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Clusters, Coxeter-sortable elements and noncrossing partitions

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ABSTRACT. We introduce Coxeter-sortable elements of a Coxeter group W. For finite W, we give bijective proofs that Coxeter-sortable elements are equinumerous with clusters and with noncrossing partitions. We characterize Coxeter-sortable elements in terms of their inversion sets and, in the classical cases, in terms of permutations.

RÉSUMÉ. Nous introduisons dans ce travail la notion d'éléments sortables pour un groupe de Coxeter W. Dans le cas où W est fini, nous montrons que les éléments sortables sont en bijection avec les clusters ainsi qu'avec les partitions non croisées. Nous donnons une caractérisation des éléments sortables au moyen de leurs ensembles d'inversion et, dans les cas classiques, en terme de permutations.

Introduction

The famous Catalan numbers can be viewed as a special case of the W-*Catalan number*, which counts various objects related to a finite Coxeter group W. In many cases, the common numerology is the only known connection between different objects counted by the W-Catalan number. One collection of objects counted by the W-Catalan number is the set of noncrossing partitions associated to W, which play a role in low-dimensional topology, geometric group theory and non-commutative probability [**17**]. Another is the set of maximal cones of the cluster fan. The cluster fan is dual to the generalized associahedron [**9, 11**], a polytope whose combinatorial structure underlies cluster algebras of finite type [**12**].

This paper connects noncrossing partitions to associahedra via certain elements of W which we call *Coxeter-sortable* elements or simply *sortable* elements. For each Coxeter element c of W, there is a set of c-sortable elements, defined in the context of the combinatorics of reduced words. We prove bijectively that sortable elements are equinumerous with clusters and with noncrossing partitions. Sortable elements and the bijections are defined without reference to the classification of finite Coxeter groups, but the proof that these maps are indeed bijections refers to the classification. The bijections are well-behaved with respect to the refined enumerations associated to the Narayana numbers and to positive clusters.

In the course of establishing the bijections, we characterize[1] the sortable elements in terms of their inversion sets. Loosely speaking, we "orient" each rank-two parabolic subgroup of W and require that the inversion set of the element be "aligned" with these orientations. In particular, we obtain a characterization of the sortable elements in types A_n, B_n and D_n as permutations.

Because sortable elements are defined in terms of reduced words, it is natural to partially order them as an induced subposet of the weak order. Indeed, the definition of sortable elements arose from the study of certain lattice quotients of the weak order called Cambrian lattices. In the sequel [**22**] to this paper, we show that sortable elements are indeed a combinatorial model for the Cambrian lattices.

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[1]This characterization is described in the full version.

Recently, Brady and Watt [**8**] observed that the cluster fan arises naturally in the context of noncrossing partitions. Their work and the present work constitute two seemingly different approaches to connecting noncrossing partitions to clusters. The relationship between these approaches is not yet understood.

The term "sortable" has reference to the special case where W is the symmetric group. For one particular choice of c, the definition of c-sortable elements of the symmetric group is as follows: Perform a bubble sort on a permutation π by repeatedly passing from left to right in the permutation and, whenever two adjacent elements are out of order, transposing them. For each pass, record the set of positions at which transpositions were performed. Then π is c-sortable if this sequence of sets is weakly decreasing in the containment order. There is also a choice of c (see Example 1.8) such that the c-sortable elements are exactly the 231-avoiding or stack-sortable permutations [**15**, Exercise 2.2.1.4–5].

1. Coxeter-sortable elements

Throughout this paper, W denotes a finite Coxeter group with simple generators S and reflections T. Some definitions apply also to the case of infinite W, but we confine the treatment of the infinite case to a series of remarks (Remarks 1.5, 2.4 and 3.5.).

The term "word" always means "word in the alphabet S." Later, we consider words in the alphabet T which, to avoid confusion, are called "T-words." A *cover reflection* of $w \in W$ is a reflection t such that $tw = ws$ with $l(ws) < l(w)$. The term "cover reflection" refers to the (right) weak order. This is the partial order on W whose cover relations are the relations of the form $w \gtrdot ws$ for $l(ws) < l(w)$, or equivalently, $w \gtrdot tw$ for a cover reflection t of w. For each $J \subseteq S$, let W_J be the subgroup of W generated by J. The notation $\langle s \rangle$ stands for the set $S - \{s\}$.

For the rest of the paper, c denotes a *Coxeter element*, that is, an element of W with a reduced word which is a permutation of S. An *orientation* of the Coxeter diagram for W is obtained by replacing each edge of the diagram by a single directed edge, connecting the same pair of vertices in either direction. Orientations of the Coxeter diagram correspond to Coxeter elements (cf. [**25**]) as follows: Given a Coxeter element c, any two reduced words for c are related by commutations of simple generators. An edge s—t in the diagram represents a pair of noncommuting simple generators, and the edge is oriented $s \longrightarrow t$ if and only if s precedes t in every reduced word for c.

We now define Coxeter-sortable elements. Fix a reduced word $a_1 a_2 \cdots a_n$ for a Coxeter element c. Write a half-infinite word

$$c^\infty = a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n \ldots$$

The *c-sorting word* for $w \in W$ is the lexicographically first (as a sequence of positions in c^∞) subword of c^∞ which is a reduced word for w. The c-sorting word can be interpreted as a sequence of subsets of S by rewriting

$$c^\infty = a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | \ldots$$

where the symbol "|" is called a *divider.* The subsets in the sequence are the sets of letters of the c-sorting word which occur between adjacent dividers. This sequence contains a finite number of non-empty subsets, and furthermore if any subset in the sequence is empty, then every later subset is also empty. For clarity in examples, we often retain the dividers when writing c-sorting words for c-sortable elements.

An element $w \in W$ is *c-sortable* if its c-sorting word defines a sequence of subsets which is decreasing under inclusion. This definition of c-sortable elements requires a choice of reduced word for c. However, for a given w, the c-sorting words for w arising from different reduced words for c are related by commutations of letters, with no commutations across dividers. In particular, the set of c-sortable elements does not depend on the choice of reduced word for c.

REMARK 1.1. The c-sortable elements have a natural search-tree structure rooted at the identity element. The edges are pairs v, w of c-sortable elements such that the c-sorting word for v is obtained from the c-sorting word for w by deleting the rightmost letter. This makes possible an efficient traversal of the set of c-sortable elements of W which, although it does not explicitly appear in what follows, allows various properties of c-sortable elements to be checked computationally in low ranks. Also, in light of the bijections of Theorems 2.1 and 3.2, an efficient traversal of the c-sortable elements leads to an efficient traversal of noncrossing partitions or of clusters.

The next two lemmas are immediate from the definition of c-sortable elements. Together with the fact that 1 is c-sortable for any c, they completely characterize c-sortability. A simple generator $s \in S$ is *initial* in (or is an *initial letter* of) a Coxeter element c if it is the first letter of some reduced word for c.

Lemma 1.2. *Let s be an initial letter of c and let $w \in W$ have $l(sw) > l(w)$. Then w is c-sortable if and only if it is an sc-sortable element of $W_{\langle s\rangle}$.*

Lemma 1.3. *Let s be an initial letter of c and let $w \in W$ have $l(sw) < l(w)$. Then w is c-sortable if and only if sw is scs-sortable.*

Remark 1.4. In the dictionary between orientations of Coxeter diagrams (i.e. quivers) and Coxeter elements, the operation of replacing c by scs corresponds to the operation on quivers which changes a source into a sink by reversing all arrows from the source. This operation was used in [**16**] in generalizing the clusters of [**11**] to Γ-clusters, where Γ is a quiver of finite type. We thank Andrei Zelevinsky for pointing out the usefulness of this operation, which plays a key role throughout the paper.

Remark 1.5. The definition of c-sortable elements is equally valid for infinite W. Lemmas 1.2 and 1.3 are valid and characterize c-sortability in the infinite case as well. However, we remind the reader that for all stated results in this paper, W is assumed to be finite.

Example 1.6. Consider $W = B_2$ with $c = s_0s_1$. The c-sortable elements are 1, s_0, s_0s_1, $s_0s_1|s_0$, $s_0s_1|s_0s_1$ and s_1. The elements $s_1|s_0$ and $s_1|s_0s_1$ are not c-sortable.

We close the section with a discussion of the sortable elements of the Coxeter group $W = A_n$, realized combinatorially as the symmetric group S_{n+1}. Permutations $\pi \in S_{n+1}$ are written in *one-line notation* as $\pi_1\pi_2\cdots\pi_{n+1}$ with $\pi_i = \pi(i)$. The simple generators of S_{n+1} are $s_i = (i\;\; i{+}1)$ for $i \in [n]$.

A *barring* of a set U of integers is a partition of that set into two sets $\overline{U}$ and $\underline{U}$. Elements of $\overline{U}$ are *upper-barred* integers denoted $\overline{i}$ and *lower-barred* integers are elements of $\underline{U}$, denoted $\underline{i}$.

Recall that orientations of the Coxeter diagram correspond to Coxeter elements. The Coxeter diagram for S_{n+1} has unlabeled edges connecting s_i to s_{i+1} for $i \in [n-1]$. We encode orientations of the Coxeter diagram for S_{n+1} as barrings of $[2, n]$ by directing $s_i \to s_{i-1}$ for every $\overline{i} \in [2, n]$ and $s_{i-1} \to s_i$ for every $\underline{i} \in [2, n]$, as illustrated in Figure 1 for $c = s_8s_7s_4s_1s_2s_3s_5s_6$ in S_9. Given a choice of Coxeter element, the corresponding barring is assumed.

barA.ps

Figure 1. Orientation and barring in S_9

We now define condition (A), which characterizes c-sortability of permutations. Condition (A) depends on the choice of c as follows: a fixed choice of c defines a barring as described above, and condition (A) depends on that fixed barring. A permutation $\pi \in S_{n+1}$ satisfies condition (A) if both of the following conditions hold:

(A1) π contains no subsequence $\overline{j}\,k\,i$ with $i < j < k$, and
(A2) π contains no subsequence $k\,i\,\underline{j}$ with $i < j < k$.

Notice that i and k appear in (A1) and (A2) without explicit barrings. This is because the barrings of i and k are irrelevant to the conditions. For example, to satisfy (A1), π may not contain any sequence of the form $\overline{j}\,k\,i$, regardless of the barrings of i and k.

Theorem 1.7. *A permutation $\pi \in S_{n+1}$ is c-sortable if and only if it satisfies condition (A) with respect to the barring corresponding to c.*

Example 1.8. For $W = S_{n+1}$ and $c = (n\;\; n{+}1)\cdots(2\;3)(1\;2)$, the c-sortable elements are exactly the 231-avoiding or stack-sortable permutations defined in [**15**, Exercise 2.2.1.4–5].

2. Sortable elements and noncrossing partitions

In this section, we define a bijection between sortable elements and noncrossing partitions. Recall that T is the set of reflections of W. Any element $w \in W$ can be written as a word in the alphabet T. To avoid confusion we always refer to a word in the alphabet T as a *T-word*. Any other use of the term "word" is assumed to refer to a word in the alphabet S. A *reduced* T-word for w is a T-word for w which has minimal length among all T-words for w. The *absolute length* of an element w of W is the length of a reduced T-word for w. This is not the usual length $l(w)$ of w, which is the length of a reduced word for w in the alphabet S.

The notion of reduced T-words leads to a prefix partial order on W, analogous to the weak order. Say $x \leq_T y$ if x possesses a reduced T-word which is a prefix of some reduced T-word for y. Equivalently, $x \leq_T y$ if every reduced T-word for x is a prefix of some reduced T-word for y. Since the alphabet T is closed under conjugation by arbitrary elements of W, the partial order $\leq_T$ is invariant under conjugation. The partial order $\leq_T$ can also be defined as a subword order: $x \leq_T y$ if and only if there is a reduced T-word for y having as a subword some reduced T-word for x. In particular, $x \leq_T y$ if and only if $x^{-1}y \leq y$.

The *noncrossing partition lattice* in W (with respect to the Coxeter element c) is the interval $[1, c]_T$, and the elements of this interval are called *noncrossing partitions*. The poset $[1, c]_T$ is graded and the rank of a noncrossing partition is its absolute length.

Let w be a c-sortable element and let $a = a_1a_2 \cdots a_k$ be a c-sorting word for w. Totally order the inversions of w such that the ith reflection in the order is $a_1a_2 \cdots a_{i-1}a_ia_{i-1} \cdots a_2a_1$. Equivalently, t is the ith reflection in the order if and only if $tw = a_1a_2 \cdots \hat{a}_i \cdots a_k$, where $\hat{a}_i$ indicates that a_i is deleted from the word. Write the set of cover reflections of w as a subsequence $t_1, t_2, \ldots, t_l$ of this order on inversions. Let nc_c be the map which sends w to the product $t_1t_2 \cdots t_l$. Recall that the construction of a c-sorting word begins with an arbitrary choice of a reduced word for c. However, since any two c-sorting words for w are related by commutation of simple generators, $\mathrm{nc}_c(w)$ does not depend of the choice of reduced word for c.

Theorem 2.1. *For any Coxeter element c, the map $w \mapsto \mathrm{nc}_c(w)$ is a bijection from the set of c-sortable elements to the set of noncrossing partitions with respect to c. Furthermore nc_c maps c-sortable elements with k descents to noncrossing partitions of rank k.*

Recall that the descents of w are the simple generators $s \in S$ such that $l(ws) < l(w)$. Recall also that these are in bijection with the cover reflections of w. The basic tool for proving Theorem 2.1 is induction on rank and length, using Lemmas 1.2 and 1.3. A more complicated induction is used to prove the existence of the inverse map.

Example 2.2. We again consider the case $W = B_2$ and $c = s_0s_1$. As a special case of the combinatorial realization of noncrossing partitions of type B given in [**24**], the noncrossing partitions in B_2 with respect to c are the centrally symmetric noncrossing partitions of the cycle shown below.

B2cycle.ps

Figure 2 illustrates the map nc_c for this choice of W and c.

w	1	$\hat{s}_0$	$s_0\hat{s}_1$	$s_0s_1\vert\hat{s}_0$	$\hat{s}_0s_1\vert s_0\hat{s}_1$	$\hat{s}_1$
$nc_c(w)$	1	s_0	$s_0s_1s_0$	$s_1s_0s_1$	$s_0 \cdot s_1$	s_1
	B2.1.ps	*B2.a.ps*	*B2.ab.ps*	*B2.aba.ps*	*B2.ababi.ps* ↓ *B2.ababii.ps*	*B2.b.ps*

Figure 2. The map nc_c

Example 2.3. Covering reflections of a permutation $\pi \in S_{n+1}$ are the transpositions corresponding to descents (pairs (π_i, π_{i+1}) with $\pi_i > \pi_{i+1}$). The map nc_c sends π to the product of these transpositions. The relations $\pi_i \equiv \pi_{i+1}$ for descents (π_i, π_{i+1}) generate an equivalence relation on $[n+1]$ which can be interpreted as a noncrossing partition (in the classical sense) of the cycle c. For $c = (n\ \ n{+}1) \cdots (2\ 3)(1\ 2)$ as in Example 1.8, this map between 231-avoiding permutations and classical (i.e. type A) noncrossing partitions is presumably known.

Remark 2.4. The definition of nc_c is valid for infinite W. However, Theorem 2.1 address the finite case only. In particular, for infinite W it is not even known whether nc_c maps c-sortable elements into the interval $[1, c]_T$.

REMARK 2.5. As a byproduct of Theorem 2.1, we obtain a canonical reduced T-word for every x in $[1,c]_T$. The letters are the canonical generators of the associated parabolic subgroup, or equivalently the cover reflections of $(\mathrm{nc}_c)^{-1}(x)$, occurring in the order induced by the c-sorting word for $(\mathrm{nc}_c)^{-1}(x)$. This is canonical, up to the choice of reduced word for c. Changing the reduced word for c alters the choice of canonical reduced T-word for x by commutations of letters.

In [**1**], it is shown that for c bipartite, the natural labeling of $[1,c]_T$ is an EL-shelling with respect to the reflection order obtained from what we here call the c-sorting word for w_0. In particular, the labels on the unique maximal chain in $[1,x]_T$ constitute a canonical reduced T-word for x. It is apparent from the proof of [**1**, Theorem 3.5(ii)] that these two choices of canonical reduced T-words are identical in the bipartite case, for W crystallographic. Presumably the same is true for non-crystallographic W.

3. Sortable elements and clusters

In this section we define c-clusters, a slight generalization (from crystallographic Coxeter groups to all finite Coxeter groups) of the Γ-clusters of [**16**]. These in turn generalize the clusters of [**11**]. The main result of this section is a bijection between c-sortable elements and c-clusters.

We build the theory of clusters within the framework of Coxeter groups, rather than in the framework of root systems. This is done in order to avoid countless explicit references to the map between positive roots and reflections in what follows. Readers familiar with root systems will easily make the translation to the language of *almost positive roots* of [**11**] and [**16**].

Let $-S$ denote the set $\{-s : s \in S\}$ of formal negatives of the simple generators of W, and let $T_{\geq -1}$ be $T \cup (-S)$. (Recall that T is the set of all reflections of W.) The notation T_J stands for $T \cap W_J$ and $(T_J)_{\geq -1}$ denotes $T_J \cup (-J)$.

For each $s \in S$, define an involution $\sigma_s : T_{\geq -1} \to T_{\geq -1}$ by

$$\sigma_s(t) := \begin{cases} -t & \text{if } t = \pm s, \\ t & \text{if } t \in (-S) \text{ and } t \neq -s, \text{ or} \\ sts & \text{if } t \in T - \{s\}. \end{cases}$$

We now define a symmetric binary relation $\|_c$ called the *c-compatibility* relation.

PROPOSITION 3.1. *There exists a unique family of symmetric binary relations $\|_c$ on $T_{\geq -1}$, indexed by Coxeter elements c, with the following properties:*

(i) *For any $s \in S$, $t \in T_{\geq -1}$ and Coxeter element c,*

$$-s \,\|_c\, t \text{ if and only if } t \in (T_{\langle s \rangle})_{\geq -1}.$$

(ii) *For any $t_1, t_2 \in T_{\geq -1}$ and any initial letter s of c,*

$$t_1 \,\|_c\, t_2 \text{ if and only if } \sigma_s(t_1) \,\|_{scs}\, \sigma_s(t_2).$$

A *c-compatible subset* of $T_{\geq -1}$ is a set of pairwise c-compatible elements of $T_{\geq -1}$. A *c-cluster* is a maximal c-compatible subset. A c-cluster is called *positive* if it contains no element of $-S$.

Let w be a c-sortable element with c-sorting word $a_1 a_2 \cdots a_k$. If $s \in S$ occurs in a then the *last reflection* for s in w is $a_1 a_2 \cdots a_j a_{j-1} \cdots a_2 a_1$, where a_j is the rightmost occurrence of s in a. If s does not occur in a then the last reflection for s in w is the formal negative $-s$. Let $\mathrm{cl}_c(w)$ be the set of last reflections of w. This is an n-element subset of $T_{\geq -1}$. This map does not depend on the choice of reduced word for c, because any two c-sorting words for w are related by commutations of simple generators.

THEOREM 3.2. *The map $w \mapsto \mathrm{cl}_c(w)$ is a bijection from the set of c-sortable elements to the set of c-clusters. Furthermore, cl_c restricts to a bijection between c-sortable elements with full support and positive c-clusters.*

The strategy for proving Theorem 3.2 is the same as for Theorem 2.1, but with fewer complications. We argue by induction on rank and length.

EXAMPLE 3.3. We continue the example of $W = B_2$ and $c = s_0 s_1$. Clusters in B_2 correspond to collections of diagonals which define centrally symmetric triangulations of the hexagon shown below.

B2seed.ps

Each element of $T_{\geq -1}$ is represented by a diameter or a centrally symmetric pair of diagonals. For details, see [**11**, Section 3.5]. Figure 3 illustrates the map cl_c on c-sortable elements for this choice of W and c.

w	1	s_0	$s_0 s_1$	$s_0 s_1 \vert s_0$	$s_0 s_1 \vert s_0 s_1$	s_1
$\mathrm{cl}_c(w)$	$-s_0, -s_1$	$s_0, -s_1$	$s_0, s_0 s_1 s_0$	$s_1 s_0 s_1, s_0 s_1 s_0$	$s_1 s_0 s_1, s_1$	$-s_0, s_1$
	B2cl.1.ps	*B2cl.a.ps*	*B2cl.ab.ps*	*B2cl.aba.ps*	*B2cl.abab.ps*	*B2cl.b.ps*

FIGURE 3. The map cl_c

EXAMPLE 3.4. By way of contrast with Example 2.3, we offer no characterization of the map cl_c on permutations satisfying (A), even in the 231-avoiding case. Such a characterization is not immediately apparent, due to the dependence of $\mathrm{cl}_c(w)$ on a specific choice of reduced word for w.

REMARK 3.5. Even for infinite W, the map cl_c associates to each c-sortable element an n-element subset of $T_{\geq -1}$. However, for infinite W, it is not even clear how c-compatibility should be defined, and in particular the proofs in this section apply to the finite case only. As mentioned in the proof of Proposition 3.1, Theorem 3.2 implies the following characterization of c-compatibility: Distinct elements t_1 and t_2 of $T_{\geq -1}$ are c-compatible if and only if there exists a c-sortable element w such that $t_1, t_2 \in \mathrm{cl}_c(w)$. Thus the map cl_c itself might conceivably provide some insight into compatibility in the infinite case.

4. Enumeration

In this section we briefly discuss the enumeration of sortable elements. The *W-Catalan number* is given by the following formula, in which h is the Coxeter number of W and the e_i are the exponents of W.

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}.$$

The values of the W-Catalan number for finite irreducible W are tabulated below.

A_n	B_n	D_n	E_6	E_7	E_8	F_4	G_2	H_3	H_4	$I_2(m)$
$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8	32	280	$m+2$

The noncrossing partitions (with respect to any Coxeter element) in an irreducible finite Coxeter group W are counted by the W-Catalan number [**3, 18, 24**]. The c-clusters (for any Coxeter element c) of an irreducible finite Coxeter group W are also counted by $\mathrm{Cat}(W)$. This follows from [**16**, Corollary 4.11] and [**11**, Proposition 3.8] for the crystallographic case, or is proved in any finite case by combining Theorems 2.1 and 3.2. We refer the reader to [**13**, Section 5.1] for a brief account of other objects counted by the W-Catalan number. By Theorem 2.1 or Theorem 3.2 we have the following.

THEOREM 4.1. *For any Coxeter element c of W, the c-sortable elements of W are counted by* $\mathrm{Cat}(W)$.

The *positive W-Catalan number* is the number of positive c-clusters (c-clusters containing no element of $-S$). The following is an immediate corollary of Theorem 3.2.

COROLLARY 4.2. *For any Coxeter element c, the c-sortable elements not contained in any proper standard parabolic subgroup are counted by the positive W-Catalan number.*

The map nc_c also respects this positive W-Catalan enumeration: The map nc_c maps the sortable elements not contained in any proper standard parabolic subgroup to the noncrossing partitions not contained in any proper standard parabolic subgroup.

The W-Narayana numbers count noncrossing partitions by their rank. That is, the kth W-Narayana number is the number of elements of $[1, c]_T$ whose absolute length is k. The following is an immediate corollary of Theorem 2.1.

COROLLARY 4.3. *For any Coxeter element c, the c-sortable elements of W which have exactly k descents are counted by the kth W-Narayana number.*

REMARK 4.4. The kth W-Narayana number is also the kth component in the h-vector of the simplicial W-associahedron. Using results from [**20**] and [**22**], one associates a complete fan to c-sortable elements. This fan has the property that any linear extension of the weak order on c-sortable elements is a shelling. In [**23**], David Speyer and the author show that the map cl_c induces a combinatorial isomorphism. Thus as a special case of a general fact explained in the discussion following [**20**, Proposition 3.5], the h-vector of Δ_c has entry h_k equal to the number of c-sortable elements with exactly k descents. This gives an alternate proof of Corollary 4.3 and, by composing bijections, a bijective explanation of why counting noncrossing partitions by rank recovers the h-vector of the W-associahedron.

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