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The combinatorics of frieze patterns and Markoff numbers

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ABSTRACT. This article, based on joint work with Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, Dylan Thurston, and Rui Viana, presents a combinatorial model based on perfect matchings that explains the symmetries of the numerical arrays that Conway and Coxeter dubbed frieze patterns. This matchings model is a combinatorial interpretation of Fomin and Zelevinsky's cluster algebras of type A. One can derive from the matchings model an enumerative meaning for the Markoff numbers, and prove that the associated Laurent polynomials have positive coefficients as was conjectured (much more generally) by Fomin and Zelevinsky. Most of this research was conducted under the auspices of REACH (Research Experiences in Algebraic Combinatorics at Harvard).

RÉSUMÉ. Cet article, basé sur un travail conjoint avec Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, Dylan Thurston, et Rui Viana presente un modèle combinatoire expliquant les symétries dans les tableaux numérique appelés motifs frieze par Conway et Coxeter. Ce modèle, basé sur les couplages parfaits, donne une interprétation combinatoire des algèbre de cluster de type A de Fomin et Zelevinksy. Ce modèle permet de fournir une interprétation énumérative des nombres Markoff, et on peut démontrer que les polynômes de Laurent associés ont des coefficients positifs, ce qui avait été conjecturé (dans un cadre plus général) par Fomin et Zelevinsky. Cette recherche s'est déroulée dans le cadre du programme REACH (Research Experiences in Algebraic Combinatorics at Harvard).

1. Introduction

A *Laurent polynomial* in the variables $x, y, \ldots$ is a polynomial in the variables $x, x^{-1}, y, y^{-1}, \ldots$. Thus the function $f(x) = (x^2+1)/x = x + x^{-1}$ is a Laurent polynomial, but the composition $f(f(x)) = (x^4+3x^2+1)/x(x^2+1)$ is not. This shows that the set of Laurent polynomials in a single variable is not closed under composition. This failure of closure also holds in the multivariate setting; for instance, if $f(x,y)$, $g(x,y)$ and $h(x,y)$ are Laurent polynomials in x and y, then we would not expect to find that $f(g(x,y), h(x,y))$ is a Laurent polynomial as well. Nonetheless, it has been discovered that, in broad class of instances (embraced as yet by no general rule), "fortuitous" cancellations occur that cause Laurentness to be preserved. This is the "Laurent phenomenon" discussed by Fomin and Zelevinsky [**13**].

Furthermore, in many situations where the Laurent phenomenon holds, there is a certain positivity phenomenon at work as well, and all the coefficients of the Laurent polynomials turn out to be positive. In these cases, the functions being composed are Laurent polynomials with positive coefficients; that is, they are expressions involving only addition, multiplication, and division. It should be noted that subtraction-free expressions do not have all the closure properties one might hope for, as the example $(x^3+y^3)/(x+y)$ illustrates: although the expression is subtraction-free, its reduced form $x^2 - xy + y^2$ is not.

Fomin and Zelevinsky have shown that a large part of the Laurentness phenomenon fits in with their general theory of cluster algebras. In this article I will discuss one important special case of the Laurentness-plus-positivity phenomenon, namely the case associated with cluster algebras of type A, discussed in detail in [**14**]. The purely combinatorial approach taken in sections 2 and 3 of my article obscures the links with deeper issues (notably the representation-theoretic questions that motivated the invention of cluster algebras), but it provides the quickest and most self-contained way to prove the Laurentness-plus-positivity assertion in this case (Theorem 3.1). The frieze patterns of Conway and Coxeter, and their link with triangulations of polygons, will play a fundamental role, as will

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perfect matchings of graphs derived from these triangulations. (For a different, more algebraic way of thinking about frieze patterns, see [**3**].)

In sections 4 and 5 of this article, two variations on the theme of frieze patterns are considered. One is the tropical analogue, which has bearing on graph-metrics in trees. The other variant is based on a recurrence that looks very similar to the frieze relation; the variant recurrence appears to give rise to tables of positive integers possessing the same glide-reflection symmetry as frieze patterns, but positivity, integrality, and symmetry are currently still unproved.

In section 6, the constructions of sections 2 and 3 are specialized to a case in which the triangulated polygons come from pairs of mutually visible points in a dissection of the plane into equilateral triangles. In this case, counting the matchings of the derived graphs gives us an enumerative interpretation of Markoff numbers (numbers satisfying the ternary cubic $x^2+y^2+z^2=3xyz$). This yields a combinatorial proof of a Laurentness assertion proved by Fomin and Zelevinsky in [**13**] (namely a special case of their Theorem 1.10) that falls outside of the framework of cluster algebras in the strict sense. Fomin and Zelevinsky proved Theorem 1.10 by use of their versatile "Caterpillar Lemma", but this proof did not settle the issue of positivity. The combinatorial approach adopted here shows that all of the Laurent polynomials that occur in the three-variable rational-function analogue of the Markoff numbers — the "Markoff polynomials" — are in fact positive (Theorem 6.2).

Section 7 concludes with some problems suggested by the main result of section 6. One can try to generalize the combinatorial picture by taking other dissections of the plane into triangles, or one can try to generalize by considering other Diophantine equations. There may be a general link here, but its nature is still obscure.

2. Triangulations and frieze patterns

A **frieze pattern** [**7**] is an infinite array such as

$$
\begin{array}{cccccccccccccc}
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \ldots \\
\ldots & & 1 & & 5 & & \frac{2}{3} & & 3 & & \frac{5}{3} & & 2 & \ldots \\
\ldots & 1 & & 4 & & \frac{7}{3} & & 1 & & 4 & & \frac{7}{3} & & \ldots \\
\ldots & & 3 & & \frac{5}{3} & & 2 & & 1 & & 5 & & \frac{2}{3} & \ldots \\
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \ldots
\end{array}
$$

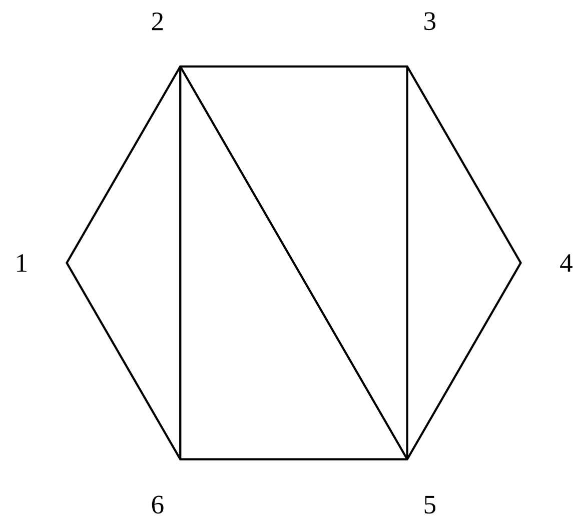
consisting of $n-1$ rows, each periodic with period n, such that all entries in the top and bottom rows are equal to 1 and all entries in the intervening rows satisfy the relation

$$
\begin{array}{ccc}
 & A & \\
B & & C \\
 & D &
\end{array}
\quad : \quad D=(BC-1)/A \ .
$$

The rationale for the term "frieze pattern" is that such an array automatically possesses glide-reflection symmetry (as found in some decorative friezes): for $1 \leq m \leq n-1$, the $n-m$th row is the same as the mth row, shifted. Hence the relation $D=(BC-1)/A$ will be referred to below as the "frieze relation" even though its relation to friezes and their symmetries is not apparent from the algebraic definition.

Frieze patterns arose from Coxeter's study of metric properties of polytopes, and served as useful scaffolding for various sorts of metric data; see e.g. [**9**] (page 160), [**10**], and [**11**]. Typically some of the entries in a frieze pattern are irrational. Conway and Coxeter completely classify the frieze patterns whose entries are positive integers, and show that these frieze patterns constitute a manifestation of the Catalan numbers. Specifically, there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labelled vertices. (In addition to [**7**], see the shorter discussion on pp. 74–76 and 96–97 of [**8**].) Note that for each fixed n, any convex n-gon would serve here just as well as the regular n-gon, since we are only viewing triangulations combinatorially.

From every triangulation T of a regular n-gon with vertices cyclically labelled 1 through n, Conway and Coxeter build an $(n-1)$-rowed frieze pattern determined by the numbers $a_1,a_2,\ldots,a_n$, where a_k is the number of triangles in T incident with vertex k. Specifically: (1) the top row of the array is $\ldots,1,1,1,\ldots$; (2) the second row (offset from the first) is $\ldots,a_1,a_2,\ldots,a_n,a_1,\ldots$ (with period n); and (3) each succeeding row (offset from the one before) is determined by the frieze relation. E.g., the triangulation

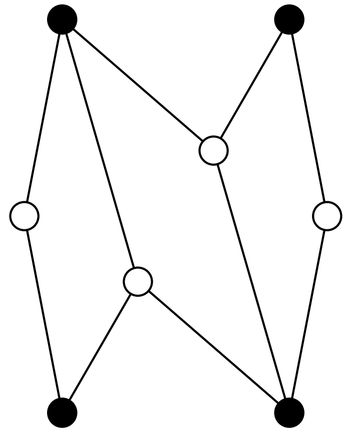


of the 6-gon determines the data $(a_1,\ldots,a_6) = (1,3,2,1,3,2)$ and 5-row frieze pattern

$$
\begin{array}{ccccccccccccccccc}
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \ldots \\
\ldots & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & \ldots \\
\ldots & 1 & & 2 & & 5 & & 1 & & 2 & & 5 & & 1 & \ldots \\
\ldots & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & \ldots \\
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \ldots
\end{array}
$$

Conway and Coxeter show that the frieze relation, applied to the initial rows $\ldots,1,1,1,\ldots$ and $\ldots,a_1,a_2,\ldots,a_n,\ldots$, yields a frieze pattern. Note that implicit in this assertion is the assertion that every entry in rows 1 through $n-3$ is non-zero (so that the recurrence $D = (BC-1)/A$ never involves division by 0). It is not a priori obvious that each of the entries in the array is positive (since the recurrence involves subtraction) or that each of the entries is an integer (since the recurrence involves division). Nor is it immediately clear why for $1 \leq m \leq n-1$, the $n-m$th row of the table given by repeated application of the recurrence should be the same as the mth row, shifted, so that in particular the $n-1$st row, like the first row, consists entirely of 1's.

These and many other properties of frieze patterns are explained by a combinatorial model of frieze patterns discovered by Carroll and Price [**5**] (based on earlier work of Itsara, Le, Musiker, Price, and Viana). Given a triangulation T as above, define a bipartite graph $G = G(T)$ whose n black vertices v correspond to the vertices of T, whose $n-2$ white vertices w correspond to the triangular faces of T, and whose edges correspond to all incidences between vertices and faces in T (that is, v and w are joined by an edge precisely if v is one of the three vertices of the triangle in T associated with w). For $i \neq j$ in the range $1,\ldots,n$, let $G_{i,j}$ be the graph obtained from G by removing black vertices i and j and all edges incident with them, and let $m_{i,j}$ be the number of perfect matchings of $G_{i,j}$ (that is, the number of ways to pair all $n-2$ of the black vertices with the $n-2$ white vertices, so that every vertex is paired to a vertex of the opposite color adjacent to it). For instance, for the triangulation T of the 6-gon defined in the preceding figure, the graph $G_{1,4}$ is



and we put $m_{1,4} = 5$ since this graph has 5 perfect matchings.

THEOREM 2.1 (Gabriel Carroll and Gregory Price [**5**]). *The Conway-Coxeter frieze pattern of a triangulation T is just the array*

$$\begin{array}{ccccccccccc}
\cdots & m_{1,2} & & m_{2,3} & & m_{3,4} & & m_{4,5} & \cdots \\
\cdots & & m_{1,3} & & m_{2,4} & & m_{3,5} & & \cdots \\
\cdots & m_{n,3} & & m_{1,4} & & m_{2,5} & & m_{3,6} & \cdots \\
\cdots & & m_{n,4} & & m_{1,5} & & m_{2,6} & & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

where here as hereafter we interpret all subscripts mod n.

Note that this claim makes the glide-reflection symmetry of frieze patterns a trivial consequence of the fact that $G_{i,j} = G_{j,i}$.

PROOF. Here is a sketch of the main steps of the proof:

(1) $m_{i,i+1} = 1$: This holds because there is a tree structure on the set of triangles in T that induces a tree structure on the set of white vertices of G. If we examine the white vertices of G, proceeding from outermost to innermost, we find that we have no freedom in how to match them with black vertices, when we keep in mind that every black vertex must be matched with a white vertex. (In fact, the same reasoning shows that $m_{i,j} = 1$ whenever the triangulation T contains a diagonal connecting vertices i and j.)

(2) $m_{i-1,i+1} = a_i$: The argument is similar, except now we have some freedom in how the ith black vertex is matched: it can be matched with any of the a_i adjacent white vertices.

(3) $m_{i,j}\, m_{i-1,j+1} = m_{i-1,j}\, m_{i,j+1} - 1$: If we move the 1 to the left-hand side, we can use (1) to write the equation in the form

$$m_{i,j}\, m_{i-1,j+1} + m_{i-1,i}\, m_{j,j+1} = m_{i-1,j}\, m_{i,j+1}.$$

This relation is a direct consequence of a lemma due to Eric Kuo (Theorem 2.5 in [**17**]), which I state here for the reader's convenience:

Condensation lemma: If a bipartite planar graph G has 2 more black vertices than white vertices, and the black vertices a,b,c,d lie in cyclic order on some face of G, then

$$m(a,c)m(b,d) = m(a,b)m(c,d) + m(a,d)m(b,c),$$

where $m(x,y)$ denotes the number of perfect matchings of the graph obtained from G by deleting vertices x and y and all incident edges.

(1) and (2) tell us that Carroll and Price's theorem applies to the first two rows of the frieze pattern, and (3) tells us (by induction) that the theorem applies to all subsequent rows. □

It should be mentioned that Conway and Coxeter give an alternative way of describing the entries in frieze patterns, as determinants of tridiagonal matrices. Note that $m_{i-1,i+1} = a_i$ which equals the determinant of the 1-by-1 matrix whose sole entry is a_i, while $m_{i-1,i+2} = a_i a_{i+1} - 1$ which equals the determinant of the 2-by-2 matrix

$$\begin{pmatrix} a_i & 1 \\ 1 & a_{i+1} \end{pmatrix}.$$

One can show by induction using Dodgson's determinant identity (for a statement and a pretty proof of this identity see [**21**]) that $m_{i-1,i+k}$ equals the determinant of the k-by-k matrix with $a_i, \dots, a_{i+k-1}$ down the diagonal, 1's in the two flanking diagonals, and 0's everywhere else. This is true for any arrays satisfying the frieze relation whose initial row consists of 1's, whether or not it is a frieze pattern. Thus, any numerical array constructed via the frieze relation from initial data consisting of a first row of 1's and a second row of positive integers will be an array of positive integers; entries in subsequent rows will be positive since they are defined by subtraction-free expressions, and they will be integers since they are equal to determinants of integer matrices. (One caveat is in order here: although the table of tridiagonal determinants always satisfies the frieze relation, it may not be possible to compute the table using just the frieze relation, since some of the expressions that arise might be indeterminate fractions of the form $0/0$.) However, for most choices of positive integers $a_1, \dots, a_n$, the resulting table of positive integers will not be an $(n-1)$-rowed frieze pattern. Indeed, Conway and Coxeter show that every $(n-1)$-rowed frieze pattern whose entries are positive integers arises from a triangulated n-gon in the fashion described above.

3. The sideways construction and its periodicity

Recall that any $(n-1)$-rowed array of real numbers that begins and ends with rows of 1's and satisfies the frieze relation in between qualifies as a frieze pattern.

Note that if the vertices $1, \dots, n$ of an n-gon lie on a circle and we let $d_{i,j}$ be the distance between points i and j, Ptolemy's theorem on the lengths of the sides and diagonals of an inscriptible quadrilateral gives us the three-term quadratic relation

$$d_{i,j}\, d_{i-1,j+1} + d_{i-1,i}\, d_{j-1,j} = d_{i-1,j}\, d_{i,j+1}$$

(with all subscripts interpreted mod n). Hence the numbers $d_{i,j}$ with $i \neq j$, arranged just as the numbers $m_{i,j}$ were, form an $(n-1)$-rowed array that almost qualifies as a frieze pattern (the array satisfies the frieze relation and has glide-reflection symmetry because $m_{i,j} = m_{j,i}$ for all i, j, but the top and bottom rows do not in general consist of 1's). The nicest case occurs when the n-gon is a regular n-gon of side-length 1; then we get a genuine frieze pattern and each row of the frieze pattern is constant.

Another source of frieze patterns is an old result from spherical geometry: the *pentagramma mirificum* of Napier and Gauss embodies the assertion that the arc-lengths of the sides in a right-angled spherical pentagram can be arranged to form the middle two rows of a four-rowed frieze pattern.

Conway and Coxeter show that frieze patterns are easy to construct if one proceeds not from top to bottom (since one is unlikely to choose numbers $a_1, \dots, a_n$ in the second row that will yield all 1's in the $(n-1)$st row) but from left to right, starting with a zig-zag of entries connecting the top and bottom rows (where the zig-zag path need not alternate between leftward steps and rightward steps but may consist of any pattern of leftward steps and rightward steps), using the sideways frieze relation

$$\begin{array}{ccc} & A & \\ B & & C \\ & D & \end{array} \quad : \quad C = (AD+1)/B$$

E.g., consider the partial frieze pattern

$$\begin{array}{cccccccccc} \dots & 1 & & 1 & & 1 & & 1 & 1 & \dots \\ & & x & & x' & & & & & \\ & y & & y' & & & & & & \\ & & z & & z' & & & & & \\ \dots & 1 & & 1 & & 1 & & 1 & 1 & \dots \end{array}$$

Given non-zero values of x, y, and z, one can successively compute $y' = (xz+1)/y$, $x' = (y+1)/x$, and $z' = (y+1)/z$, obtaining a new zig-zag of entries x', y', z' connecting the top and bottom rows. For generic choices of non-zero x, y, z, one has x', y', z' non-zero as well, so the procedure can be repeated, yielding further zig-zags of entries. Happily (and perhaps surprisingly), after six iterations of the procedure one will recover the original numbers x, y, z six places to the right of their original position (unless one has unluckily chosen x, y, z so as to cause one to encounter an indeterminate expression of the form $0/0$ from the recurrence).

To dodge the issue of indeterminate expressions, we embrace indeterminacy by regarding x, y, z as formal quantities, not specific numbers, so that x', y', z', etc. become rational functions of x, y, and z. Then our recurrence ceases to be problematic. Indeed, one finds that the rational functions that arise are of a special kind, namely, Laurent polynomials with positive coefficients.

We can see why this is so by incorporating weighted edges into our matchings model. Returning to the triangulated hexagon from section 2, associate the values x, y, and z with the diagonals joining vertices 2 and 6, vertices 2 and 5, and vertices 3 and 5, respectively. Call these the formal weights of the diagonals. Also assign weight 1 to each of the 6 sides of the hexagon. Now construct the graph G as before, only this time assigning weights to all the edges. Specifically, if v is a black vertex of G that corresponds to a vertex of the n-gon and w is a white vertex of G that corresponds to a triangle in the triangulation T that has v as one of its three vertices (and has v' and v'' as the other two vertices), then the edge in G that joins v and w should be assigned the weight of the side or diagonal in T that joins v' and v''. We now define $W_{i,j}$ as the sum of the weights of all the perfect matchings of the graph $G_{i,j}$ obtained by deleting vertices i and j (and all their incident edges) from G, where the weight of a perfect matching is the product of the weights of its constituent edges, and we define $M_{i,j}$ as $W_{i,j}$ divided by the product of the weights of all the diagonals (this product is xyz in our running example). These $M_{i,j}$'s, which are rational functions of x, y, and z, generalize the numbers denoted by $m_{i,j}$ earlier, since we recover the $m_{i,j}$'s from the $M_{i,j}$'s by setting $x = y = z = 1$. It is clear that each $W_{i,j}$ is a polynomial with positive coefficients, so each $M_{i,j}$ is a Laurent polynomial with positive coefficients. And, because of the normalization (division by xyz), we have gotten each $M_{i,i+1}$ to equal 1. So the table of rational functions $M_{i,j}$ is exactly what we get by running our recurrence from left to right. When we pass from x, y, z to x', y', z', we are effectively rotating our triangulation by one-sixth of a full turn; six iterations bring us back to where we started.

We have proved:

THEOREM 3.1. *Given any assignment of formal weights to $n-3$ entries in an $(n-1)$-rowed table that form a zig-zag joining the top row (consisting of all 1's) to the bottom row (consisting of all 1's), there is a unique assignment of rational functions to all the entries in the table so that the frieze relation is satisfied. These rational functions of the original $n-3$ variables have glide-reflection symmetry that gives each row period n. Furthermore, each of the rational functions in the table is a Laurent polynomial with positive coefficients.*

Note that a zig-zag joining the top row to the bottom row corresponds to a triangulation T whose dual tree is just a path. Not every triangulation is of this kind. In general, the entries in a frieze pattern that correspond to the diagonals of a triangulation T do not form a zig-zag path, so it is not clear from the frieze pattern how to extend the known entries to the unknown entries. In such a case, it is best to refer directly to the triangulation itself, and to use a generalization of the frieze relation, namely the (formal) Ptolemy relation [**5**]

$$M_{i,j}M_{k,l} + M_{j,k}M_{i,l} = M_{i,k}M_{j,l}$$

where i, j, k, l are four vertices of the n-gon listed in cyclic order. Since every triangulation of a convex n-gon can be obtained from every other by means of flips that replace one diagonal of a quadrilateral by the other diagonal, we can iterate the Ptolemy relation so as to solve for all of the $M_{i,j}$'s in terms of the ones whose values were given.

Up until now we have allowed the diagonals, but not the sides, of our n-gon to have indeterminate weights; that is, the sides have all had weight 1. We can remedy this seeming lack of generality by noting that if we multiply the weights of the three sides of any triangle in the triangulation T by some constant c, the effect is to multiply by c the weights of three edges of the graph G, namely, the three edges incident with the white vertex w associated with T. This has the effect of multiplying the weight of every perfect matching of every graph $G_{i,j}$ by c, and such a scaling has no effect on the Laurentness phenomenon.

Our combinatorial construction of Laurent polynomials associated with the diagonals of an n-gon is essentially nothing more than the type A case (more precisely, the A_{n-3} case) of the cluster algebra construction of Fomin and Zelevinsky [**14**]. The result that our matchings model yields, stated in a self-contained way, is as follows:

THEOREM 3.2. *Given any assignment of formal weights $x_{i,j}$ to the $2n-3$ edges of a triangulated convex n-gon, there is a unique assignment of rational functions to all $n(n-3)/2$ diagonals of the n-gon such that the rational* *functions assigned to the four sides and two diagonals of any quadrilateral determined by four of the n vertices satisfies the Ptolemy relation. These rational functions of the original* $2n-3$ *variables are Laurent polynomials with positive coefficients.*

The formal weights are precisely the cluster variables in the cluster algebra of type A_{n-3}, and the triangulations are the clusters. The periodicity phenomenon is a special case of a more general periodicity conjectured by Zamolodchikov and proved in the type A case independently by Frenkel and Szenes and by Gliozzi and Tateo; see [**14**] for details.

4. The tropical analogue

Since the sideways frieze relation involves only subtraction-free expressions in the cluster variables, our whole picture admits a tropical analogue (for background on tropical mathematics, see [**19**]) in which multiplication is replaced by addition, division by subtraction, addition by max, and 1 by 0. (One could use min instead of max, but max will be more useful for us.) In this new picture, the Ptolemy relation

$$d_{i,j}\,d_{k,l}+d_{j,k}\,d_{i,l}=d_{i,k}\,d_{j,l}$$

becomes the ultrametric relation

$$\max(d_{i,j}+d_{k,l},d_{j,k}+d_{i,l})=d_{i,k}+d_{j,l}.$$

Metrics satisfying this relation arise from finite collections of non-intersecting arcs that join points on the sides of the n-gon (not vertices) in pairs (which we will call finite laminations). For any pair of vertices i,j, we define $d_{i,j}$ as the smallest possible number of intersections between a path in the n-gon from i to j and the arcs in the finite lamination (we choose the path so as to avoid crossing any arc in the finite lamination more than once). Then these quantities $d_{i,j}$ satisfy the ultrametric relation. As in the non-tropical case, we can find all the quantities $d_{i,j}$ once we know the values associated with the sides of the n-gon and the diagonals belonging to some triangulation.

For an alternative picture, one can divide the laminated n-gon into a finite number of sub-regions, each of which is bounded by pieces of the boundary of the n-gon and/or arcs of the finite lamination; the vertices of the n-gon correspond to n special sub-regions (some of which may coincide with one another, if there is no arc in the finite lamination separating the associated vertices of the n-gon). Then the dual of this dissection of the n-gon is a tree with n specified leaf vertices (some of which may coincide), and $d_{i,j}$ is the graph-theoretic distance between leaf i and leaf j (which could be zero). We see that if we know $2n-3$ of these leaf-to-leaf distances, and the $2n-3$ pairs of leaves correspond to the sides and diagonals of a triangulated n-gon, then all of the other leaf-to-leaf distances can be expressed as piecewise-linear functions of the $2n-3$ specified distances. (For more on the graph metric on trees, see [**2**].)

5. A variant

Before leaving the topic of frieze patterns, I mention an open problem concerning a variant of Conway and Coxeter's definition, in which the frieze relation is replaced by the relation

$$\begin{array}{ccc} & A & \\ B & C & D \\ & E & \end{array} \quad : \; E=(BD-C)/A$$

and its sideways version

$$\begin{array}{ccc} & A & \\ B & C & D \\ & E & \end{array} \quad : \; D=(AE+C)/B\ .$$

Here, too, it appears that we can construct arrays that have the same sort of symmetries as frieze patterns by starting with a suitable zig-zag of entries (where successive downwards steps can go left, right, or straight) and proceeding from left to right. E.g., consider the partial table

$$\begin{array}{ccccccc} \ldots & 1 & 1 & 1 & 1 & 1 & \ldots \\ & & A & D & x & & \\ & B & E & y & & & \\ & & C & F & z & & \\ \ldots & 1 & 1 & 1 & 1 & 1 & \ldots \end{array}$$

where $A,\ldots,F$ are pre-specified, and where we compute $y = (AC+E)/B$, $x = (y+D)/A$, $z = (y+F)/C$, etc. Then one can check that after exactly fourteen iterations of the procedure, one gets back the original numbers (in their original order). Moreover, along the way one sees Laurent polynomials with positive coefficients.

Define a "double zig-zag" to be a subset of the entries of an $(n-2)$-rowed table consisting of a pair of adjacent entries in each of the middle $n-4$ rows, such that the pair in each row is displaced with respect to the pair in the preceding and succeeding rows by at most one position.

CONJECTURE: Given any assignment of formal weights to the $2(n-4)$ entries in a double zig-zag in an $(n-2)$-rowed table, there is a unique assignment of rational functions to all the entries in the table so that the variant frieze relation is satisfied. These rational functions of the original $2(n-4)$ variables have glide-reflection symmetry that gives each row period $2n$. Furthermore, each of the rational functions in the table is a Laurent polynomial with positive coefficients.

There ought to be a way to prove this by constructing the numerators of these Laurent polynomials as sums of weights of perfect matchings of some suitable graph (or perhaps sums of weights of combinatorial objects more general than perfect matchings), and the numerators undoubtedly contain abundant clues as to how this can be done.

For $n = 5,6,7,8$, it appears that the number of positive integer arrays satisfying the variant frieze relation is respectively 1, 5, 51, 868. This variant of the Catalan sequence does not appear to have been studied before. However, it should be said that these numbers were not computed in a rigorous fashion. Indeed, it is not clear that there really is a variant of the Catalan sequence operating here; that is to say, it is conceivable that beyond some point, the sequence becomes infinite (i.e., for some n there could be infinitely many $(n-2)$-rowed positive integer arrays satisfying the variant frieze relation).

6. Markoff numbers

A **Markoff triple** is a triple (x,y,z) of positive integers satisfying $x^2+y^2+z^2 = 3xyz$; e.g., the triple (2,5,29). A **Markoff number** is a positive integer that occurs in at least one such triple.

Writing the Markoff equation as $z^2-(3xy)z+(x^2+y^2) = 0$, a quadratic equation in z, we see that if (x,y,z) is a Markoff triple, then so is (x,y,z'), where $z' = 3xy-z = (x^2+y^2)/z$, the other root of the quadratic in z. (z' is positive because $z' = (x^2+y^2)/z$, and is an integer because $z' = 3xy-z$.) Likewise for x and y.

The following claim is well-known (for an elegant proof, see [**1**]): Every Markoff triple (x,y,z) can be obtained from the Markoff triple $(1,1,1)$ by a sequence of such exchange operations, in fact, by a sequence of exchange operations that leaves two numbers alone and increases the third. E.g., $(1,1,1) \to (2,1,1) \to (2,5,1) \to (2,5,29)$.

Create a graph whose vertices are the Markoff triples and whose edges correspond to the exchange operations $(x,y,z) \to (x',y,z)$, $(x,y,z) \to (x,y',z)$, $(x,y,z) \to (x,y,z')$ where $x' = \frac{y^2+z^2}{x}$, $y' = \frac{x^2+z^2}{y}$, $z' = \frac{x^2+y^2}{z}$. This 3-regular graph is connected (see the claim in the preceding paragraph), and it is not hard to show that it is acyclic. Hence the graph is the 3-regular infinite tree.

This tree can be understood as the dual of the triangulation of the upper half plane by images of the modular domain under the action of the modular group. Concretely, we can describe this picture by using Conway's terminology of "lax vectors", "lax bases", and "lax superbases" ([**6**]).

A **primitive** vector $\mathbf{u}$ in a lattice L is one that cannot be written as $k\mathbf{v}$ for some vector $\mathbf{v}$ in L, with $k > 1$. A **lax vector** is a primitive vector defined only up to sign; if $\mathbf{u}$ is a primitive vector, the associated lax vector is written $\pm\mathbf{u}$. A **lax base** for L is a set of two lax vectors $\{\pm\mathbf{u},\pm\mathbf{v}\}$ such that $\mathbf{u}$ and $\mathbf{v}$ form a basis for L. A **lax superbase** for L is a set of three lax vectors $\{\pm\mathbf{u},\pm\mathbf{v},\pm\mathbf{w}\}$ such that $\pm\mathbf{u}\pm\mathbf{v}\pm\mathbf{w} = \mathbf{0}$ (with appropriate choice of signs) and any two of $\mathbf{u},\mathbf{v},\mathbf{w}$ form a basis for L.

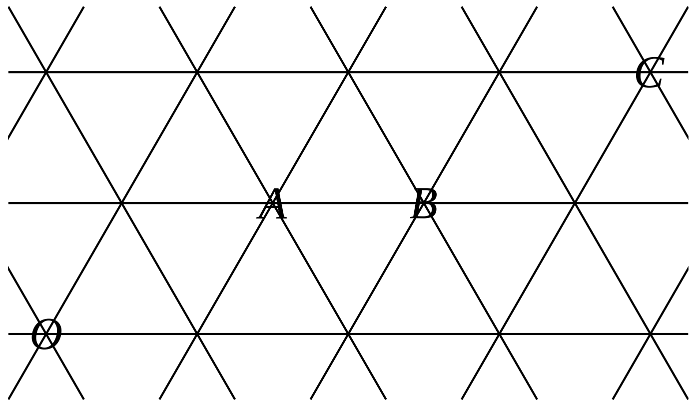
Each superbase $\{\pm\mathbf{u},\pm\mathbf{v},\pm\mathbf{w}\}$ contains the three bases $\{\pm\mathbf{u},\pm\mathbf{v}\}$, $\{\pm\mathbf{u},\pm\mathbf{w}\}$, $\{\pm\mathbf{v},\pm\mathbf{w}\}$ and no others. In the other direction, each base $\{\pm\mathbf{u},\pm\mathbf{v}\}$ is in the two superbases $\{\pm\mathbf{u},\pm\mathbf{v},\pm(\mathbf{u}+\mathbf{v})\}$, $\{\pm\mathbf{u},\pm\mathbf{v},\pm(\mathbf{u}-\mathbf{v})\}$ and no others.

The **topograph** is the graph whose vertices are lax superbases and whose edges are lax bases, where each superbase is incident with the three bases in it. This gives a 3-valent tree whose vertices correspond to the lax superbases of L, whose edges correspond to the lax bases of L, and whose "faces" correspond to the lax vectors in L.

The lattice L that we will want to use is the triangular lattice $L = \{(x,y,z) \in \mathbb{Z}^3 \,:\, x+y+z = 0\}$ (or $\mathbb{Z}^3/\mathbb{Z}\mathbf{v}$ where $\mathbf{v} = (1,1,1)$, if you prefer).

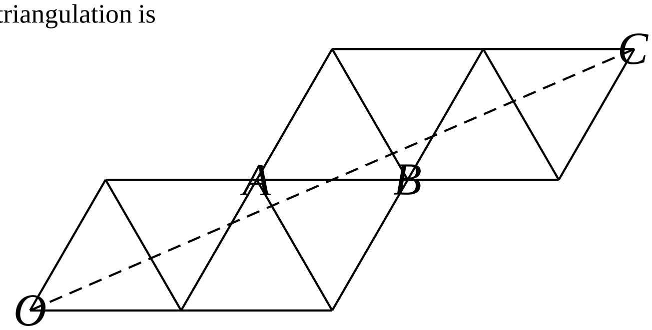
Using this terminology, I can now state the main idea of this section: Unordered Markoff triples are associated with lax superbases of the triangular lattice, and Markoff numbers with lax vectors of the triangular lattice. For example, the unordered Markoff triple $2,5,29$ will correspond to the lax superbase $\{\pm\mathbf{u},\pm\mathbf{v},\pm\mathbf{w}\}$ where $\mathbf{u} = \vec{OA}$,

$\mathbf{v} = \vec{OB}$, and $\mathbf{w} = \vec{OC}$, with O, A, B, and C forming a fundamental parallelogram for the triangular lattice, as shown below.



The Markoff numbers 1, 2, 5, and 29 will correspond to the primitive vectors $\vec{AB}$, $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$.

To find the Markoff number associated with a primitive vector $\vec{OX}$, take the union R of all the triangles that segment OX passes through. The underlying lattice provides a triangulation of R. E.g., for the vector $\mathbf{u} = \vec{OC}$ from the previous figure, the triangulation is



Turn this into a planar bipartite graph as in Part I, let $G(\mathbf{u})$ be the graph that results from deleting vertices O and C, and let $M(\mathbf{u})$ be the number of perfect matchings of $G(\mathbf{u})$. (If $\mathbf{u}$ is a shortest vector in the lattice, put $M(\mathbf{u}) = 1$.)

THEOREM 6.1 (Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, and Rui Viana). *If $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a lax superbase of the triangular lattice, then $(M(\mathbf{u}),M(\mathbf{v}),M(\mathbf{w}))$ is a Markoff triple. Every Markoff triple arises in this fashion. In particular, if $\mathbf{u}$ is a primitive vector, then $M(\mathbf{u})$ is a Markoff number, and every Markoff number arises in this fashion.*

(The association of Markoff numbers with the topograph is not new; what is new is the combinatorial interpretation of the association, by way of perfect matchings.)

PROOF. The base case, with

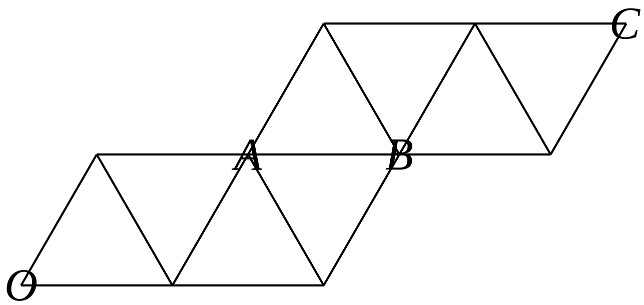
$$(M(\mathbf{e}_1),M(\mathbf{e}_2),M(\mathbf{e}_3)) = (1,1,1),$$

is clear. The only non-trivial part of the proof is the verification that

$$M(\mathbf{u}+\mathbf{v}) = (M(\mathbf{u})^2 + M(\mathbf{v})^2)/M(\mathbf{u}-\mathbf{v}).$$

E.g., in the picture below, we need to verify that

$$M(\vec{OC})M(\vec{AB}) = M(\vec{OA})^2 + M(\vec{OB})^2.$$



But if we rewrite the desired equation as

$$M(\vec{OC})M(\vec{AB}) = M(\vec{OA})M(\vec{BC}) + M(\vec{OB})M(\vec{AC})$$

we see that this is just Kuo's lemma. $\square$

Remark 1: Some of the work done by the REACH students used a square lattice picture; this way of interpreting the Markoff numbers combinatorially was actually discovered first, in 2001–2002 (see [**4**]).

Remark 2: the original combinatorial model for the Conway-Coxeter numbers (found by Price) involved paths, not perfect matchings. Carroll turned this into a perfect matchings model, which made it possible to arrive at the matchings model of Itsara, Le, Musiker, and Viana via a different route.

More generally, one can put $M(\mathbf{e}_1) = x$, $M(\mathbf{e}_2) = y$, and $M(\mathbf{e}_3) = z$ (with $x, y, z > 0$) and recursively define

$$M(\mathbf{u}+\mathbf{v}) = (M(\mathbf{u})^2 + M(\mathbf{v})^2)/M(\mathbf{u}-\mathbf{v}).$$

Then for all primitive vectors $\mathbf{u}$, $M(\mathbf{u})$ is a Laurent polynomial in x, y, z; that is, it can be written in the form $P(x,y,z)/x^a y^b z^c$, where $P(x,y,z)$ is an ordinary polynomial in x, y, z (with non-zero constant term). The numerator $P(x,y,z)$ of each Markoff polynomial is the sum of the weights of all the perfect matchings of the graph $G(\mathbf{u})$, where edges have weight x, y, or z according to orientation. The triples $X = M(\mathbf{u})$, $Y = M(\mathbf{v})$, $Z = M(\mathbf{w})$ of rational functions associated with lax superbases are solutions of the equation

$$X^2 + Y^2 + Z^2 = \frac{x^2 + y^2 + z^2}{xyz} XYZ.$$

We have seen that these numerators $P(x,y,z)$ are polynomials with positive coefficients. This proves the following theorem:

THEOREM 6.2. *Consider the initial triple (x,y,z), along with any triple of rational functions in x, y, and z that can be obtained from the initial triple by a sequence of operations of the form $(X,Y,Z) \mapsto (X',Y,Z)$, $(X,Y,Z) \mapsto (X,Y',Z)$, or $(X,Y,Z) \mapsto (X,Y,Z')$, where $X' = (Y^2+Z^2)/X$, $Y' = (X^2+Z^2)/Y$, and $Z' = (X^2+Y^2)/Z$, Every rational function of x, y, and z that occurs in such a triple is a Laurent polynomial with positive coefficients.*

Fomin and Zelevinsky proved in [**13**] (Theorem 1.10) that the rational functions $X(x,y,z), Y(x,y,z), Z(x,y,z)$ are Laurent polynomials, but their methods did not prove positivity. An alternative proof of positivity, based on topological ideas, was given by Dylan Thurston [**20**].

It can be shown that if $\mathbf{u}$ inside the cone generated by $+\mathbf{e}_1$ and $-\mathbf{e}_3$, then $a < b > c$ and $(c+1)\mathbf{e}_1 - (a+1)\mathbf{e}_3 = \mathbf{u}$. (Likewise for the other sectors of L.) This implies that all the "Markoff polynomials" $M(\mathbf{u})$ are distinct (aside from the fact that $M(\mathbf{u}) = M(-\mathbf{u})$), and thus $M(\mathbf{u})(x,y,z) \neq M(\mathbf{v})(x,y,z)$ for all primitive vectors $\mathbf{u} \neq \pm\mathbf{v}$ as long as (x,y,z) lies in a dense G_δ set of real triples. This fact can be used to show [**20**] that, for a generic choice of hyperbolic structure on the once-punctured torus, no two simple geodesics have the same length.

7. Other directions for exploration

7.1. Other ternary cubics. Neil Herriot (another member of REACH) showed [**15**] that if we replace the triangular lattice used above by the tiling of the plane by isosceles right triangles (generated from one such triangle by repeated reflection in the sides), superbases of the square lattice correspond to triples (x,y,z) of positive integers satisfying either

$$x^2 + y^2 + 2z^2 = 4xyz$$

or

$$x^2 + 2y^2 + 2z^2 = 4xyz.$$

(Note that these two Diophantine equations are essentially equivalent, as the map $(x,y,z) \mapsto (2z,y,x)$ gives a bijection between solutions to the former and solutions to the latter.) This result, considered in conjunction with the result on Markoff numbers, raises the question of whether there might be some more general combinatorial approach to ternary cubic equations of similar shape.

Rosenberger [**18**] showed that there are exactly three ternary cubic equations of the shape $ax^2+by^2+cz^2 = (a+b+c)xyz$ for which all the positive integer solutions can be derived from the solution $(x,y,z) = (1,1,1)$ by means of the exchange operations $(x,y,z) \to (x',y,z)$, $(x,y,z) \to (x,y',z)$, and $(x,y,z) \to (x,y,z')$, with $x' = (by^2+cz^2)/ax$, $y' = (ax^2+cz^2)/by$, and $z' = (ax^2+by^2)/cz$. These three ternary cubic equations are

$$x^2+y^2+z^2 = 3xyz,$$

$$x^2+y^2+2z^2 = 4xyz,$$

and

$$x^2+2y^2+3z^2 = 6xyz.$$

Note that the triples of coefficients that occur here — (1,1,1), (1,1,2), and (1,2,3) — are precisely the triples that occur in the classification of finite reflection groups in the plane. Specifically, the ratios 1:1:1, 1:1:2, and 1:2:3 describe the angles of the three triangles — the 60-60-60 triangle, the 45-45-90 triangle, and the 30-60-90 triangle — that arise as the fundamental domains of the three irreducible two-dimensional reflection groups.

Since the solutions to the ternary cubic $x^2+y^2+z^2 = 3xyz$ describe properties of the tiling of the plane by 60-60-60 triangles, and solutions to the ternary cubic $x^2+y^2+2z^2 = 4xyz$ describe properties of the tiling of the plane by 45-45-90 triangles, the solutions to the ternary cubic $x^2+2y^2+3z^2 = 6xyz$ "ought" to be associated with some combinatorial model involving the reflection-tiling of the plane by 30-60-90 triangles. Unfortunately, the most obvious approach (based on analogy with the 60-60-60 and 45-45-90 cases) does not work. So we are left with two problems that may or may not be related: first, to find a combinatorial interpretation for the integers (or, more generally, the Laurent polynomials) that arise from solving the ternary cubic $x^2+2y^2+3z^2 = 6xyz$; and second, to find algebraic recurrences that govern the integers (or, more generally, the Laurent polynomials) that arise from counting (or summing the weights of) perfect matchings of graphs derived from the reflection-tiling of the plane by 30-60-90 triangles.

If there is a way to make the analogy work, one might seek to extend the analysis to other ternary cubics. It is clear how this might generalize on the algebraic side. On the geometric side, one might drop the requirement that the triangle tile the plane by reflection, and insist only that each angle be a rational multiple of 360 degrees. There is a relatively well-developed theory of "billiards flow" in such a triangle (see e.g. [**16**]) where a particle inside the triangle bounces off the sides following the law of reflection (angle of incidence equals angle of reflection) and travels along a straight line in between bounces. The path of such a particle can be unfolded by repeatedly reflecting the triangular domain in the side that the particle is bouncing off of, so that the unfolded path of the particle is just a straight line in the plane. Of special interest in the theory of billiards are trajectories joining a corner to a corner (possibly the same corner or possibly a different one); these are called saddle connections. The reflected images of the triangular domain form a triangulated polygon, and the saddle connection itself is a combinatorial diagonal of this polygon. It is unclear whether the combinatorics of such triangulations might contain dynamical information about the billiards flow, but if this prospect were to be explored, enumeration of matchings on the derived bipartite graphs would be one thing to try.

7.2. More variables. Another natural variant of the Markoff equation is $w^2+x^2+y^2+z^2 = 4wxyz$ (one special representative of a broader class called Markoff-Hurwitz equations; see [**1**]). The Laurent phenomenon applies here too: The four natural exchange operations convert an initial formal solution (w,x,y,z) into a quadruple of Laurent polynomials. (This is a special case of Theorem 1.10 in [**13**].)

Furthermore, the coefficients of these Laurent polynomials appear to be positive, although this has not been proved.

The numerators of these Laurent polynomials ought to be weight-enumerators for some combinatorial model, but I have no idea what this model looks like.

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