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A new construction of the Loday-Ronco algebra

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ABSTRACT. We provide a new construction of the Loday-Ronco algebra by realizing it in terms of non-commutative polynomials in infinitely many variables. This construction relies on a bijection between words and labeled binary trees which can be regarded as a kind of degenerate Robinson-Schensted correspondence and leads to a new Knuth type correspondence involving binary trees.

RÉSUMÉ. Nous donnons une nouvelle construction de l'algèbre de Loday et Ronco en termes de polynômes non-commutatifs en une infinité de variables. Cette construction repose sur une bijection entre les mots et les arbres binaires étiquetés qui permet de définir une correspondence de type Robinson-Schensted dégénérée et aboutit à la construction d'une nouvelle correspondence de type Knuth mettant en jeu les arbres binaires.

1. Introduction

We give a new construction of the Loday-Ronco algebra of the plane binary trees, also known as the free dendriform dialgebra on one generator (see [**8**]). We first use, in Section 3, the argument given in [**9**] on dendriform trialgebras in order to prove that the algebra of non-commutative polynomials in infinitely variables can be endowed with the structure of a dendriform dialgebra. We then state that the sub-dialgebra generated by the sum of the letters is free as a dendriform dialgebra. To prove this statement, we introduce in Section 4 a bijection between words and labeled binary trees, which leads to a degenerate kind of Robinson-Schensted correspondence, reminiscent of the degenerate correspondence with ribbons and quasi-ribbon diagrams in [**5**], and dual to the Sylvester Schensted Algorithm of [**2**] as explained in Section 5. This leads, in Section 6, to a new Knuth type correspondence between integer matrices and some pairs of labeled binary trees. In Section 7 we define a family of elements indexed by binary trees that permits to prove that our dendriform dialgebra on one generator is free, using a bijection between binary trees and its elements.

2. Preliminaries and Notations

In this paper, $\mathbb{K}$ stands for a field of any characteristic. Let $A = \{a_1, a_2, ...\}$ be a totally ordered (infinite) alphabet and denote by A^* the free monoid on A. The map $max : A^* \to A$ maps a word w to its greatest letter, according to the total order of the alphabet A. We denote by $Std(w)$ the standardized word of $w \in A^*$ defined as follows.

DEFINITION 2.1. Let $w = w_1 \cdots w_n \in A^*$ and $Std(w) = w'_1 \cdots w'_n$. Then, $\forall i, j \in [1, n]$ with $i \neq j$:

- if $w_i > w_j$ then $w'_i > w'_j$,
- if $w_i = w_j$ with $i > j$, then $w'_i > w'_j$,

such that $Std(w)$ is a permutation.

For example $Std(abcadbcaa) = 157296834$. For a word $w \in A^*$ and a subset B of A, $w|_B$ stands for the subword of w obtained by erasing the letters which are not in B. The evaluation of a word w is the vector $ev(w) = (|w|_{a_1}, |w|_{a_2}, ...)$.

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We will denote by BT the set of all plane binary trees and LBT stands for the set of labeled plane binary trees. We denote by *Shape* the map that for a labeled binary tree forgets its labels and returns a binary tree of the same shape.

Let w be an element of A^* without repetition. Its decreasing tree $\mathcal{T}(w)$ is an element of LBT obtained as follows: the root is labeled by the greatest letter, n of w, and if $w = unv$, where u and v are words without repetition, the left subtree is $\mathcal{T}(u)$ and the right subtree is $\mathcal{T}(v)$. Moreover, we associate the empty tree to the empty word.

Let w be an element of A^*, we will denote by $\mathcal{B}(w)$ its associated binary search tree. It is obtained by reading w from *right-to-left*, each letter being inserted into a binary search tree in the following way: if the tree is empty, one creates a node labeled by the letter; otherwise, this letter is recursively inserted into the left (resp. right) subtree if it is smaller or equal than (resp. greater than) the root. Exemples will be given further.

A biletter on A is a pair $(a, b) \in A \times A$ which we will write $\begin{bmatrix} a \\ b \end{bmatrix}$ for convenience. A biword $\begin{bmatrix} u \\ v \end{bmatrix}$ on A^* is a concatenation of biletters $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\ldots\begin{bmatrix} u_n \\ v_n \end{bmatrix}$. We denote by $\begin{bmatrix} u' \\ v' \end{bmatrix}$ the nondecreasing rearrangement of $\begin{bmatrix} u \\ v \end{bmatrix}$ for the lexicographic order with priority on the top row, and by $\begin{bmatrix} u'' \\ v'' \end{bmatrix}$ the nondecreasing rearrangement for the lexicographic order with priority on the bottom row. Let $\langle\rangle$ denote the linear map from $\mathbb{K}[[A, B]]$ to $\mathbb{K}\langle\langle A\rangle\rangle \otimes \mathbb{K}\langle\langle B\rangle\rangle$, defined by $\langle\begin{pmatrix} u \\ v \end{pmatrix}\rangle = u'' \otimes v'$, with $u \in A$ and $v \in B$.

3. The free dendriform dialgebra embedded in words

Following a suggestion of [**9**], we define the following operations on words.

Definition 3.1. For all $u, v \in A^+$,

$$u \leftharpoonup v := \begin{cases} uv & \text{if } max(u) > max(v) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

$$u \rightharpoondown v := \begin{cases} uv & \text{if } max(u) \leq max(v) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Clearly, the usual operation of concatenation $\cdot$ on A^* can be written this way:

$$\cdot \;=\; \leftharpoonup + \rightharpoondown \quad . \tag{3.3}$$

Proposition 3.1. *$(\bigoplus_{n\geq 0} \mathbb{K}[A], \leftharpoonup, \rightharpoondown)$ is a dendriform dialgebra, in the sense of* [**7**].

Proof – Since (3.3) holds by definition, we only have to check the following three relations:

$$\begin{cases} (u \leftharpoonup v) \leftharpoonup w &= u \leftharpoonup (v \cdot w)\,, & (i) \\ (u \rightharpoondown v) \leftharpoonup w &= u \rightharpoondown (v \leftharpoonup w)\,, & (ii) \\ (u \cdot v) \rightharpoondown w &= u \rightharpoondown (v \rightharpoondown w)\,, & (iii) \end{cases} \tag{3.4}$$

whith $u, v, w \in A^+$. Notice first that for all these relations, there are only two possible values for each side, which are 0 and uvw.

(i) We first prove that

$$(u \leftharpoonup v) \leftharpoonup w = uvw \Longleftrightarrow u \leftharpoonup (v \cdot w) = uvw\,. \tag{3.5}$$

By definition, we have $(u \leftharpoonup v) \leftharpoonup w = uvw$ if and only if $max(u) > max(v)$ and $max(uv) > max(w)$. Since if $max(u) > max(v)$ then $max(uv) = max(u)$, we have a necessary and sufficient condition

$$max(u) > max(v) \;\wedge\; max(u) > max(w)\,. \tag{3.6}$$

On the right-hand side, we have $u \leftharpoonup (v \cdot w) = uvw$ if and only if

$$max(u) > max(vw)\,. \tag{3.7}$$

Since (3.6) and (3.7) are clearly equivalent, we have proved assertion (3.5).

(ii) We have $(u \rightharpoondown v) \leftharpoondown w = uvw$ if and only if

$$max(u) \leq max(v) \ \wedge \ max(w) < max(uv) \,. \tag{3.8}$$

But since $max(u) \leq max(v)$, assertion (3.8) is equivalent to

$$max(u) \leq max(v) \ \wedge \ max(w) < max(v) \,. \tag{3.9}$$

Moreover, $u \rightharpoondown (v \leftharpoondown w) = uvw$ if and only if

$$max(w) < max(v) \ \wedge \ max(u) \leq max(vw) \,,$$

which can be rewritten as

$$max(w) < max(v) \ \wedge \ max(u) \leq max(v) \,, \tag{3.10}$$

due to $max(w) < max(v)$. It results that $(u \rightharpoondown v) \leftharpoondown w = uvw$ if and only if $u \rightharpoondown (v \leftharpoondown w) = uvw$, by equivalence of assertions (3.9) and (3.10).

(iii) We have $u \rightharpoondown (v \rightharpoondown w) = uvw$ if and only if

$$max(u) \leq max(vw) \ \wedge \ max(v) \leq max(w) \,,$$

which can immediately be rewritten as

$$max(u) \leq max(w) \ \wedge \ max(v) \leq max(w) \,, \tag{3.11}$$

due to $max(v) \leq max(w)$. Moreover, we have $(u \cdot v) \rightharpoondown w = uvw$ if and only if $max(uv) \leq max(w)$, which is equivalent to (3.11). Hence it results $(u \cdot v) \rightharpoondown w = uvw$ if and only if $u \rightharpoondown (v \rightharpoondown w) = uvw$. ∎

Consider now the sub-dialgebra $\mathfrak{D}$ of $(\bigoplus_{n \geq 0} \mathbb{K}[A], \leftharpoondown, \rightharpoondown)$ generated by

$$P_\bullet := \sum_{a \in A} a \,.$$

There are two basis elements in the homogeneous component of degree 2 of $\mathfrak{D}$:

$$P_\bullet \leftharpoondown P_\bullet = \sum_{a<b} ba \,,$$

$$P_\bullet \rightharpoondown P_\bullet = \sum_{a \leq b} ab \,.$$

There are only five independent basis elements in the homogeneous component of degree 3 of $\mathfrak{D}$:

$$P_\bullet \leftharpoondown (P_\bullet \leftharpoondown P_\bullet) = \sum_{a<b<c} cba \,, \tag{3.12}$$

$$P_\bullet \rightharpoondown (P_\bullet \leftharpoondown P_\bullet) = \sum_{a<b;\ a' \leq b} a'ba \,, \tag{3.13}$$

$$P_\bullet \leftharpoondown (P_\bullet \rightharpoondown P_\bullet) = \sum_{a \leq b<c} cab \,, \tag{3.14}$$

$$(P_\bullet \leftharpoondown P_\bullet) \rightharpoondown P_\bullet = \sum_{a<b \leq c} bac \,, \tag{3.15}$$

$$(P_\bullet \rightharpoondown P_\bullet) \rightharpoondown P_\bullet = \sum_{a \leq b \leq c} abc \,, \tag{3.16}$$

since following equalities hold:

$$(P_\bullet \leftharpoondown P_\bullet) \leftharpoondown P_\bullet = \sum_{a<b;\ a'<b} baa' = (3.12) + (3.14) \,,$$

$$(P_\bullet \to P_\bullet) \leftarrow P_\bullet = \sum_{a \leq b;\ a' < b} aba' = (3.13)\,,$$

$$P_\bullet \to (P_\bullet \to P_\bullet) = \sum_{a \leq b;\ a' \leq b} a'ab = (3.15) + (3.16)\,,$$

which are exactly relations (3.4). We can now state our main result.

THEOREM 3.2. *$\mathfrak{D}$ is the free dendriform dialgebra on one generator.*

The remainder of this article will provide appropriate tools to prove this statement and exhibit some interesting remarks about them.

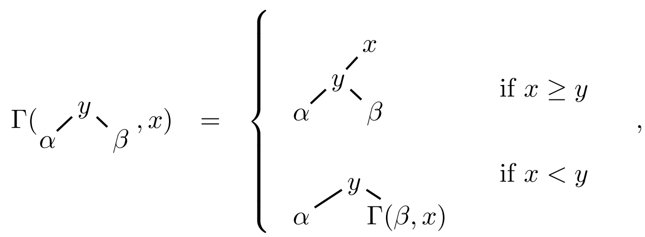
4. Algorithm Ψ

We first describe an algorithm Ψ that associates with a word a labelled plane binary tree. Then it will be possible to associate a plane binary tree with a word by considering only the shape of the labelled tree produced by algorithm Ψ. To this purpose we introduce a map

$$\Gamma : LBT \times A \longrightarrow LBT\,,$$

which can be recursively defined as follows:

(4.1)



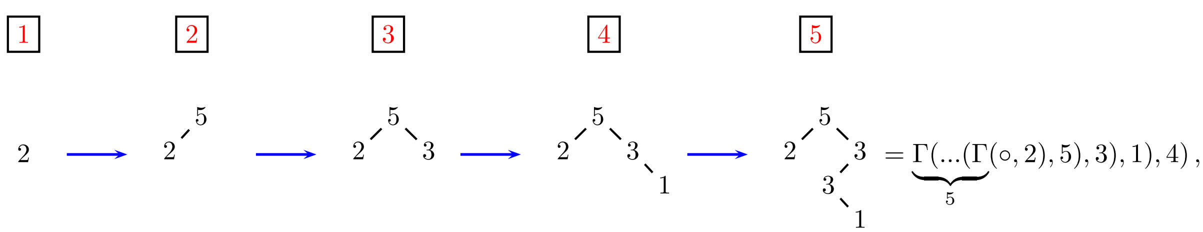
$$\Gamma(\circ, x) \quad = \quad x\,,$$

where ∘ stands for the empty tree and $\alpha, \beta \in LBT$.

DEFINITION 4.1. Consider the following function Ψ:

$$\Psi : \quad A^* \longrightarrow LBT\,, \qquad w = w_1 w_2 ... w_n \longmapsto \begin{cases} \Gamma(\Psi(w_1...w_{n-1}), w_n) & \text{if } n \geq 2\,, \\ \Gamma(\circ, w_1) & \text{if not}\,. \end{cases} \tag{4.2}$$

For example, using the alphabet $\mathbb{N}_{>0}$ with the natural order on integers, we apply Ψ to the word 25313 as follows:



starting with $\Gamma(\circ, 2)$.

PROPOSITION 4.1. *Let $w \in A^*$. Then, $Shape(\Psi(w)) = Shape(\Psi(Std(w)))$.*

Proof – We proceed by induction. The initial case, considering a word of size 1, is obvious since there is only one tree of size 1. Assuming that this property is satisfied for words of length $n-1$, we check that it is true for words of length n due to Definition 2.1 and the inductive definition (4.1). ∎

We recall that $\mathcal{T}$ stands for the decreasing tree algorithm which associates to a permutation its decreasing tree.

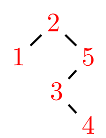
PROPOSITION 4.2. $\forall \sigma \in \mathfrak{S}, \Psi(\sigma) = \mathcal{T}(\sigma)$.

Proof – We proceed by induction. This property is obvious for the permutation of size 1. Assume that this property is satisfied for permutations of all sizes smaller than n and $\sigma = \sigma_1 \cdot max(\sigma) \cdot \sigma_2$ is a permutation of size n. By the inductive Definition 4.1 it is clear that the root of $\Psi(\sigma)$ is labeled by $max(\sigma)$, and that the left subtree of the root of $\Psi(\sigma)$ will be $\mathcal{T}(\sigma_1)$ by the inductive hypothesis, and similarly the right subtree of the root of $\Psi(\sigma)$ will be $\mathcal{T}(\sigma_2)$. ∎

Hence algorithm Ψ is clearly a generalization on words of the well-known decreasing tree algorithm for permutations. From Propositions 4.2 and 4.1 the following result is immediate.

PROPOSITION 4.3. *The algorithm Ψ is injective.*

Let $w \in A^*$, we now consider the labeled binary tree having the same shape as $\Psi(w)$ and for which the label of each node is the step of its insertion in the tree $\Psi(w)$. We will denote it by $\psi(w)$. From the previous calculation of $\Psi(25313)$ we get the following tree:



.

We notice that $\psi(25313)$ is the binary search tree of $Std(25313)^{-1} = 41352$. We develop a new Schensted-like correspondence and a new Schensted-Knuth-like correspondence from this consideration.

5. The Co-Sylvester Schensted Algorithm ($CSSA$)

The Sylvester Schensted Algorithm (SSA) has been introduced in [**2**]. From Algorithms Ψ and ψ we give a dual correspondence of SSA.

DEFINITION 5.1. We note $CSSA$ the *Co-Sylvester Schensted Algorithm* which sends a word $w \in A^*$ to the pair

$$(\Psi(w), \psi(w)) \,.$$

Algorithm $CSSB$ sends this pair to the word obtained by reading the labels of $\Psi(w)$ in the order of the corresponding labels in $\psi(w)$.

We know that

LEMMA 5.2 ([**2**]). *Let w be a word and $\sigma = Std(w)$. Then*

$$Shape(\mathcal{B}(w)) = Shape(\mathcal{B}(\sigma)) = Shape(\mathcal{T}(\sigma^{-1}))\,.$$

We generalize this result in terms of biwords.

LEMMA 5.3. *For any biword* $\begin{bmatrix} u \\ v \end{bmatrix}$ *, $Shape(\Psi(v')) = Shape(\mathcal{B}(u''))$.*

Proof – Using notations of Section 2, the standardization being compatible under transposition of two letters it follows that $Std(v)' = Std(v')$ and $Std(u)'' = Std(u'')$. It is well-known that the inverse of a permutation $\begin{bmatrix} Id \\ \sigma \end{bmatrix}$ is the biword $\begin{bmatrix} \sigma^{-1} \\ \sigma^{\uparrow} = Id \end{bmatrix}$ where $w^{\uparrow}$ denotes the nondecreasing rearrangement of a word w. Then, since $Std(v)' = (Std(u)'')^{-1}$, we have

$$Std(v') = (Std(u''))^{-1}\,.$$

Hence, using Lemma 5.2 and Proposition 4.1 we obtain these equivalences:

$$Shape(\Psi(v')) = Shape(\mathcal{T}(Std(v'))) = Shape(\mathcal{B}((Std(v'))^{-1})) = Shape(\mathcal{B}(u''))\,.$$

■

This allows us to state an analogous of the Schützenberger Theorem on tableaux for binary trees.

Theorem 5.4. *$\forall \sigma \in \mathfrak{S}$, $\psi(\sigma) = \mathcal{B}(\sigma^{-1})$.*

Proof – By definition of ψ, $\psi(\sigma)$ have the same shape as $\Psi(\sigma) = \mathcal{T}(\sigma)$. Moreover, by induction it is also clear that $\psi(\sigma)$ is a binary search tree. Indeed, assuming the inductive hypothesis for trees with $k-1$ nodes, the k-th node is, by definition of the algorithm ψ, labeled by k whereas the $k-1$ remaining nodes are labeled by elements of $[1,(k-1)]$. Moreover, by recursive definition (4.1) two cases arise:

- The k-th node is inserted at the root and then it is still a binary search tree since the $k-1$ remaining nodes are in its left subtree.
- The k-th node is inserted somewhere in the right subtree, and so it is still a binary search tree since it is greater than the root, and by inductive hypothesis.

Nevertheless, in general there is not a single permutation σ' such that $\mathcal{B}(\sigma') = T$, with T a given binary search tree (see [**2**] for the exact description of such sets of permutations). But it is clear, by definition of ψ, that at each step k of the insertion algorithm $CSSA$, $\Psi(\sigma_1\sigma_2...\sigma_k)$ and $\psi(\sigma_1\sigma_2...\sigma_k)$ have the same shape. Moreover setting

$$\begin{bmatrix} u(k)' \\ v(k)' \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_k \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} u(k)'' \\ v(k)'' \end{bmatrix} = \begin{bmatrix} \sigma^{-1}|_{[1,k]} \\ 1\,2\cdots k \end{bmatrix}\ ,$$

from Lemma 5.3 we have that at each step of the insertion algorithm, $\Psi(\sigma_1\sigma_2\cdots\sigma_k)$ and $\mathcal{B}(\sigma^{-1}|_{[1,k]})$ have the same shape. Hence, at the last step, $\psi(\sigma) = \mathcal{B}(\sigma^{-1})$. ■

Proposition 5.1. *$\forall\, w \in A^*, \psi(w) = \psi(Std(w))$.*

Proof – From Definition 2.1 and Proposition 4.1, it is immediate since at each step k, $\Psi(w_1\cdots w_k)$ and $\Psi(Std(w_1\cdots w_k))$ have the same shape. ■

Hence from Theorem 5.4 and Proposition 5.1 we immediately obtain the following result on words.

Corollary 5.5. *$\forall\, w \in A^*$, $\psi(w) = \mathcal{B}(Std(w)^{-1})$.*

From Theorem 5.4 and Proprosition 4.2 it is straightforward that $CSSA$ is the dual Schensted-like correspondence of SSA of [**2**] in the following sense.

Proposition 5.2. *$\forall \sigma \in \mathfrak{S}$,*

$$CSSA(\sigma) = (\Psi(\sigma), \mathcal{B}(\sigma^{-1})) \Longleftrightarrow SSA(\sigma^{-1}) = (\mathcal{B}(\sigma^{-1}), \Psi(\sigma))\,.$$

It is interesting to notice that these two correspondences look quite similar to the two Robinson-Schensted type correspondences on ribbons and quasi-ribbons, introduced in [**5**].

6. The Sylvester Schensted-Knuth correspondence

We first recall that there is an easy bijection between integer matrices and commutative biwords on A^* which consists to repeat m_{ij} times the biletter $\begin{pmatrix} \mathrm{i} \\ \mathrm{j} \end{pmatrix}$ for a matrix $M = (m_{ij})_{(i,j)\in[1,n]\times[1,m]}$ of dimensions $n \times m$. For example commutative biwords $\begin{pmatrix} 1111222333 \\ 1113123133 \end{pmatrix}$ and $\begin{pmatrix} 1112321233 \\ 1111123333 \end{pmatrix}$ (which are equal) have the same corresponding matrix which is :

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

Definition 6.1. Let $\begin{pmatrix} u \\ v \end{pmatrix}$ be a commutative biword and $\begin{bmatrix} u' \\ v' \end{bmatrix}$, $\begin{bmatrix} u'' \\ v'' \end{bmatrix}$ be the two biwords associated with $\begin{bmatrix} u \\ v \end{bmatrix}$ as explained in Section 2 . We note κ_S the *Sylvester Schensted-Knuth correspondence* defined as follows:

$$\kappa_S \begin{pmatrix} u \\ v \end{pmatrix} := (\Psi(v'), \mathcal{B}(u'')) .$$

This definition holds since for any biletter permutation of $\begin{bmatrix} u \\ v \end{bmatrix}$, v' and u'' remain the same. Moreover by Lemma 5.3 the following Proposition is straightforward.

Proposition 6.1. *For any commutative biword* $\begin{pmatrix} u \\ v \end{pmatrix}$, $Shape(\Psi(v')) = Shape(\mathcal{B}(u''))$.

We notice that $CSSA$ is recovered by encoding a word w by $\begin{pmatrix} 1 & 2 & \cdots & m \\ w_1 & w_2 & \cdots & w_m \end{pmatrix}$.

In order to prove that κ_S is a bijection, we proceed as Lascoux, Leclerc and Thibon did in [**6**] for the usual Knuth correspondence [**4**].

Theorem 6.2. *The algorithm* κ_S *is a bijection.*

Proof – Using again arguments of the proof of Lemma 5.3 we obtain $\mathcal{B}(Std(u'')) = \mathcal{B}(Std(v')^{-1})$. Then, applying Theorem 5.4 we have that $\mathcal{B}(Std(v')^{-1}) = \psi(Std(v'))$ and from Corollary 5.5 we get:

$$\mathcal{B}(Std(u'')) = \psi(v') .$$

This means that $\mathcal{B}(u'')$ is the unique binary search tree of evaluation $ev(u'')$ such that $\mathcal{B}(Std(u'')) = \psi(v')$. ■

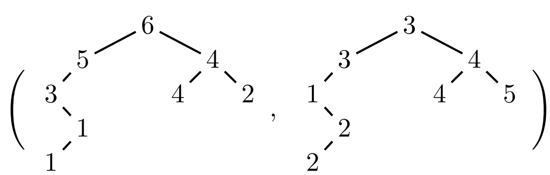
An easy remark is the following:

$$\kappa_S \begin{pmatrix} u \\ v \end{pmatrix} = (\Psi(v'), \mathcal{B}(u'')) \Longleftrightarrow \kappa_S \begin{pmatrix} v \\ u \end{pmatrix} = (\Psi(u''), \mathcal{B}(v')) ,$$

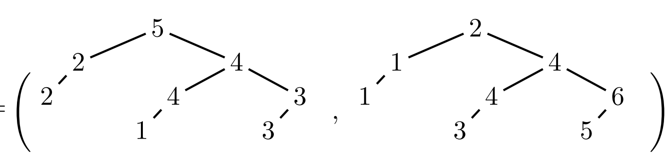
which generalizes results of Section 5. Nevertheless the symmetry of the usual Knuth correspondence is broken for P and Q symbols. At last, we give a full example of our construction.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 21335424 \\ 13652414 \end{bmatrix}, \quad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 12233445 \\ 31156442 \end{bmatrix} \quad \begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} 22514433 \\ 11234456 \end{bmatrix},$$

$(\Psi(v'), \mathcal{B}(u''))$ =

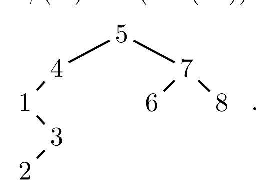


$(\Psi(u''), \mathcal{B}(v'))$ =



Since $Std(22514433) = 23816745$, we can check that $\psi(v') = \mathcal{B}(Std(u''))$:

$\psi(31156442) = \mathcal{B}(23816745) =$



This bijection leads to a Cauchy formula type for binary trees.

Proposition 6.2. *Let A and B be two non-commutative alphabets, such that A and B are commuting one with the other and words on alphabet B are quotiented by the sylvester congruence (see* [**2**]*). Then,*

$$\left\langle \overrightarrow{\prod_{a\in A,\, b\in B}} \frac{1}{1-ab} \right\rangle = \sum_{T\in BT} P_T(A)\otimes Q_T(B)\,,$$

where $(P_T)_{T\in BT}$ comes from Definition 7.1 and $(Q_T)_{T\in BT}$ is its dual basis introduced in [**2**].

Proof – We recall that the sylvester monoid introduced in [**2**] is the monoid such that two words having the same shape through algorithm $\mathcal{B}$ are equal. This proof needs the Free Cauchy identity mentionned in [**3**] and to be fully introduced in [**1**]:

$$\left\langle \overrightarrow{\prod_{a\in A,\, b\in B}} \frac{1}{1-ab} \right\rangle = \sum_{Std(v)=Std(u)^{-1}} u\otimes v\,. \tag{6.1}$$

Since v is an element of the sylvester monoid and by definition of $(Q_T)_{T\in BT}$ (see [**2**]), by Theorem 6.2 and by Proposition 6.1, right-hand side of Equation (6.1) can be rewritten as

$$\sum_{T\in BT}\left(\sum_{Shape(\Psi(w))=T} w\right)\otimes Q_T\,.$$

Using Definition 7.1 we obtain the desired equality. ■

7. Back to dendriform structure

Definition 7.1. Let $T\in BT$. We define

$$P_T := \sum_{w;\ Shape(\Psi(w))=T} w\,. \tag{7.1}$$

As a special case we recover:

$$P_\bullet = \sum_{a\in A} a\,. \tag{7.2}$$

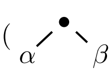
We now provide an algorithm that associates a plane binary tree to an element of the dendriform algebra $\mathfrak{D}$.

Definition 7.2. We consider

$$\Phi : BT \longrightarrow (\mathbb{K}[A], \leftharpoonup, \rightharpoondown)\,,$$

whose recursive definition is the following:

(7.3)

$\Phi($  $) = (\Phi(\alpha) \rightharpoondown (P_\bullet \leftharpoonup \Phi(\beta)))\,,$ (*i*)

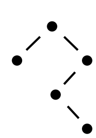
$\Phi($  $) = (\Phi(\alpha) \rightharpoondown P_\bullet)\,,$ (*ii*)

$\Phi($  $) = (P_\bullet \leftharpoonup \Phi(\beta))\,,$ (*iii*)

$\Phi(\bullet) = P_\bullet\,,$ (*iv*)

where $\alpha, \beta \in BT$.

For example:

$\Phi($  $) = (P_\bullet \rightharpoondown (P_\bullet \leftharpoonup ((P_\bullet \leftharpoonup P_\bullet) \rightharpoondown P_\bullet)))\,.$

LEMMA 7.3. *Let $T \in BT$. Then,* $P_T = \Phi(T)$ *.*

Proof – We proceed by induction. The initial case is immediate by (*iv*) of Definition 7.2. Assume that this property is satisfied for trees of size $n-1$. We consider a tree T of size n, with $n \geq 2$. Three kinds of trees are possible, according to wether their roots have only a non-empty right subtree or only a non-empty left subtree or finally have both left and right subtrees are non-empty. These cases correspond to $(i), (ii)$ and (iii) of (7.2).

(*i*) By Definition 7.1 this means that for all words appearing in the sum P_T, the $|\alpha| + 1$ letter is greater or equal than its $|\alpha|$ first letters and greater to its $|\beta|$ last letters. This is the exact meaning of the right-hand side of (*i*) using (3.1), (3.2), remembering relation (*ii*) (associativity) of (3.4) and assuming inductive hypothesis.

(*ii*) By Definition 7.1 this means that for all words appearing in the sum P_T, their last letter is greater or equal than all others letters appearing in it. Then, by (3.2) and by induction hypothesis we have proved this case.

(*iii*) By Definition 7.1 this means that for all words appearing in the sum P_T, their first letter is greater than all others letters appearing in it. Hence by (3.1) and by induction hypothesis we have proved this case.

■

We now are able to provide the proof of Theorem 3.2:

Proof of Theorem 3.2 – Since the Loday-Ronco algebra of plane binary trees is the free dendriform algebra on one generator, we only have to check that the Hilbert serie of this subalgebra $\mathfrak{D}$ generated by $P_\bullet$ is counted by Catalan numbers. To this purpose we consider the familly of $(P_T)_{T\in BT}$. By Lemma 7.3 they are clearly elements of $\mathfrak{D}$. Moreover the intersection of any pairs of them is always empty, by construction. Then $(P_T)_{T\in BT}$ are linearly independent. ■

References

[1] G. Duchamp, F. Hivert, J.-C. Novelli, J.-Y. Thibon, Noncommutative symmetric functions VII (under construction).
[2] F. Hivert, J.-C. Novelli, J.-Y. Thibon, The algebra of binary search trees, Theoret. Computer Sci. 339 (2005), 129–165.
[3] F. Hivert, J.-C. Novelli, J.-Y. Thibon, Un analogue du monoïde plaxique pour les arbres binaires de recherche, C.R. Acad. Sci. Paris Sr I Math. 332 (2002) 577-580.
[4] D. Knuth, "Permutations, matrices, and generalized Young tableaux", Pacific J. Math. 34 (1970), 709-727.
[5] D. Krob and J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at q = 0, J. Alg. Comb. 6 (1997), 339–376.
[6] A. Lascoux, B. Leclerc, and J.-Y. Thibon, The Plactic Monoid, in: Algebraic Combinatorics on Words, M. Lothaire, Univ. Press Cambridge (2001).
[7] J.-L. Loday, Dialgebras and related operads, Springer Lecture Notes in Mathematics 1763 (2001), 7-66.
[8] J.-L. Loday, M. O. Ronco, Hopf algebra of the planar binary trees, Adv. Math. 139 (1998), no. 2, 293–309.
[9] J.-C Novelli and J.-Y. Thibon, Construction of dendriform trialgebras, preprint math.CO/0510218.

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