



Strong Descent Numbers and Turán Type Theorems (Extended Abstract)

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ABSTRACT. For a permutation π in the symmetric group S_n let the *total degree* be its valency in the Hasse diagram of the strong Bruhat order on S_n , and let the *down degree* be the number of permutations which are covered by π in the strong Bruhat order. The maxima of the total degree and the down degree and their values at a random permutation are computed. Proofs involve variants of a classical theorem of Turán from extremal graph theory.

RÉSUMÉ. Pour une permutation π du groupe symétrique S_n , on définit le degré total de π comme sa valence dans le diagramme de Hasse de l'ordre de Bruhat fort sur S_n , et le degré bas de π comme le nombre de permutations couvertes par π dans l'ordre de Bruhat fort. Nous calculons les valeurs maximales pour le degré total et le degré bas, ainsi que leur valeurs pour une permutation aléatoire. Nos démonstrations utilisent des variantes d'un théorème de Turán provenant de la théorie des graphes extrémaux.

1. The Down, Up and Total Degrees

DEFINITION 1.1. For a permutation $\pi \in S_n$ let the down degree $d_-(\pi)$ be the number of permutations in S_n which are covered by π in the strong Bruhat order. Let the up degree $d_+(\pi)$ be the number of permutations which cover π in this order. The total degree of π is the sum

$$d(\pi) := d_-(\pi) + d_+(\pi),$$

i.e., the valency of π in the Hasse diagram of the strong Bruhat order.

Explicitly, for $1 \leq a < b \leq n$ let $t_{a,b} = t_{b,a} \in S_n$ be the transposition interchanging a and b , and for $\pi \in S_n$ let

$$\ell(\pi) := \min\{k \mid \pi = s_{i_1} s_{i_2} \cdots s_{i_k}\}$$

be the length of π with respect to the standard Coxeter generators $s_i = t_{i,i+1}$ ($1 \leq i < n$) of S_n . Then

$$\begin{aligned} d_-(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) - 1\} \\ d_+(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) + 1\} \\ d(\pi) = d_-(\pi) + d_+(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) \pm 1\} \end{aligned}$$

For the general definitions and other properties of the weak and strong Bruhat orders see, e.g., [10, Ex. 3.75] and [3, §§2.1, 3.1].

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We shall describe $\pi \in S_n$ by its sequence of values $[\pi(1), \dots, \pi(n)]$.

OBSERVATION 1.2. π covers σ in the strong Bruhat order on S_n if and only if there exist $1 \leq i < k \leq n$ such that

- (1) $b := \pi(i) > \pi(k) =: a$.
- (2) $\sigma = t_{a,b}\pi$, i.e., $\pi = [\dots, b, \dots, a, \dots]$ and $\sigma = [\dots, a, \dots, b, \dots]$.
- (3) There is no $i < j < k$ such that $a < \pi(j) < b$.

COROLLARY 1.3. For every $\pi \in S_n$

$$d_-(\pi) = d_-(\pi^{-1}).$$

EXAMPLE 1.4. In S_3 , $d_-[123] = 0$, $d_-[132] = d_-[213] = 1$, and $d_-[321] = d_-[231] = d_-[312] = 2$. On the other hand, $d[321] = d[123] = 2$ and $d[213] = d[132] = d[312] = d[231] = 3$.

REMARK 1.5. The classical descent number of a permutation π in the symmetric group S_n is the number of permutations in S_n which are covered by π in the (right) weak Bruhat order. Thus, the down degree may be considered as a “strong descent number”.

DEFINITION 1.6. For $\pi \in S_n$ denote

$$D_-(\pi) := \{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) - 1\},$$

the strong descent set of π .

EXAMPLE 1.7. The strong descent set of $\pi = [7, 9, 5, 2, 3, 8, 4, 1, 6]$ is

$$D_-(\pi) = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,5}, t_{3,5}, t_{4,5}, t_{4,8}, t_{5,7}, t_{5,9}, t_{6,7}, t_{6,8}, t_{8,9}\}.$$

REMARK 1.8. Generalized pattern avoidance, involving strong descent sets, was applied by Yong and Woo [12] to determine which Schubert varieties are Gorenstein.

PROPOSITION 1.9. The strong descent set $D_-(\pi)$ uniquely determines the permutation π .

PROOF. By induction on n . The claim clearly holds for $n = 1$.

Let π be a permutation in S_n , and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained by deleting the value n from π . Note that, by Observation 1.2,

$$D_-(\bar{\pi}) = D_-(\pi) \setminus \{t_{a,n} \mid 1 \leq a < n\}.$$

By the induction hypothesis $\bar{\pi}$ is uniquely determined by this set. Hence it suffices to determine the position of n in π .

Now, if $j := \pi^{-1}(n) < n$ then clearly $t_{\pi(j+1),n} \in D_-(\pi)$. Moreover, by Observation 1.2, $t_{a,n} \in D_-(\pi) \implies a \geq \pi(j+1)$. Thus $D_-(\pi)$ determines

$$\bar{\pi}(j) = \pi(j+1) = \min\{a \mid t_{a,n} \in D_-(\pi)\},$$

and therefore determines j . Note that this set of a 's is empty if and only if $j = n$. This completes the proof. □

2. Maximal Down Degree

In this section we compute the maximal value of the down degree on S_n and find all the permutations achieving the maximum. We prove

PROPOSITION 2.1. For every positive integer n

$$\max\{d_-(\pi) \mid \pi \in S_n\} = \lfloor n^2/4 \rfloor.$$

REMARK 2.2. The same number appears as the order dimension of the strong Bruhat poset [8]. An upper bound on the maximal down degree for finite Coxeter groups appears in [5, Prop. 3.4].

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For the proof of Proposition 2.1 we need a classical theorem of Turán.

DEFINITION 2.3. *Let $r \leq n$ be positive integers. The Turán graph $T_r(n)$ is the complete r -partite graph with n vertices and all parts as equal in size as possible, i.e., each size is either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Denote by $t_r(n)$ the number of edges of $T_r(n)$.*

THEOREM 2.4. [11] [4, IV, Theorem 8] (**Turán's Theorem**)

- (1) *Every graph of order n with more than $t_r(n)$ edges contains a complete subgraph of order $r + 1$.*
- (2) *$T_r(n)$ is the unique graph of order n with $t_r(n)$ edges that does not contain a complete subgraph of order $r + 1$.*

We shall apply the special case $r = 2$ (due to Mantel) of Turán's theorem to the following graph.

DEFINITION 2.5. *The strong descent graph of $\pi \in S_n$, denoted $\Gamma_-(\pi)$, is the undirected graph whose set of vertices is $\{1, \dots, n\}$ and whose set of edges is*

$$\{\{a, b\} \mid t_{a,b} \in D_-(\pi)\}.$$

By definition, the number of edges in $\Gamma_-(\pi)$ equals $d_-(\pi)$.

REMARK 2.6. *Permutations for which the strong descent graph is connected are called indecomposable. Their enumeration was studied in [6]; see [7, pp. 7–8]. The number of components in $\Gamma_-(\pi)$ is equal to the number of global descents in πw_0 (where $w_0 := [n, n-1, \dots, 1]$), which were introduced and studied in [2, Corollaries 6.3 and 6.4].*

LEMMA 2.7. *For every $\pi \in S_n$, the strong descent graph $\Gamma_-(\pi)$ is triangle-free.*

PROOF. Assume that $\Gamma_-(\pi)$ contains a triangle. Then there exist $1 \leq a < b < c \leq n$ such that $t_{a,b}, t_{a,c}, t_{b,c} \in D_-(\pi)$. By Observation 1.2,

$$t_{a,b}, t_{b,c} \in D_-(\pi) \implies \pi^{-1}(c) < \pi^{-1}(b) < \pi^{-1}(a) \implies t_{a,c} \notin D_-(\pi).$$

This is a contradiction. □

PROOF. (OF PROPOSITION 2.1) By Theorem 2.4(1) together with Lemma 2.7, for every $\pi \in S_n$

$$d_-(\pi) \leq t_2(n) = \lfloor n^2/4 \rfloor.$$

Equality holds since

$$d_-(\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n, 1, 2, \dots, \lfloor n/2 \rfloor) = \lfloor n^2/4 \rfloor.$$

□

Next we classify (and enumerate) the permutations which achieve the maximal down degree.

LEMMA 2.8. *Let $\pi \in S_n$ be a permutation with maximal down degree. Then π has no decreasing subsequence of length 4.*

PROOF. Assume that $\pi = [\dots d \dots c \dots b \dots a \dots]$ with $d > c > b > a$ and $\pi^{-1}(a) - \pi^{-1}(d)$ minimal. Then $t_{a,b}, t_{b,c}, t_{c,d} \in D_-(\pi)$ but, by Observation 1.2, $t_{a,d} \notin D_-(\pi)$. It follows that $\Gamma_-(\pi)$ is not a complete bipartite graph, since $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$ are edges but $\{a, d\}$ is not. By Lemma 2.7, combined with Theorem 2.4(2), the number of edges in $\Gamma_-(\pi)$ is less than $\lfloor n^2/4 \rfloor$. □

PROPOSITION 2.9. *For every positive integer n*

$$\#\{\pi \in S_n \mid d_-(\pi) = \lfloor n^2/4 \rfloor\} = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Each such permutation has the form

$$\pi = [t + m + 1, t + m + 2, \dots, n, t + 1, t + 2, \dots, t + m, 1, 2, \dots, t],$$

where $m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ and $1 \leq t \leq n - m$. Note that $t = n - m$ (for m) gives the same permutation as $t = 0$ (for $n - m$ instead of m).

PROOF. It is easy to verify the claim for $n \leq 3$. Assume $n \geq 4$.

Let $\pi \in S_n$ with $d_-(\pi) = \lfloor n^2/4 \rfloor$. By Theorem 2.4(2), $\Gamma_-(\pi)$ is isomorphic to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Since $n \geq 4$, each side of the graph contains at least two vertices. Let $1 = a < b$ be two vertices on one side, and $c < d$ two vertices on the other side of the graph. Since $t_{b,c}, t_{b,d} \in D_-(\pi)$, there are three possible cases:

- (1) $b < c$, and then $\pi = [\dots c \dots d \dots b \dots]$ (since $\pi = [\dots d \dots c \dots b \dots]$ contradicts $t_{b,d} \in D_-$).
- (2) $c < b < d$, and then $\pi = [\dots d \dots b \dots c \dots]$.
- (3) $d < b$, and then $\pi = [\dots b \dots c \dots d \dots]$ (since $\pi = [\dots b \dots d \dots c \dots]$ contradicts $t_{b,c} \in D_-$).

The same also holds for a instead of b , but then cases 2 and 3 are impossible since $a = 1 < c$. Thus necessarily c appears before d in π , and case 2 is therefore impossible for *any* b on the same side as $a = 1$. In other words: no vertex on the same side as $a = 1$ is intermediate, either in position (in π) or in value, to c and d .

Assume now that n is even. The vertices not on the side of 1 form (in π) a block of length $n/2$ of numbers which are consecutive in value as well in position. They also form an increasing subsequence of π , since $\Gamma_-(\pi)$ is bipartite. The numbers preceding them are all larger in value, and are increasing; the numbers succeeding them are all smaller in value, are increasing, and contain 1. It is easy to check that each permutation π of this form has maximal $d_-(\pi)$. Finally, π is completely determined by the length $1 \leq t \leq n/2$ of the last increasing subsequence.

For n odd one obtains a similar classification, except that the length of the side not containing 1 is either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. This completes the proof. □

3. Maximal Total Degree

Obviously, the maximal value of the total degree $d = d_- + d_+$ cannot exceed $\binom{n}{2}$, the total number of transpositions in S_n . This is slightly better than the bound $2\lfloor n^2/4 \rfloor$ obtainable from Proposition 2.1. The actual maximal value is smaller.

THEOREM 3.1. *For $n \geq 2$, the maximal total degree in the Hasse diagram of the strong Bruhat order on S_n is*

$$\lfloor n^2/4 \rfloor + n - 2.$$

In order to prove this result, associate with each permutation $\pi \in S_n$ a graph $\Gamma(\pi)$, whose set of vertices is $\{1, \dots, n\}$ and whose set of edges is

$$\{\{a, b\} \mid \ell(t_{a,b}\pi) - \ell(\pi) = \pm 1\}.$$

This graph has many properties; e.g., it is K_5 -free and is the edge-disjoint union of two triangle-free graphs on the same set of vertices. However, these properties are not strong enough to imply the above result. A property which does imply it is the following bound on the minimal degree.

LEMMA 3.2. *There exists a vertex in $\Gamma(\pi)$ with degree at most $\lfloor n/2 \rfloor + 1$.*

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PROOF. Assume, on the contrary, that each vertex in $\Gamma(\pi)$ has at least $\lfloor n/2 \rfloor + 2$ neighbors. This applies, in particular, to the vertex $\pi(1)$. Being the first value of π , the neighborhood of $\pi(1)$ in $\Gamma(\pi)$, viewed as a subsequence of $[\pi(2), \dots, \pi(n)]$, consists of a shuffle of a decreasing sequence of numbers larger than $\pi(1)$ and an increasing sequence of numbers smaller than $\pi(1)$. Let a be the rightmost neighbor of $\pi(1)$. The intersection of the neighborhood of a with the neighborhood of $\pi(1)$ is of cardinality at most two. Thus the degree of a is at most

$$n - (\lfloor n/2 \rfloor + 2) + 2 = \lceil n/2 \rceil \leq \lfloor n/2 \rfloor + 1,$$

which is a contradiction. □

PROOF. (OF THEOREM 3.1) First note that, by definition, the total degree of $\pi \in S_n$ in the Hasse diagram of the strong Bruhat order is equal to the number of edges in $\Gamma(\pi)$. We will prove that this number $e(\Gamma(\pi)) \leq \lfloor n^2/4 \rfloor + n - 2$, by induction on n .

The claim is clearly true for $n = 2$. Assume that the claim holds for $n - 1$, and let $\pi \in S_n$. Let a be a vertex of $\Gamma(\pi)$ with minimal degree, and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained from π by deleting the value a (and decreasing by 1 all the values larger than a). Then

$$e(\Gamma(\bar{\pi})) \geq e(\Gamma(\pi) \setminus a),$$

where the latter is the number of edges in $\Gamma(\pi)$ which are not incident with the vertex a . By the induction hypothesis and Lemma 3.2,

$$\begin{aligned} e(\Gamma(\pi)) &= e(\Gamma(\pi) \setminus a) + d(a) \leq e(\Gamma(\bar{\pi})) + d(a) \\ &\leq \lfloor (n-1)^2/4 \rfloor + (n-1) - 2 + \lfloor n/2 \rfloor + 1 \\ &= \lfloor n^2/4 \rfloor + n - 2. \end{aligned}$$

Equality holds since, letting $m := \lfloor n/2 \rfloor$,

$$e(\Gamma([m+1, m+2, \dots, n, 1, 2, \dots, m])) = \lfloor n^2/4 \rfloor + n - 2.$$

□

THEOREM 3.3.

$$\#\{\pi \in S_n \mid d(\pi) = \lfloor n^2/4 \rfloor + n - 2\} = \begin{cases} 2, & \text{if } n = 2; \\ 4, & \text{if } n = 3 \text{ or } n = 4; \\ 8, & \text{if } n \geq 6 \text{ is even}; \\ 16, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

The extremal permutations have one of the following forms:

$$\pi_0 := [m+1, m+2, \dots, n, 1, 2, \dots, m] \quad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}),$$

and the permutations obtained from π_0 by one or more of the following operations:

$$\begin{aligned} \pi &\mapsto \pi^r := [\pi(n), \pi(n-1), \dots, \pi(2), \pi(1)] && \text{(reversing } \pi), \\ \pi &\mapsto \pi^s := \pi \cdot t_{1,n} && \text{(interchanging } \pi(1) \text{ and } \pi(n)), \\ \pi &\mapsto \pi^t := t_{1,n} \cdot \pi && \text{(interchanging 1 and } n \text{ in } \pi). \end{aligned}$$

PROOF. It is not difficult to see that all the specified permutations are indeed extremal, and their number is as claimed (for all $n \geq 2$).

The claim that there are no other extremal permutations will be proved by induction on n . For small values of n (say $n \leq 4$) this may be verified directly. Assume now that the claim holds for some $n \geq 4$, and let $\pi \in S_{n+1}$ be extremal. Following the proof of Lemma 3.2, let a be a vertex of $\Gamma(\pi)$ with degree at most $\lfloor (n+1)/2 \rfloor + 1$, which is either $\pi(1)$ or its rightmost neighbor. As in the proof of Theorem 3.1, let $\bar{\pi} \in S_n$ be the permutation obtained from π by deleting the value a

(and decreasing by 1 all the values larger than a). All the inequalities in the proof of Theorem 3.1 must hold as equalities, namely: $e(\Gamma(\pi) \setminus a) = e(\Gamma(\bar{\pi}))$, $d(a) = \lfloor (n+1)/2 \rfloor + 1$, and $\bar{\pi}$ is extremal in S_n . By the induction hypothesis, $\bar{\pi}$ must have one of the prescribed forms. In all of them, $\{\bar{\pi}(1), \bar{\pi}(n)\} = \{m, m+1\}$ is an edge of $\Gamma(\bar{\pi})$. Therefore the corresponding edge $\{\pi(1), \pi(n+1)\}$ (or $\{\pi(2), \pi(n+1)\}$ if $a = \pi(1)$, or $\{\pi(1), \pi(n)\}$ if $a = \pi(n+1)$) is an edge of $\Gamma(\pi) \setminus a$, namely of $\Gamma(\pi)$. If $a \neq \pi(1), \pi(n+1)$ then $\pi(n+1)$ is the rightmost neighbor of $\pi(1)$, contradicting the choice of a . If $a = \pi(n+1)$ we may use the operation $\pi \mapsto \pi^r$. Thus we may assume from now on that $a = \pi(1)$.

Let $N(a)$ denote the set of neighbors of a in $\Gamma(\pi)$. Assume first that

$$\bar{\pi} = \pi_0 = [m+1, m+2, \dots, n, 1, 2, \dots, m] \quad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}).$$

Noting that $\lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor$ and keeping in mind the decrease in certain values during the transition $\pi \mapsto \bar{\pi}$, we have the following cases:

(1) $a > m+1$: in this case $1, \dots, m \notin N(a)$, so that

$$d(a) \leq n - m \leq \lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor < \lfloor (n+1)/2 \rfloor + 1.$$

Thus π is not extremal.

(2) $a < m$: in this case $m+3, \dots, n+1, m+1 \notin N(a)$, so that

$$d(a) \leq 1 + (m-1) \leq \lceil n/2 \rceil < \lfloor (n+1)/2 \rfloor + 1.$$

Again, π is not extremal.

(3) $a \in \{m, m+1\}$: in this case

$$d(a) = 1 + m \leq \lfloor (n+1)/2 \rfloor + 1,$$

with equality if and only if $m = \lfloor (n+1)/2 \rfloor$. This gives $\pi \in S_{n+1}$ of the required form (either π_0 or π_0^s).

A similar analysis for $\bar{\pi} = \pi_0^s$ gives extremal permutations only for $a \in \{m+1, m+2\}$ and $d(a) = 3$, so that $n = 4$ and $\bar{\pi} = [2413] \in S_4$. The permutations obtained are $\pi = [32514]$ and $\pi = [42513]$, which are $\pi_0^{rt}, \pi_0^{rst} \in S_5$, respectively.

The other possible values of $\bar{\pi}$ are obtained by the $\pi \mapsto \pi^r$ and $\pi \mapsto \pi^t$ operations from the ones above, and yield analogous results. □

4. Expectation

THEOREM 4.1. *For every positive integer n , the expected down degree of a random permutation in S_n is*

$$E_{\pi \in S_n} d_-(\pi) = \sum_{i=2}^n \sum_{j=2}^i \sum_{k=2}^j \frac{1}{i \cdot (k-1)} = (n+1) \sum_{i=1}^n \frac{1}{i} - 2n.$$

For a proof see [1].

COROLLARY 4.2. *As $n \rightarrow \infty$,*

$$E_{\pi \in S_n} d_-(\pi) = n \ln n + O(n)$$

and

$$E_{\pi \in S_n} d(\pi) = 2n \ln n + O(n).$$

5. Generalized Down Degrees

DEFINITION 5.1. For $\pi \in S_n$ and $1 \leq r < n$ let

$$D_-^{(r)}(\pi) := \{t_{a,b} \mid \ell(\pi) > \ell(t_{a,b}\pi) > \ell(\pi) - 2r\}$$

the r -th strong descent set of π .

Define the r -th down degree as

$$d_-^{(r)}(\pi) := \#D_-^{(r)}(\pi).$$

EXAMPLE 5.2. The first strong descent set and down degree are those studied in the previous section; namely, $D_-^{(1)}(\pi) = D_-(\pi)$ and $d_-^{(1)}(\pi) = d_-(\pi)$.

The $(n - 1)$ -st strong descent set is the set of inversions:

$$D_-^{(n-1)}(\pi) = \{t_{a,b} \mid a < b, \pi^{-1}(a) > \pi^{-1}(b)\}.$$

Thus

$$d_-^{(n-1)}(\pi) = \text{inv}(\pi),$$

the inversion number of π .

OBSERVATION 5.3. For every $\pi \in S_n$ and $1 \leq a < b \leq n$, $t_{a,b} \in D_-^{(r)}(\pi)$ if and only if $\pi = [\dots, b, \dots, a, \dots]$ and there are less than r letters between the positions of b and a in π whose value is between a and b .

DEFINITION 5.4. The r -th strong descent graph of $\pi \in S_n$, $\Gamma_r(\pi)$ is the graph whose set of vertices is $\{1, \dots, n\}$ and whose set of edges is

$$\{\{a, b\} \mid t_{a,b} \in D_-^{(r)}(\pi)\}.$$

The following lemma generalizes Lemma 2.7.

LEMMA 5.5. For every $\pi \in S_n$, the graph $\Gamma_r(\pi)$ contain no subgraph isomorphic to the complete graph K_{r+2} .

For a proof see [1].

COROLLARY 5.6. For every $1 \leq r < n$,

$$\max\{d_-^{(r)}(\pi) \mid \pi \in S_n\} \leq t_{r+1}(n) \leq \binom{r+1}{2} \left(\frac{n}{r+1}\right)^2.$$

PROOF. Combining Turán’s Theorem together with Lemma 5.5. □

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