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Green polynomials at roots of unity and its application

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Abstract. We consider Green polynomials at roots of unity. We obtain a recursive formula for Green polynomials at appropriate roots of unity, which is described in a combinatorial manner. The coefficients of the recursive formula are realized by the number of permutations satisfying a certain condition, which leads to interpretation of a combinatorial property of certain graded modules of the symmetric group in terms of representation theory.

Résumé. Nous étudions les polynômes de Green évalués en les racines de l'unité. Nous obtenons une formule récursive pour ces polynômes en certaines racines de l'unité, que nous décrivons combinatoirement. Les coefficients de cette formule récursive énumèrent certaines permutations, ce qui permet d'interpréter une propriété combinatoire de certains modules du groupe symétrique, en termes de la théorie de la représentation.

1. Introduction

The Green polynomials $Q^{\mu}_{\rho}(q)$ at roots of unity are considered. We handle Green polynomials $Q^{\mu}_{\rho}(q)$ of type A for any partition μ, and consider the behavior of them at l-th roots of unity ζ_l, where l is not larger than the maximum multiplicity M_μ of μ. We describe a certain recursive formula of Green polynomials $Q^{\mu}_{\rho}(q)$ at $q = \zeta_l$ for the partition ρ satisfying $Q^{\mu}_{\rho}(\zeta_l) \neq 0$. The results of Lascoux-Leclerc-Thibon on Hall-Littlewood functions at roots of unity play an important role in the argument. Our result includes the result of Lascoux-Leclerc-Thibon on Green polynomials as a special case.

We also consider the recursive formula in terms of representation theory of the symmetric group S_n. It is known that the Green polynomials give the graded characters of a family of graded representations of the symmetric group, called the DeConcini-Procesi-Tanisaki algebras, which includes the coinvariant algebra as a special case. The DeConcini-Procesi-Tanisaki algebra R_μ was first introduced by C. DeConcini and C. Procesi [**DP**] as an algebraic model of the cohomology ring of a certain subvariety of the flag variety parametrized by a partition μ, and T. Tanisaki [**T**] gives simple generators of the defining ideal of the algebra, described by combinatorial information on the partition μ. The DeConcini-Procesi-Tanisaki algebra R_μ has a structure of graded S_n-modules, and the Green polynomial $Q^{\mu}_{\rho}(q)$ gives its graded character values at the conjugacy class of which cycle type is ρ. The recursive formula is equivalent to some representation theoretical interpretation of a certain combinatorial property on the Hilbert polynomial $\mathrm{Hilb}_{R_\mu}(q)$ of R_μ, that is, $\mathrm{Hilb}_{R_\mu}(q)$ has l-th roots of unity ζ_l^j $(j = 1, 2, \ldots, l-1)$ as its zeros for each positive integer l not larger than the maximum multiplicity M_μ of μ. This property of the Hilbert polynomial is equivalent to the fact that the direct sums $R_\mu(k; l)$ $(k = 0, 1, \ldots, l-1)$ of the homogeneous components of R_μ of which degrees are congruent to k modulo l, have the same dimension. The recursive formula shows that there exists a subgroup $H_\mu(l)$ of S_n and $H_\mu(l)$-modules $Z_\mu(k; l)$ of equal dimension such that each $R_\mu(k; l)$ is induced from the corresponding $H_\mu(l)$-modules $Z_\mu(k; l)$ for each $k = 0, 1, \ldots, l-1$, which could be regarded as a representation theoretical interpretation of the property 'coincidence of dimensions'. This work is a sequel of [Mt, MN1, MN2].

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2. Preliminaries

We follow [M] for fundamental notation. Let n be a positive integer and μ a partition of n. Define M_μ to be the maximum multiplicity of the partition μ:

$$M_\mu := \max\{m_1(\mu), m_2(\mu), \cdots, m_n(\mu)\},$$

where $m_i = m_i(\mu)$ denotes the multiplicity of i in the sequence μ. Let μ and ρ be partitions and let q be an indeterminate. The Green polynomial $X^\mu_\rho(q)$ is defined to be the coefficients of the Hall-Littlewood function $P_\mu(x;q)$ in the linear expansion

$$p_\rho(x) = \sum_\mu X^\mu_\rho(q) P_\mu(x;q),$$

where $p_\rho(x)$ denotes the power-sum function corresponding to the partition ρ, and the sum is over partitions μ of the same size as ρ. We also define the polynomial $Q^\mu_\rho(q)$ for partitions μ and ρ of the same size by

$$Q^\mu_\rho(q) = q^{n(\mu)} X^\mu_\rho(q^{-1}),$$

where $n(\mu) = \sum_{i\geq 1}(i-1)\mu_i$ if $\mu = (\mu_1, \mu_2, \dots)$. The polynomial $Q^\mu_\rho(q)$ is also called the Green polynomial. The Green polynomial $Q^\mu_\rho(q)$ is a polynomial with integer coefficients whose degree is $n(\mu)$, which was introduced by J. A. Green [**Gr**] to describe irreducible character values of the general linear group $GL_n(\mathbf{F}_q)$ over a finite field $\mathbf{F}_q$.

Let $\varphi_r(q)$ be the polynomial $(1-q)(1-q^2)\cdots(1-q^r)$, and $b_\mu(q)$ the polynomial

$$b_\mu(q) = \prod_{i\geq 1} \varphi_{m_i(\mu)}(q),$$

where $m_i(\mu)$ is the multiplicity of i in the partition μ. Define

$$Q_\mu(x;q) = b_\mu(q) P_\mu(x;q),$$

which are referred to, as well as the P_μ, as Hall-Littlewood functions. If we replace the variables $x = (x_1, x_2, \dots)$ of $Q_\mu(x;q)$ by

$$x/(1-q) = (x_1, x_2, \dots; qx_1, qx_2, \dots; q^2x_1, q^2x_2, \dots),$$

then we obtain the *modified* Hall-Littlewood function, which is denoted by

$$Q'_\mu(x;q) \left(= Q_\mu\left(\frac{x}{1-q}; q\right)\right).$$

Equivalently, it is also defined by replacing $p_k(x)$ by $p_k(x)/(1-t^k)$ after expressing $Q_\mu(x;t)$ as a polynomial in $\{p_k(x) | k \geq 1\}$. It is known (see, e.g., [**DLT**]) that the Green polynomial $X^\mu_\rho(q)$ is obtained as the inner product value

$$X^\mu_\rho(x) = \langle Q'_\mu(x;q), p_\rho(x)\rangle$$

of the modified Hall-Littlewood function $Q'_\mu(x;q)$ and the power-sum function $p_\rho(x)$. The inner product $\langle \cdot, \cdot \rangle$ of the ring $\Lambda[q]$ is defined by $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$, where s_λ denotes the Schur function corresponding to the partition λ, and $\delta_{\lambda\mu}$ the Kronecker delta.

In the rest of this section, we recall results on (modified) Hall-Littlewood functions at roots of unity due to Lascoux-Leclerc-Thibon [**LLT**]. Let $\mu \vdash n$ be a partition, l an integer such that $2 \leq l \leq M_\mu$ be fixed, and $m_i(\mu) = lq_i + r_i$, $0 \leq r_i \leq l-1$, for each i. Set $q = q_1 + 2q_2 + \cdots + nq_n$ and $r = r_1 + 2r_2 + \cdots + nr_n$. Let $\tilde{\mu}(l)$ and $\bar{\mu}(l)$ be the partitions

$$\tilde{\mu}(l) := (1^{lq_1} 2^{lq_2} \cdots n^{lq_n})$$

and

$$\bar{\mu}(l) := (1^{r_1} 2^{r_2} \cdots n^{r_n}).$$

It is clear that the partition μ decomposes into the disjoint union $\mu = \tilde{\mu}(l) \cup \bar{\mu}(l)$. Also define

$$\tilde{\mu}(l)^{1/l} := (1^{q_1} 2^{q_2} \cdots n^{q_n}),$$

which is a partition of q.

Example 2.1. If $\mu = (3,3,3,2,2,1)$, then $M_\mu = 3$. Let $l = 2$ be fixed. Then $\tilde{\mu}(l) = (3,3,2,2)$, $\bar{\mu}(l) = (3,1)$, and $\mu = (3,3,2,2) \cup (3,1)$. Also the partition $\tilde{\mu}(l)^{1/l}$ is $(3,2)$.

Let μ be a partition, and l a positive integer such that $l \le M_\mu$. The modified Hall-Littlewood function $Q'_\mu(x;q)$ at $q = \zeta_l$, a primitive l-th root of unity, is factorized in such a way that is consistent with the decomposition of the partition $\mu = \tilde{\mu}(l) \cup \bar{\mu}(l)$.

PROPOSITION 2.1 ([LLT, Theorem 2.1.]). $Q'_\mu(x;\zeta_l) = Q'_{\bar{\mu}(l)}(x;\zeta_l) \prod_{i\ge 1} \left(Q'_{(i^l)}(x;\zeta_l) \right)^{q_i}$.

EXAMPLE 2.2. Let $\mu = (3,3,3,2,1,1,1,1,1)$ and $l = 2$. Then $\bar{\mu}(l) = (3,2,1)$, and we have

$$Q'_{(3,3,3,2,1,1,1,1,1)}(x;\zeta_2) = Q'_{(3,2,1)}(x;\zeta_2) Q'_{(3^2)}(x;\zeta_2) \left(Q'_{(1^2)}(x;\zeta_2) \right)^2 .$$

PROPOSITION 2.2 ([LLT, Theorem 2.2.]). $Q'_{(i^l)}(x;\zeta_l) = (-1)^{(l-1)i}(p_l \circ h_i)(x)$, *where* $(p_l \circ h_i)(x)$ *denotes the plethysm.*

REMARK 2.3. Note that

$$(p_l \circ h_i)(x) = \sum_{\lambda \vdash i} z_\lambda^{-1} p_{l\lambda}(x), \tag{2.1}$$

Thus we have for example $Q'_{(3^2)}(x;\zeta_2) = (-1)^{(2-1)3}(p_2 \circ h_3)(x) = -z_{(3)}^{-1}p_{(6)}(x) - z_{(2,1)}^{-1}p_{(4,2)} - z_{(1,1,1)}^{-1}p_{(2,2,2)}(x)$.

It follows from Proposition 2.1, Proposition 2.2 and (2.1) that the Green polynomial corresponding to a rectangular partition $\mu = (r^k)$ at a primitive k-th root of unity is described by a certain 'smaller' Green polynomial.

PROPOSITION 2.3 ([LLT, Theorem 3.2.]). *Let* $\mu = (r^k)$ *be a rectangular partition,* ζ_k *a primitive* k*-th root of unity. If* $m_i(\mu) \ge 1$ *for some* $i \ge 1$ *divisible by* k*, then it holds that*

$$X^\mu_\rho(\zeta_k) = (-1)^{(k-1)j} k X^{((r-j)^k)}_{\rho\setminus\{i\}}(\zeta_k), \tag{2.2}$$

where $i = jk$.

If we rewrite the identity (2.2) in terms of the polynomial $Q^\mu_\rho(x)$, then the sign $(-1)^{(k-1)j}$ is vanished and we have [Mt, Lemma 7 or Proposition 5]

$$Q^\mu_\rho(\zeta_k) = kQ^{((r-j)^k)}_{\rho\setminus\{i\}}(\zeta_k).$$

Applying this identity repeatedly, we also have

$$Q^\mu_\rho(\zeta_k) = k^{l(\rho)},$$

if the partition ρ consists of multiples of k.

3. Roots of unity

Let μ be a partition of n l a positive integer such that $2 \le l \le M_\mu$ be fixed, and $m_i(\mu) = lq_i + r_i$, $0 \le r_i \le l-1$, for each i. Set $q = q_1 + 2q_2 + \cdots + nq_n$ and $r = r_1 + 2r_2 + \cdots + nr_n$. Let $\tilde{\mu}(l)$, $\bar{\mu}(l)$, and $\tilde{\mu}(l)^{1/l}$ be as in the previous section. We define 'partitions of a partition'as follows. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_d)$ be a partition of n. A *partition* of the partition ν is by definition a sequence of partitions

$$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$$

such that $\lambda^{(i)} \vdash \nu_i$ for each $i = 1, 2, \ldots, d$, which is denoted by $\lambda \vdash \nu$. We distinguish any nontrivial permutation of $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ from the original one. For example, we consider that the following two partitions $((2),(1,1)), ((1,1),(2))$ are different as partitions of $(2,2)$. The *length* $l(\lambda)$ of $\lambda \vdash \nu$ is defined by

$$l(\lambda) = \sum_{i=1}^{d} l(\lambda^{(i)}),$$

and the *size* $|\lambda|$ is defined by the sum of sizes of the components $\lambda^{(i)}$ of λ, which is equal to $n = |\nu|$. Also define

$$z_\lambda := \prod_{i\ge 1} z_{\lambda^{(i)}},$$

where z_π is defined by

$$z_\pi = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$$

for a partition $\pi = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \vdash n$ of a positive integer as usual. Let $\nu = (\nu_i)$ be a partition of n and $\lambda = (\lambda^{(i)})$ a partition of ν. Let

$$m_k(\lambda) := \sum_{i=1}^{d} m_k(\lambda^{(i)})$$

for each possible $k \geq 1$. Then define

$$m_\lambda := \prod_{k \geq 1} \binom{m_k(\lambda)}{m_k(\lambda^{(1)}), m_k(\lambda^{(2)}), \ldots, m_k(\lambda^{(d)})}.$$

Also, for each positive integer l, let $l\lambda$ denotes the partition whose components are those of λ multiplied by l.

Example 3.1. Let $\nu = (4, 2)$. Then the partitions λ of ν are $((4), (2))$, $((3, 1), (2))$, $((2, 2), (1, 1))$, $((2, 1, 1), (1, 1))$ and so on. Suppose that $\lambda = ((2, 1, 1), (2)) \vdash \nu$. Then m_λ is computed as follows: $m_{((2,1,1),(2))} = \binom{m_1(\lambda)}{m_1(\lambda^{(1)}), m_1(\lambda^{(2)})} \binom{m_2(\lambda)}{m_2(\lambda^{(1)}), m_2(\lambda^{(2)})} = \binom{2}{2,0}\binom{2}{1,1} = 2$. For the same λ, if $l = 2$ for example, the partition $l\lambda = 2\lambda$ is $(4, 4, 2, 2)$.

Let ρ be a partition and ν a *subpartition* of ρ, i.e., $m_i(\nu) \leq m_i(\rho)$ for each possible $i \geq 1$. Then we define the *binomial coefficient* $\binom{\rho}{\nu}$ by

$$\binom{\rho}{\nu} := \prod_{i \geq 1} \binom{m_i(\rho)}{m_i(\nu)}.$$

Let μ be a partition, and l an integer such that $2 \leq l \leq M_\mu$ be fixed. For a partition ν of $|\tilde{\mu}(l)|$, define

$$C(\nu, \mu; l) := \sum_{\substack{\pi \vdash \tilde{\mu}(l)^{1/l} \\ l\pi = \nu}} m_\pi.$$

If there exists no $\pi \vdash \tilde{\mu}(l)^{1/l}$ such that $l\pi = \nu$, then $C(\nu, \mu; l) = 0$.

Example 3.2. Let $\mu = (5, 4, 4, 2, 2, 1)$, and l such that $2 \leq l \leq M_\mu$ fixed, say $l = 2$. Then $\tilde{\mu}(l) = (4, 4, 2, 2)$ and $\tilde{\mu}(l)^{1/2} = (4, 2)$. Suppose that $\nu = (4, 4, 4) \vdash |\tilde{\mu}(l)|$. Then there exists only one $\pi \vdash \tilde{\mu}(l)^{1/2}$ such that $2\pi = \nu$, i.e., $\pi = ((2, 2), (2))$. Hence $C(\nu, \mu; 2) = m_{((2,2),(2))} = \binom{3}{2,1} = 3$. On the other hand, if $\nu = (4, 4, 2, 2)$, then there exist two $\pi \vdash (4, 2)$ such that $2\pi = \nu$, i.e., $\pi = ((2, 2), (1, 1)), ((2, 1, 1), (2))$. Hence we have $C(\nu, \mu; 2) = m_{((2,2),(1,1))} + m_{((2,1,1),(2))} = \binom{2}{0,2}\binom{2}{2,0} + \binom{2}{2,0}\binom{2}{1,1} = 1 + 2 = 3$. On the other hand, in the case where $\tilde{\mu}(l)$ is given by $(4, 4)$ for $l = 2$ and $\nu = (4, 2, 2)$, the partitions $\pi \vdash \tilde{\mu}(l)^{1/l}$ satisfying $l\pi = \nu$ are $\pi = ((2), (1, 1)), ((1, 1)(2))$. Since we distinguish these two partitions, $C(\nu, \mu; l)$ is obtained by $m_{((2),(1,1))} + m_{((1,1),(2))} = 1 + 1 = 2$.

Now we can state our main result, which retrieves LLT's result, Proposition 2.3, if we consider the case where μ is a rectangle and $l = M_\mu$. Proposition 2.1 and Proposition 2.2 play a crucial role in the proof.

Theorem 3.3. *Let $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ be a partition of n, a positive integer $l = 1, 2, \ldots, M_\mu$ fixed, and ζ_l an l-th primitive root of unity. Let $m_i = lq_i + r_i$, $0 \leq r_i \leq l - 1$, for each $i = 1, 2, \ldots, n$. Let $r = r_1 + 2r_2 + \cdots + nr_n$, and $\bar{\mu}(l) = (i^{r_i}) \vdash r$.*

Then we have:

1. $Q^\mu_\rho(\zeta_l) \neq 0 \Longrightarrow \rho = l\tilde{\rho} \cup \bar{\rho}$ *for some* $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ *and* $\bar{\rho} \vdash r$.
2. *For such a partition $\rho = l\tilde{\rho} \cup \bar{\rho}$, it holds that:*

$$Q^\mu_\rho(\zeta_l) = \sum_{\substack{\nu \vdash |\tilde{\mu}(l)| \\ \nu \subset \rho}} \binom{\rho}{\nu} C(\nu, \mu; l) l^{l(\nu)} Q^{\bar{\mu}(l)}_{\rho \backslash \nu}(\zeta_l).$$

Example 3.4. Let $\mu = (5, 4, 4, 2, 2, 1) \vdash 18$ and $l = 2$. In this case, we have $\widetilde{\mu(2)} = (4, 4, 2, 2)$ and $\widetilde{\mu(2)}^{1/2} = (4, 2)$. Suppose that $\rho = (4, 4, 2, 2) \cup (4, 2) = (4, 4, 4, 2, 2, 2)$. Then subpartitions ν of ρ which satisfy $\nu \vdash |\widetilde{\mu(2)}| = 12$ are $\nu = (4, 4, 4), (4, 4, 2, 2)$. Consider the case where $\nu = (4, 4, 4)$. Then

$\binom{\rho}{\nu} = \binom{3}{0}\binom{3}{3} = 1$. There exists only one $\lambda \vdash \widetilde{\mu(2)}^{1/2} = (4,2)$ such that $2\lambda = (4,4,4)$, i.e., $\lambda = ((2,2),(2))$, and we have $m_\lambda = \binom{2+1}{2,1} = 3$. Thus $C(\nu,\mu;2) = 3$. If $\nu = (4,4,2,2)$, then $\binom{\rho}{\nu} = \binom{3}{2}\binom{3}{2} = 9$. The corresponding λ's satisfying $2\lambda = \nu$ are $\lambda = ((2,2),(1,1)),((2,1,1),(2))$, and $m_{((2,2),(1,1))} = \binom{2}{0,2}\binom{2}{2,0} = 1$, $m_{((2,1,1),(2))} = \binom{2}{2,0}\binom{2}{1,1} = 2$. Hence we have $C(\nu,\mu;2) = 3$ in this case. Thus we have $Q^{(5,4,4,2,2,1)}_{(4,4,4,2,2,2)}(\zeta_2) = \binom{\rho}{(4,4,4)}C((4,4,4),\mu;2)2^{l(4,4,4)}Q^{\tilde{\mu}(l)}_{\rho\backslash(4,4,4)}(\zeta_2) + \binom{\rho}{(4,4,2,2)}C((4,4,2,2),\mu;2)2^{l(4,4,2,2)}Q^{\tilde{\mu}(l)}_{\rho\backslash(4,4,2,2)}(\zeta_2) = 1 \times 3 \times 8Q^{(5,1)}_{(2,2,2)}(\zeta_2) + 9 \times 3 \times 16Q^{(5,1)}_{(4,2)}(\zeta_2)$.

4. Permutation enumeration

In this section, we shall give a combinatorial characterization of the coefficients

$$\binom{\rho}{\nu}C(\nu,\mu;l)l^{l(\nu)},$$

in the preceding formula. Let μ be a partition of a positive integer n, and an integer $l \in \{2,3,\dots,M_\mu\}$ fixed. We define a product of cyclic permutations $a = a_\mu(l)$ corresponding to μ and l as follows. To avoid abuse of notation, we shall see the definition by the following example. It is clear from the definition that the element $a_\mu(l)$ has the order l.

Example 4.1 (Definition of $a_\mu(l)$). Let $\mu = (3,3,2,2,2,1)$ and $l = 2(\leq M_\mu = 3)$. We fix the numbering of the Young diagram of μ as follows:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \\ 9 & 10 & \\ 11 & 12 & \\ 13 & & . \end{array}$$

Corresponding to the number $l = 2$, we extract subtableaux

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}, \quad \begin{array}{cc} 7 & 8 \\ 9 & 10 \end{array}$$

Then the cyclic permutation product $a_\mu(2)$ is defined by using the letters corresponding to $\tilde{\mu}(l)$ as follows:

$$a_\mu(2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 7 & 8 & 9 & 10 \\ 9 & 10 & 7 & 8 \end{pmatrix}$$

□

Let $n = ql + r$, $0 \leq r \leq l-1$. Recall that $\tilde{\mu}(l)$ is a partition of $n - r$. Let $S_{\tilde{\mu}(l)}$ be the Young subgroup which permutes the letters corresponding to $\tilde{\mu}(l)$ in the preceding tableau, and let S_r be the subgroups which permutes the remaining letters. It is obvious that elements of these groups commute with each other. In the preceding definition (Example 11), these groups are the following:

$$S_{\tilde{\mu}(l)} = S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9,10\}},$$
$$S_r = S_{\{11,12,13\}},$$

where $\tilde{\mu}(l) = (3,3,2,2)$, $r = 3$ and $S_{\{i,j,\dots,k\}}$ denotes the symmetric group of the letters $\{i,j,\dots,k\}$. Consider the subgroup of S_n

$$H_\mu(l) := \left(S_{\tilde{\mu}(l)} \times S_r\right) \rtimes \langle a_\mu(l)\rangle = \left(S_{\tilde{\mu}(l)} \rtimes \langle a_\mu(l)\rangle\right) \times S_r.$$

The following lemma is proved by straightforward computation.

Lemma 4.2. *The cycle types ρ of elements of the subgroup $H_\mu(l)$ are of the form*

$$\rho = l\tilde{\rho} \cup \bar{\rho},$$

where $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$. Conversely, if ρ is a partition of such a form, then there exists an element of $H_\mu(l)$ whose cycle type is ρ.

Example 4.3. Consider the case $\mu = (3,3,2,2,2,1)$ and $l = 2$. Then the corresponding cyclic permutation product is $a_\mu(2) = (1,4)(2,5)(3,6)(7,9)(8,10)$. If we consider $w = (1,2)(7,8)a_\mu(2)(11,13) \in H_\mu(2)$, then $w = (1,4,2,5)(3,6)(7,9,8,10)(11,13)$ and its cycle type is $(4,4,2,2,1)$, which is the union of $(4,4,2)$ and $(2,1)$. The partition $(4,4,2)$ is written in the form $(4,4,2) = 2((2,1),(2))$ for $((2,1),(2)) \vdash (3,2) = \tilde{\mu}(l)^{1/2}$. Conversely, if we consider $\rho = 2((2,1),(1,1)) \cup (3) = (4,3,2,2,2)$, then choose $\tau_1 = (1,2) \in S_{\tilde{\mu}(l)}$ and $\tau_2 = (11,12,13) \in S_r$ for example. It is easy to see that the cycle type of $w = \tau_1\tau_2 a_\mu(2)$ coincides with ρ. □

A direct but a little complicated enumeration shows the following proposition. Remark that $l(\lambda) = l(k\lambda)$ for any partition λ and any positive integer k.

Proposition 4.1. *Let $\mu \vdash n$ be a partition, $l = 2,3,\dots,M_\mu$ fixed, and $a = a_\mu(l)$ the cyclic permutation product corresponding to μ and l. Let $\rho \vdash n$ be a partition of the form $\rho = l\tilde{\rho} \cup \bar{\rho}$ where $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$. Suppose that $w \in S_n$ be a permutation whose cycle type is ρ. Then it follows that*

$$\binom{\rho}{l\tilde{\rho}} C(l\tilde{\rho},\mu;l)l^{l(\tilde{\rho})} = \sharp\{\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r | w\sigma a^{-1} \equiv \sigma \ mod\ S_{\tilde{\mu}(l)} \times S_r\}.$$

Example 4.4. Let $\mu = (2,2,2,2,2,1)$ and $l = 2,\dots,M_\mu(=5)$ be fixed, say $l = 2$. Then the corresponding product of cyclic permutations is $a = (13)(24)(57)(68)$. The subgroups $S_{\tilde{\mu}(l)}$ and $S_r = S_3$ are $S_{\{1,2\}} \times S_{\{3,4\}} \times S_{\{5,6\}} \times S_{\{7,8\}}$ and $S_{\{9,10,11\}}$ respectively. Let us consider the case $w = (12)a(9,10) = (1324)(57)(68)(9,10)$ ($\tau_1 = (12)$, $\tau_2 = (9,10)$). The cycle type ρ of w is $\rho = (4,2,2,2,1)$. If we let $\tilde{\rho} = ((2),(1,1)) \vdash \tilde{\mu}(l)^{1/2} = (2,2)$ and $\bar{\rho} = (2,1) \vdash r = 3$, we have $\rho = 2\tilde{\rho} \cup \bar{\rho}$. Then it follows that

$$\sum_{\substack{\lambda \vdash \tilde{\mu}(l)^{1/2} = (2,2) \\ 2\lambda = (4,2,2)}} m_\lambda = m_{((2),(1,1))} + m_{((1,1),(2))} = 2,$$

and $\binom{\rho}{l\tilde{\rho}} = \binom{2+1}{2} = 3$. Thus we have

$$\sharp\{\sigma \in S_{11}/S_{(2^4)} \times S_3 | w\sigma a^{-1} \equiv \sigma \text{ mod } S_{(2^4)} \times S_3\} = \binom{3}{2}\left(m_{((2),(1,1))} + m_{((1,1),(2))}\right)2^3 = 48.$$

5. Representation theory of the symmetric group

In this final section, we understand the main result in terms of representation theory of the symmetric group.

It is known that the Green polynomial $Q^\mu_\rho(q)$ gives the graded character value of a certain graded S_n-module, called the *DeConcini-Procesi-Tanisaki algebra* [DP]. The DeConcini-Procesi-Tanisaki algebras R_μ are defined for each partition μ of n, and afford a family of graded representations of S_n. We denote by

$$R_\mu = \bigoplus_{d \geq 0} R^d_\mu$$

its grading. Geometrically, the algebra R_μ is isomorphic to the cohomology ring

$$H^*(X_\mu, \mathbf{C})$$

of the fixed point subvariety X_μ of the flag variety, corresponding to the partition μ. In this point of view, the representation of S_n afforded by R_μ is called the Springer representation [S, L]. As an S_n-module, R_μ is isomorphic to the induced representation $\mathrm{Ind}^{S_n}_{S_\mu} 1$.

The *graded character* $\mathrm{char}_q R_\mu$ of the graded module R_μ, evaluated on the conjugacy class corresponding to $\rho \vdash n$, is by definition a polynomial in q

$$\mathrm{char}_q R_\mu(\rho) = \sum_{d \geq 0} q^d \mathrm{char} R^d_\mu(\rho)$$

with integer coefficients. It is known that it coincides with the Green polynomial

$$Q^\mu_\rho(q) = \mathrm{char}_q R_\mu(\rho)$$

for each $\rho \vdash n$.

The aim of this section is to rephrase the recursive formula of the Green polynomials $Q^{\mu}_{\rho}(q)$ in the main theorem, in terms of the graded algebra R_μ. The formula gives a representation theoretical interpretation of a certain combinatorial property of the algebra R_μ. By considering behavior of the Hilbert polynomial

$$\mathrm{Hilb}_\mu(q) = \sum_{d\geq 0} q^d \dim R^d_\mu$$

of the graded module R_μ at roots of unity, we can show that R_μ has the following property. Let M_μ be the maximum multiplicity of μ, and let an integer $l \in \{2, 3, \ldots, M_\mu\}$ be fixed. For each $k = 0, 1, \ldots, l-1$, define

$$R_\mu(k; l) := \bigoplus_{d\equiv k \bmod l} R^d_\mu.$$

It is clear that these $R_\mu(k; l)$'s are S_n-submodules of R_μ. Then it follows that

Proposition 5.1. *The dimensions of the submodules $R_\mu(k; l)$ $(k = 0, 1, \ldots, l-1)$ coincides with each other.*

This is a consequence of the fact that the Hilbert polynomial $\mathrm{Hilb}_\mu(q)$ has the roots of unity ζ_l^j for each $j = 1, 2, \ldots, l-1$ as its zeros.

Our problem is to give an interpretation to this property "coincidence of dimensions" in terms of representation theory, that is, constructing a subgroup $H(l)$ and its modules $Z(k; l)$ $(k = 0, 1, \ldots, l-1)$ of equal dimension such that

$$R_\mu(k; l) \cong_{S_n} \mathrm{Ind}^{S_n}_{H(l)} Z(k; l), \quad k = 0, 1, \ldots, l-1.$$

Since the dimension of the induced representation $\mathrm{Ind}^{S_n}_{H(l)} Z(k; l)$ is $\dim Z(k; l)|S_n|/|H(l)|$, we can convince ourselves that these isomorphisms are representation theoretical interpretation of the coincidence of dimensions. Let $\mu \vdash n$ be a partition, $l \in \{2, 3, \ldots, M_\mu\}$ fixed, $a = a_\mu(l)$ the cyclic permutation product corresponding to μ and l, and $C_l = \langle a \rangle$ the cyclic subgroup of S_n generated by a. Recall that the subgroup $H_\mu(l)$ is defined by $H_\mu(l) = \big(S_{\tilde{\mu}(l)} \rtimes C_l\big) \times S_r$. Consider, for each $k = 0, 1, \ldots, l-1$, $H_\mu(l)$-modules $Z_\mu(k; l)$ defined as follows:

$$Z_\mu(k; l) = \bigoplus_{d=1}^{n(\bar{\mu}(l))} \varphi_l^{(k-d)} \otimes R^d_{\bar{\mu}(l)},$$

where $\varphi_l^{(r)}$ is the irreducible representation of the cyclic group $C_l = \langle a \rangle$ such that $a \longmapsto \zeta_l^r$. The Young subgroup $S_{\tilde{\mu}(l)}$ acts trivially on $Z_\mu(k; l)$. Since $\varphi_l^{(r)}$'s are one dimensional, the dimension of $Z_\mu(k; l)$ is equal to $\dim R_{\bar{\mu}(l)}$ for each k. We shall show that

$$R_\mu(k; l) \cong_{S_n} \mathrm{Ind}^{S_n}_{H_\mu(l)} Z_\mu(k; l), \quad k = 0, 1, \ldots, l-1.$$

Actually, we shall show a certain $S_n \times C_l$-module isomorphism between R_μ and $\mathrm{Ind}^{S_n}_{S_{\tilde{\mu}(l)}\times S_r} R_{\bar{\mu}(l)}$, originally suggested by T. Shoji, which is equivalent to those isomorphisms.

We define $S_n \times C_l$-modules structures on R_μ and $\mathrm{Ind}^{S_n}_{S_{\tilde{\mu}(l)}\times S_r} R_{\bar{\mu}(l)}$ as follows. In both cases, the S_n-actions are natural ones. The action of C_l on R_μ is defined by

$$a.x = \zeta_l^d x, \quad x \in R^d_\mu.$$

Recall that the induced modules $\mathrm{Ind}^{S_n}_{S_{\tilde{\mu}(l)}\times S_r} R_{\bar{\mu}(l)}$ has the following realization:

$$\mathrm{Ind}^{S_n}_{S_{\tilde{\mu}(l)}\times S_r} R_{\bar{\mu}(l)} = \bigoplus_{\sigma\in S_n/S_{\tilde{\mu}(l)}\times S_r} \sigma \otimes R_{\bar{\mu}(l)}.$$

Then the C_l-action is defined by

$$a.\sigma \otimes x = \sigma a^{-1} \otimes a.x, \quad \sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r,\ x \in R_{\bar{\mu}(l)}.$$

It is easy to see that the S_n-action and the C_l-action commute on each module. These two $S_n \times C_l$-modules are isomorphic, which is proved by comparing the characters of these modules.

Theorem 5.1. *Let μ be a partition of a positive integer n, and l an integer such that $2 \le l \le M_\mu$ fixed. Suppose that $n = ql + r$, $0 \le r \le l-1$, and let C_l be the cyclic group generated by the element $a = a_\mu(l)$. Then there exists an isomorphism of $S_n \times C_l$-modules*

$$R_\mu \cong \mathrm{Ind}_{S_{\bar{\mu}(l)} \times S_r}^{S_n} R_{\bar{\mu}(l)}. \tag{5.1}$$

If we consider the eigenspace decomposition of the action of a in the $S_n \times C_l$-isomorphism (5.1), then we obtain a representation theoretical interpretation of the property, coincidence of dimension, of the algebra R_μ.

Proposition 5.2. *Let $\mu \vdash n$ be partition and an integer $l \in \{2, 3, \ldots, M_\mu\}$ fixed. Then there exist $H_\mu(l)$-modules $Z_\mu(k; l)$ $(k = 0, 1, \ldots, l-1)$ of equal dimension such that*

$$R_\mu(k; l) \cong_{S_n} \mathrm{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)$$

for each $k = 0, 1, \ldots, l-1$.

Example 5.2. Let $\mu = (5, 4, 4, 2, 2, 1)$ and $l = 2$. Then $\tilde{\mu}(2) = (4, 4, 2, 2)$, $\bar{\mu}(l) = (5, 1)$, and

$$a = a_\mu(2) = \begin{pmatrix} 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 10 & 11 & 12 & 13 & 6 & 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 14 & 15 & 16 & 17 \\ 16 & 17 & 14 & 15 \end{pmatrix}.$$

The dimensions of $R_\mu(k; 2)$, $k = 0, 1$, equals $\dim R_\mu/2 = \binom{18}{5,4,4,2,2,1}/2 = 18!/5!4!4!2!2!1!2$. The subgroup $H_\mu(2)$ is defined by $H_\mu(2) = S_{\mu(2)} \rtimes \langle a \rangle \times S_6$, where $S_{\mu(2)} = S_{\{6,7,8,9\}} \times S_{\{10,11,12,13\}} \times S_{\{14,15\}} \times S_{\{16,17\}}$ and $S_r = S_{\{1,2,3,4,5,18\}}$ $(r = 3)$. Define $H_\mu(2)$-modules $Z_\mu(k; l)$ $(k = 0, 1)$ by $Z_\mu(k; 2) := \bigoplus_{d \equiv k \bmod 2} \varphi_2^{(k-d)} \otimes R_{\bar{\mu}(l)}^d$. These spaces are considered as $H_\mu(2)$-modules, where $S_{\tilde{\mu(2)}}$ acts on them trivially. The dimension of these modules are both equal to $\dim R_{\bar{\mu}(l)} = \binom{6}{5,1} = 6!/5!1!$. Then, for each $k = 0, 1$, we have an isomorphism of S_{18}-modules $R_\mu(k; 2) \cong \mathrm{Ind}_{(S_{(4,4,2,2)} \rtimes C_2) \times S_6}^{S_{18}} Z_\mu(k; 2)$. The induced modules are of dimension $18!/4!4!2!2!6!2 \times 6!/5!1! = 18!/5!4!4!2!2!1!2 = \dim R_\mu(k; 2)$ for each $k = 0, 1$.

Remark 5.3. Recently, the author was informed by T. Shoji that the problem considered in this section is given an affirmative answer in a largely generalized setting [**Sh**].

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