ *is maximal among*

$$\{F \cap \sigma \mid F \text{ is a facet of } \Delta \text{ such that } \sigma \nsubseteq F \text{ and } F \cup \sigma \neq [n]\},$$

then $\widetilde{F} = F \cap \sigma$ *is a facet of* Δ_σ*.*

(B) If a facet F *of* Δ *satisfies* $F \cup \sigma = [n]$*, then* $\widetilde{F} = (F \cap \sigma) \cup \{v\}$ *is a facet of* Δ_σ*.*

EXAMPLE 3.7. Let Δ be a simplicial complex on $\{1,2,3,4,5,6,7\}$ with facets $12367, 12346, 13467, 34567, 13457, 14567, 12345$ and let $\sigma = \{1,2,3,4,5\}$. The open interval $(U_{67}, \hat{1})$ is shown in Figure 2. Then Δ_F is a simplicial complex on $\{1,2,3,4,5,v\}$ and its facets are $123v, 1234, 134v, 345v, 1345, 145v$. The proper part of the intersection lattice L_{Δ_F} is shown in Figure 3 and it is easy to see that its order complex is homotopy equivalent to $(U_{67}, \hat{1})$.

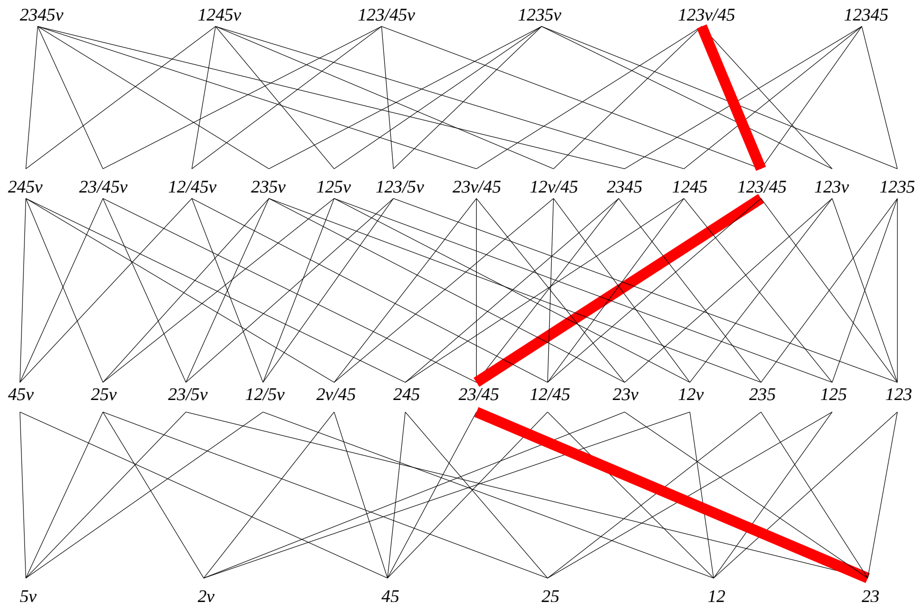

FIGURE 3. The interval $(\hat{0}, \hat{1})$ in L_{Δ_F}

In general, the simplicial complex Δ_σ of Lemma 3.6 is not shellable, even though Δ is shellable (see Example 1.2). However, the next lemma shows that Δ_F is shellable if F is the *last* facet in the shelling order.

LEMMA 3.8. *Let Δ be a shellable simplicial complex on $[n]$ such that $\dim \Delta \leq n-3$ and let F be the last facet in a shelling order of Δ. Then Δ_F is shellable.*

PROOF. Using the notation of Lemma 3.6, a shelling order for Δ_F is the ordering of facets of type (A) in any order, followed by the facets of type (B) according to the order of the corresponding facets of Δ. $\square$

EXAMPLE 3.9. The simplicial complex Δ in Example 3.7 is shellable with a shelling $12367, 12346, 13467, 34567, 13457, 14567, 12345$. Since $1234, 1345$ are facets of Δ_F of type (A) and $123v, 134v, 345v, 145v$ are facets of Δ_F of type (B), $1234, 1345, 123v, 134v, 345v, 145v$ is a shelling of Δ_F.

We next construct the saturated chains appearing in the statement of Theorem 3.1.

3.1. Constructing the chains $C_{D,\omega}$. Let Δ be a shellable simplicial complex on $[n]$ with $\dim \Delta \leq n-3$ and let $U_{\bar{\sigma}}$ is a subspace in L_Δ. Let $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ be a shelling-trapped decomposition of $\bar{\sigma}$ and let ω be a permutation on $[p-1]$. We define a chain $C_{D,\omega}$ in $[\hat{0}, U_{\bar{\sigma}}]$ as follows:

(1) By Lemma 3.2, the interval $[\hat{0}, U_{\bar{\sigma}}]$ contains $U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}$ and the interval $[U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}, U_{\bar{\sigma}}]$ is isomorphic to the set partition lattice Π_p. It is well known that the order complex of $\overline{\Pi}_p = \Pi_p - \{\hat{0}, \hat{1}\}$ is homotopy equivalent to a wedge of $(p-1)!$ spheres of dimension $p-3$ and there is a saturated chain C_ω in Π_p for each permutation ω of $[p-1]$ such that removing $\{\overline{C}_\omega = C_\omega - \{\hat{0}, \hat{1}\} | \omega \in \mathfrak{S}_{p-1}\}$ from the order complex of $\overline{\Pi}_p$ gives a contractible subcomplex (see [**1**, Example 2.9]). Identify $U_{\bar{\sigma}_1}, \cdots, U_{\bar{\sigma}_p}$ with $1, \ldots, p$ in this order and take the saturated chain $\widetilde{C}_\omega$ in $[U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}, U_{\bar{\sigma}}]$ which corresponds to the chain C_ω in Π_p.

(2) By Lemma 3.4,
$$[\hat{0}, U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}] \cong [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$
Since Δ is shellable and $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all j, one can see that $[\hat{0}, U_{\bar{\sigma}_j}]$ has a subinterval $[U_{\overline{F}_{i_j}}, U_{\bar{\sigma}_j}]$ which is isomorphic to the boolean algebra of the set of order $|\bar{\sigma}_j| - |\overline{F}_{i_j}|$. Thus
$$[U_{\overline{F}_{i_1}} \cap \cdots \cap U_{\overline{F}_{i_p}}, U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}]$$

is isomorphic to
$$[U_{\overline{F}_{i_1}}, U_{\bar{\sigma}_1}] \times \cdots \times [U_{\overline{F}_{i_p}}, U_{\bar{\sigma}_p}]$$
and hence is isomorphic to the boolean algebra of the set of order $\sum_{j=1}^{p} \left(|\bar{\sigma}_j| - |\overline{F}_{i_j}|\right)$. Take any saturated chain $\widetilde{C}$ in
$$[U_{\overline{F}_{i_1}} \cap \cdots \cap U_{\overline{F}_{i_p}}, U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}].$$

(3) Define a saturated chain $C_{D,\omega}$ by
$$\hat{0} \prec U_{\overline{F}_{i_p}} \prec U_{\overline{F}_{i_p}} \cap U_{\overline{F}_{i_{p-1}}} \prec \cdots \prec U_{\overline{F}_{i_p}} \cap \cdots \cap U_{\overline{F}_{i_1}}$$
followed by the chains $\widetilde{C}$ and $\widetilde{C}_\omega$ (where $\prec$ means the covering relation in L_Δ).

Let
$$\overline{C}_{D,\omega} = C_{D,\omega} - \{\hat{0}, U_{\bar{\sigma}}\}.$$
Then $\overline{C}_{D,\omega} \in \Delta(\hat{0}, U_{\bar{\sigma}})$.

Note that the length of this chain $\overline{C}_{D,\omega}$ is
$$\begin{aligned} l(\overline{C}_{D,\omega}) &= p + \sum_{j=1}^{p} \left(|\bar{\sigma}_j| - |\overline{F}_{i_j}|\right) + (p-1) - 2 \\ &= p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3. \end{aligned}$$

Example 3.10. Let Δ be the shellable simplicial complex in Example 3.7. Then one can see that
$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$
is a shelling-trapped decomposition of $\{1,2,3,4,5,6,7\}$. Let ω be a permutation in $\mathfrak{S}_2$ with $\omega(1) = 2$ and $\omega(2) = 1$. Then the maximal chain C_ω in Π_3 corresponding to ω is $(1 \mid 2 \mid 3) - (1 \mid 23) - (123)$. By identifying U_{45}, U_{123}, U_{67} with $1, 2, 3$ in this order, one can get
$$\widetilde{C}_\omega = U_{45} \cap U_{123} \cap U_{67} \prec U_{45} \cap U_{12367} \prec U_{1234567}.$$
Since $[U_{45} \cap U_{23} \cap U_{67}, U_{45} \cap U_{123} \cap U_{67}]$ is isomorphic to a boolean algebra of the set of order 1, one can take
$$\widetilde{C} = U_{45} \cap U_{23} \cap U_{67} \prec U_{45} \cap U_{123} \cap U_{67}.$$
Thus $C_{D,\omega}$ is the chain
$$\begin{gathered} \hat{0} \prec U_{67} \prec U_{23} \cap U_{67} \prec U_{45} \cap U_{23} \cap U_{67} \\ \prec U_{45} \cap U_{123} \cap U_{67} \prec U_{45} \cap U_{12367} \prec U_{1234567}. \end{gathered}$$
The chain $\overline{C}_{D,\omega}$ is represented by thick lines in Figure 2.

The following lemma gives the relationship between the shelling-trapped decompositions of $[n]$ containing F and the shelling-trapped decompositions of $F \cup \{v\}$.

Lemma 3.11. *Let Δ be a shellable simplicial complex on $[n]$ such that $\dim \Delta \le n-3$ and let F be the last facet in the shelling order of Δ.*

Then there is a one-to-one correspondence between the set of all pairs (D, ω) of shelling-trapped decompositions D of $[n]$ over Δ containing F and $\omega \in \mathfrak{S}_{|D|-1}$, and the set of all pairs $(\widetilde{D}, \tilde{\omega})$ of shelling-trapped decompositions $\widetilde{D}$ of $F \cup \{v\}$ over Δ_F and $\tilde{\omega} \in \mathfrak{S}_{|\widetilde{D}|-1}$. Moreover, one can choose $\overline{C}_{D,\omega}$ and $\overline{C}_{\widetilde{D},\tilde{\omega}}$ so that $\overline{C}_{D,\omega} - U_{\overline{F}}$ corresponds to $\overline{C}_{\widetilde{D},\tilde{\omega}}$ under the homotopy equivalence in Theorem 3.6.

Example 3.12. Let Δ be the shellable simplicial complex in Example 3.7. In Example 3.10, we had
$$\begin{aligned} C_{D,\omega} = &\hat{0} \prec U_{67} \prec U_{23} \cap U_{67} \prec U_{45} \cap U_{23} \cap U_{67} \\ &\prec U_{45} \cap U_{123} \cap U_{67} \prec U_{45} \cap U_{12367} \prec U_{1234567} \end{aligned}$$
for a shelling-trapped decomposition
$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$
of $\{1,2,3,4,5,6,7\}$ and a permutation ω in $\mathfrak{S}_2$ with $\omega(1) = 2$ and $\omega(2) = 1$.

Decomposition	Facets
1234	$G_3 = \emptyset \subseteq \overline{1234} = \emptyset \subseteq F_3 = 23$
1234	$G_5 = \emptyset \subseteq \overline{1234} = \emptyset \subseteq F_5 = 34$
$24 \cup 13$	$G_2 = 1 \subseteq \overline{24} = 13 \subseteq F_2 = 13$, $G_4 = 2 \subseteq \overline{13} = 24 \subseteq F_4 = 24$
$34 \cup 12$	$G_1 = 12 \subseteq \overline{34} = 12 \subseteq F_1 = 12$, $G_5 = \emptyset \subseteq \overline{12} = 34 \subseteq F_5 = 34$

TABLE 2. Shelling-trapped decompositions of $\bar{\sigma} = 1234$

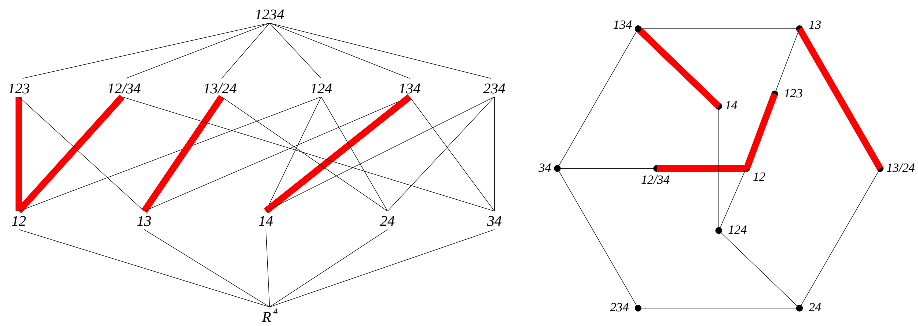

FIGURE 4. The intersection lattice for Δ and the order complex for its proper part

Since $67 = \overline{F}_7$, the corresponding shelling-trapped decomposition $\widetilde{D}$ of $\{1,2,3,4,5,v\}$ is

$$\widetilde{D} = \{(45, \widetilde{F}_1 = 123v), (123v, \widetilde{F}_6 = 145v)\}$$

and the corresponding permutation $\tilde{\omega} \in \mathfrak{S}_1$ is the identity.

The corresponding chain $\overline{C}_{\widetilde{D},\tilde{\omega}}$ is

$$\hat{0} \prec U_{23} \prec U_{45} \cap U_{23} \prec U_{45} \cap U_{123} \prec U_{45} \cap U_{123v}.$$

PROOF SKETCH OF THEOREM 3.1. One can consider the following decomposition of $\widehat{\Delta}(\overline{L})$:

$$\widehat{\Delta}(\overline{L}) = \widehat{\Delta}(\overline{L} - \{H\}) \cup \widehat{\Delta}(\overline{L}_{\geq H}),$$

where $\widehat{\Delta}(\overline{L} - \{H\})$ is obtained by removing all chains $\overline{C}_{D,\omega}$ not containing H from $\overline{L} - \{H\}$ and $\widehat{\Delta}(\overline{L}_{\geq H})$ is obtained by removing $\overline{C}_{D,\omega}$ and $\overline{C}_{D,\omega} - H$ from $\overline{L}_{\geq H}$ for all $\overline{C}_{D,\omega}$ containing H. Then one can show that all three spaces $\widehat{\Delta}(\overline{L} - \{H\})$, $\widehat{\Delta}(\overline{L}_{\geq H})$ and their intersection are contractible, and hence $\widehat{\Delta}(\overline{L})$ is also contractible. $\square$

EXAMPLE 3.13. Let Δ be a simplicial complex with a shelling

$$F_1 = 12, \quad F_2 = 13, \quad F_3 = 23, \quad F_4 = 24, \quad F_5 = 34.$$

Then

$$G_1 = 12, \quad G_2 = 1, \quad G_3 = \emptyset, \quad G_4 = 2, \quad G_5 = \emptyset.$$

Let $\bar{\sigma} = 1234$. Then there are four possible shelling-trapped decompositions of $\bar{\sigma}$ (see Table 2). Thus $\Delta(\hat{0}, U_{1234})$ is homotopy equivalent to a wedge of four circles. The intersection lattice and the order complex for its proper part are shown in Figure 4. Note that the chains and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines.

4. The homology of the singularity link of $\mathcal{A}_\Delta$

In this section, we give the corollary about the homotopy type of the singularity link of $\mathcal{A}_\Delta$ when Δ is shellable. Also we give the homology version of the corollary.

Ziegler and Živaljević [**14**] show the following theorem about the homotopy type of $\mathcal{V}^{\circ}_{\mathcal{A}}$.

Theorem 4.1. *For every subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$,*

$$\mathcal{V}^\circ_{\mathcal{A}} \simeq \bigvee_{x \in L_{\mathcal{A}} - \{\hat{0}\}} (\Delta(\hat{0}, x) * \mathbb{S}^{\dim(x)-1}).$$

From this and our results in Section 3, one can deduce the following.

Corollary 4.2. *Let Δ be a shellable simplicial complex on $[n]$ with $\dim \Delta \leq n-3$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension*

$$n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$$

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$.

Remark 4.3. The following theorem is a homology version of this corollary.

Theorem 4.4. *Let Δ be a shellable simplicial complex on $[n]$ with $\dim \Delta \leq n-3$ and $F_1, \ldots, F_q$ be the shelling order on facets of Δ. Then $\dim_{\mathbb{F}} H_i(\mathcal{V}^\circ_{\mathcal{A}_\Delta}; \mathbb{F})$ is the number of ordered shelling-trapped decompositions $((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))$ with $i = n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$.*

This last result can be proven without Theorem 3.1 by combining

(1) a result of Peeva, Reiner and Welker [**12**, Theorem 1.3],
(2) results of Herzog, Reiner and Welker [**8**, Theorem 4, Theorem 9],
(3) the theory of *Golod rings*.

It is what motivated Corollary 4.2 and eventually Theorem 1.1.

5. $K(\pi, 1)$ examples from matroids

Davis, Januszkiewicz and Scott [**5**] show the following theorem.

Theorem 5.1. *Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{A}$ be any arrangement of codimension-2 intersection subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$.*

Remark 5.2. In order to apply this to diagonal arrangements, we need to consider hyperplane arrangements $\mathcal{H}$ which are subarrangements of the braid arrangement $\mathcal{B}_n$ and also simplicial. It turns out (and we omit the straightforward proof) that all such arrangements $\mathcal{H}$ are direct sums of smaller braid arrangements. So we only consider $\mathcal{H} = \mathcal{B}_n$ itself here.

Corollary 5.3. *Let $\mathcal{A}$ be diagonal arrangement of codimension 2 subspaces inside $\mathcal{H} = \mathcal{B}_n$, so that*

$$\mathcal{A} = \{U_{ijk} \mid \{i,j,k\} \in T_{\mathcal{A}}\},$$

for some collection $T_{\mathcal{A}}$ of 3-element subsets of $[n]$. Then $\mathcal{A}$ satisfies the hypothesis of Theorem 5.1 (and hence $\mathcal{M}_{\mathcal{A}}$ is $K(\pi,1)$) if and only if every permutation w in $\mathfrak{S}_n$ has at least one triple in $T_{\mathcal{A}}$ consecutive.

Proof. It is easy to see that there is a bijection with chambers of $\mathcal{B}_n$ and permutations $w = w_1 \cdots w_n$ in $\mathfrak{S}_n$. Moreover, each chamber has the form $x_{w_1} > \cdots > x_{w_n}$ with bounding hyperplanes $x_{w_1} = x_{w_2}, \ldots, x_{w_{n-1}} = x_{w_n}$ and intersects the 3-equal subspaces of the form $x_{w_i} = x_{w_{i+1}} = x_{w_{i+2}}$ for $i = 1, 2, \ldots, n-2$. $\square$

A rich source of shellable complexes are the *matroid complexes* $\mathcal{I}(M)$, that is the independent sets of a matroid M. If $\Delta = \mathcal{I}(M)$ for some matroid M, then facets of Δ are bases of M. Therefore

$$\begin{aligned} \mathcal{A}_\Delta &= \{U_{ijk} \mid \{i,j,k\} = [n] - B \text{ for some } B \in \mathcal{B}(M)\} \\ &= \{U_{ijk} \mid \{i,j,k\} \in \mathcal{B}(M^\perp)\}, \end{aligned}$$

where $M^\perp$ is the dual matroid of M.

Definition 5.4. Say a rank 3 matroid M on $[n]$ is *DJS* if its bases $\mathcal{B}(M)$ satisfies the condition of Corollary 5.3.

Note that a matroid M which is DJS gives rise to a diagonal arrangement $\mathcal{A}_\Delta$ for $\Delta = \mathcal{I}(M^\perp)$ which has $\mathcal{M}_{\mathcal{A}_\Delta}$ $K(\pi,1)$ and with the homotopy type of $L_\Delta, \mathcal{V}^\circ_{\mathcal{A}_\Delta}$ all predicted by Theorem 3.1. Unfortunately, the following example shows that matroids are not always DJS in general.

EXAMPLE 5.5. Let Δ be the boundary of an octahedron. Then it is a simplicial complex on $\{1,2,3,4,5,6\}$ whose facets are $123, 134, 145, 125, 236, 346, 456$ and 256. It is easy to see that it is vertex-decomposable, hence is shellable. Also note that Δ is the independent set complex $\mathcal{I}(M)$ of a matroid M of rank 3 which has three distinct parallel classes $\{1,6\}, \{2,4\}$ and $\{3,5\}$. But,

$$T_{\mathcal{A}_\Delta} = \{123, 134, 145, 125, 236, 346, 456, 256\}$$

and $w = 124356$ is a permutation that does not satisfy the condition of Corollary 5.3.

Thus we look for some subclasses of matroids which are DJS. The following two propositions give some rank 3 matroids which are DJS.

PROPOSITION 5.1. *Let M be a rank 3 matroid on the ground set $[n]$ with no circuits of size 3. Let $P_1, \ldots, P_k$ be distinct parallel classes which have more than one element and let N be the set of all elements which are not parallel with anything else. Then, M is DJS if and only if* $\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k < |N| - 2$.

A simplicial complex Δ on $[n]$ is *shifted* if, for any face of Δ, replacing any vertex i by a vertex $j(< i)$ gives another face in Δ. The *Gale ordering* on all k element subsets of $[n]$ is given by $\{x_1 < \cdots < x_k\}$ is less than $\{y_1 < \cdots < y_k\}$ if

$$x_i \leq y_i \text{ for all } i \text{ and } \{x_1, \ldots, x_k\} \neq \{y_1, \ldots, y_k\}.$$

Then it is known that shifted complexes are exactly the order ideals of Gale ordering. Klivans [**10**] shows the following theorem.

THEOREM 5.6. *Let M be a rank 3 loop-coloop free matroid on the ground set $[n]$ such that $\mathcal{I}(M)$ is also shifted. Then its bases are the principal order ideal generated by $\{a,b,n\}$ in the Gale ordering such that $1 < a < b < n$. Moreover, M has the following form:*

(1) elements $b+1, b+2, \ldots, n$ form the unique non-trivial parallel class.
(2) elements $a+1, a+2, \ldots, n$ form a rank 2 flat, and this is the only rank 2 flat which can contain more than two parallelism classes.

From this, one can see the following.

PROPOSITION 5.2. *Let M be the rank 3 matroid on the ground set $[n]$ corresponding to the principal order ideal generated by $\{a,b,n\}$. Then, M is DJS if and only if* $\lfloor \frac{n-b}{2} \rfloor < a$.

PROBLEM: Characterize the rank 3 matroids which are DJS.

References

[1] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), pp. 159-183.
[2] A. Björner and L. Lovász, Linear decision trees, subspace arrangements and Möbius functions, *J. Amer. Math. Soc.* **7** (1994), 677-706.
[3] A. Björner, L. Lovász and A. Yao, Linear decision trees: volume estimates and topological bounds, 24th ACM Symp. on Theory of Computing (May 1992), ACM Press, N.Y., 1992, 170-177.
[4] A. Björner and V. Welker, The homology of "k-equal" manifolds and related partition lattices, *Advances in Math.* **110** (1995), 277-313.
[5] M. Davis, T. Januszkiewicz and R. Scott, Nonpositive curvature of blow-ups, *Selecta Math. (N.S.)* **4** (1998), 491–547.
[6] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962), 111-118.
[7] M. Goresky and R. D. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas (3)], 14. Springer-Verlag, Berlin, 1988.
[8] J. Herzog, V. Reiner and V. Welker, Componentwise linear ideals and Golod rings, *Michigan Math. J.* **46** (1999), 211-223.
[9] M. Khovanov, Real $K(\pi,1)$ arrangements from finite root systems, *Math. Res. Lett.* **3** (1996), 261-274.
[10] C. Klivans, Shifted matroid complexes, preprint.
[11] D. Kozlov, A class of hypergraph arrangements with shellable intersection lattice, *J. Combin. Theory Ser. A* **86** (1999), 169-176.
[12] I. Peeva, V. Reiner and V. Welker, Cohomology of real diagonal subspace arrangements via resolutions, *Compositio Mathematica* **117** (1999), 99-115.
[13] J. W. Walker, Topology and combinatorics of ordered sets, Thesis, M.I.T., Cambridge, MA, 1981.

[14] G. M. Ziegler and R. T. Živaljević, Homotopy types of subspace arrangements via diagrams of spaces, *Math. Ann.* **295** (1993), 527-548.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: swkim@math.umn.edu