Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Matrix compositions

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Abstract. In this paper we study the class of *m-row matrix compositions* (briefly, m-compositions), i.e. matrices with nonnegative integer entries, having m rows, and whose columns are different from the zero vector. We provide enumeration results, combinatorial identities, and various combinatorial interpretations. In particular we extend to the m-dimensional case most of the combinatorial properties of ordinary compositions.

Résumé. Dans cet article nous étudions la classe des *compositions de matrices de m-lignes* (appelées simplement m-compositions), dont les éléments sont des entiers positifs ou nuls, et sans vecteur colonne nul. Nous présentons, outre des interprétations combinatoires, leur énumération ainsi que des identités combinatoires. En particulier nous étendons au cas m-dimensionnel la plupart des propriétés combinatoires des compositions usuelles d'entiers.

1. Introduction

A *composition* (sometimes called *ordered partition*) of a natural number n is any k-tuple $\gamma = (x_1, \dots, x_k)$ of positive integers such that $x_1 + \cdots + x_k = n$. The elements x_i, k and n are the parts, the length and the order of γ, respectively. It is well known that there are $\binom{n-1}{k-1}$ compositions of length k of n and 2^{n-1} compositions of n, when $n \geq 1$. Compositions are very well known combinatorial objects [**1, 9, 13**] and several of their properties have been studied in some recent papers, as in [**7, 10, 14, 15, 17, 18, 23**].

In [**12**] the authors extended the definition of ordinary compositions introducing 2-*compositions* in order to have a bijection between this class and the class of L-convex polyominoes. Such an extension to the bidimensional case can be immediately generalized to the m-dimensional case. Indeed, for any positive integer m, an *m-row matrix composition*, or *m-composition* for short, is an $m \times k$ matrix with nonnegative integer entries

$$M = \begin{bmatrix} x_{11} & \dots & x_{1k} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mk} \end{bmatrix}$$

whose columns are different from the zero vector. We say that the number k of columns is the *length* of the composition. Moreover we say that M is an *m-composition* of a nonnegative integer n if the sum of all its elements is exactly n. We will write $\sigma(M)$ for the sum of all the elements of the matrix M. For instance, there are seven 2-compositions of 2:

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The aim of this paper is to study the class of m-compositions by several points of view, and to extend to the m-dimensional case most of the combinatorial properties of ordinary compositions.

2000 *Mathematics Subject Classification.* Primary 05A17; Secondary 05A15.
Key words and phrases. compositions, integer partitions, enumeration.

We also remark that our matrix compositions are very similar to the *vector compositions* [**1**, p.57] defined by P. A. MacMahon [**20, 21, 22**] and studied for instance in [**2, 3, 4**]. Another extension of ordinary compositions is described in [**19**].

2. Combinatorial identities

As a first step we present several identities about m-compositions, obtained using elementary combinatorial arguments. Since some of the proofs in this section are rather simple, sometimes they will only be sketched.

Let us start by recalling some basic definitions and properties of multisets. A *multiset* on a set X is a function $\mu : X \to \mathbb{N}$. The *multiplicity* of an element $x \in X$ is $\mu(x)$. The *order* of μ is the sum $\mathrm{ord}(\mu)$ of the multiplicities of the elements of X, i.e. $\mathrm{ord}(\mu) = \sum_{x\in X} \mu(x)$. The number of all multisets of order k on a set of size n is the *multiset coefficient*

$$\left(\!\!\binom{n}{k}\!\!\right) = \frac{n^{\overline{k}}}{k!} = \frac{n(n+1)\dots(n+k-1)}{k!}\,.$$

Let $\mathcal{C}^{(m)}_{n,k}$ be the set of all m-compositions of n of length k and let $c^{(m)}_{n,k} = |\mathcal{C}^{(m)}_{n,k}|$. Similarly let $\mathcal{C}^{(m)}_{n}$ be the set of all m-compositions of n and let $c^{(m)}_{n} = |\mathcal{C}^{(m)}_{n}|$. Let us observe that any $M \in \mathcal{C}^{(m)}_{n+m,k+1}$ can be decomposed into two parts: the first column, equivalent to a multiset of $[m] = \{1,\dots,m\}$ of nonzero order i and the rest of the matrix, that is any m-composition of $n+m-i$ of length k. Hence it follows the recurrence:

$$c^{(m)}_{n+m,k+1} = \sum_{i=1}^{n+m-k} \left(\!\!\binom{m}{i}\!\!\right) c^{(m)}_{n+m-i,k}\,. \tag{2.1}$$

The same argument yields the identity

$$c^{(m)}_{n+m} = \sum_{i=1}^{n+m} \left(\!\!\binom{m}{i}\!\!\right) c^{(m)}_{n+m-i}\,. \tag{2.2}$$

Now we will use some arguments based on the Principle of Inclusion-Exclusion. Let A_i be the set of all m-compositions M of $n+m$ with a positive entry in position $i1$. Then, since the first column of M is different from the zero vector, it follows that $\mathcal{C}^{(m)}_{n+m} = A_1 \cup \ldots \cup A_m$ and from the Principle of Inclusion-Exclusion

$$c^{(m)}_{n+m} = |A_1 \cup \ldots \cup A_m| = \sum_{\substack{S\subseteq[m]\\ S\neq\emptyset}} (-1)^{|S|-1} \left|\bigcap_{i\in S} A_i\right|\,.$$

The set $\bigcap_{i\in S} A_i$ is formed of all the m-compositions $M = [x_{i,j}]$ of $n+m$ having positive entries in the first column in the positions indexed by S. If we replace each element $x_{i,1}$, $i \in S$, with $x_{i,1} - 1$, we have two cases: the first column of M is the zero vector or it is not. In the first case removing the first column we have an m-compositions of $n+m-|S|$, while in the second case we just have an m-composition of $n+m-|S|$. Hence

$$\left|\bigcap_{i\in S} A_i\right| = 2c^{(m)}_{n+m-|S|}\,.$$

Since this result depends only on the size of S it follows that

$$c^{(m)}_{n+m} = 2\sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c^{(m)}_{n+m-i}\,. \tag{2.3}$$

For instance for $m = 2,3,4$ we have the recurrences

$$c^{(2)}_{n+2} = 4c^{(2)}_{n+1} - 2c^{(2)}_{n}, \qquad c^{(3)}_{n+3} = 6c^{(3)}_{n+2} - 6c^{(3)}_{n+1} + 2c^{(3)}_{n}, \qquad c^{(4)}_{n+4} = 8c^{(4)}_{n+3} - 12c^{(4)}_{n+2} + 8c^{(4)}_{n+1} - 2c^{(4)}_{n}\,.$$

We remark that the recurrence $c^{(2)}_{n+2}$ was first obtained in [**12**]. Exactly with the same argument we can obtain the following recurrence

$$c^{(m)}_{n+m,k+1} = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c^{(m)}_{n+m-i,k} + \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c^{(m)}_{n+m-i,k+1}\,. \tag{2.4}$$

m/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	2	4	8	16	32	64	128	256	512	1024
2	1	2	7	24	82	280	956	3264	11144	38048	129904	443520
3	1	3	15	73	354	1716	8318	40320	195444	947380	4592256	22260144
4	1	4	26	164	1031	6480	40728	255984	1608914	10112368	63558392	399478064
5	1	5	40	310	2395	18501	142920	1104060	8528890	65885880	508970002	3931805460
6	1	6	57	524	4803	44022	403495	3698352	33898338	310705224	2847860436	26102905368

FIGURE 1. Table of the numbers $c_n^{(m)}$, with $m = 0, \ldots, 6$.

Let A_i be the set of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ with the i-th column is equal to the zero vector such that $\sigma(M) = n$. Then $\mathcal{C}_{n,k}^{(m)} = A_1' \cap \ldots \cap A_k'$ and from the Principle of Inclusion-Exclusion

$$c_{n,k}^{(m)} = |A_1' \cap \ldots \cap A_k'| = \sum_{S \subseteq [k]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right| .$$

The set $\bigcap_{i \in S} A_i$ is formed of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ with the zero vector in each column indexed by the elements of S. It corresponds to the set of all multisets of order n on a set of size $mk - m|S|$ and so

$$\left| \bigcap_{i \in S} A_i \right| = \left(\!\!\binom{m(k-|S|)}{n}\!\!\right) .$$

Since this result depends only on the size of S it follows that

$$c_{n,k}^{(m)} = \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right) (-1)^i . \tag{2.5}$$

Moreover

$$c_n^{(m)} = \sum_{k=0}^{n} c_{n,k}^{(m)} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right) (-1)^i . \tag{2.6}$$

This argument can be easily generalized as follows. Consider the set $\mathcal{C}_k^{(m)}(r_1, \ldots, r_m)$ of all m-compositions of length k where the i-th row has sum equal to r_i, for each $i = 1, \ldots, m$, and let $c_k^{(m)}(r_1, \ldots, r_m)$ be its cardinality. Now let A_i denote the set of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ having the i-th column equal to the zero vector, and row-sums $r_1, \ldots, r_m$. Then $\mathcal{C}_k^{(m)}(r_1, \ldots, r_m) = A_1' \cap \ldots \cap A_k'$, and from the Principle of Inclusion-Exclusion

$$c_k^{(m)}(r_1, \ldots, r_m) = |A_1' \cap \ldots \cap A_k'| = \sum_{S \subseteq [k]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right| .$$

The set $\bigcap_{i \in S} A_i$ is formed of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ with the zero vector in each column indexed by the elements of S. The i-th row of such a matrix M corresponds to a multiset of order r_i on a set of size $k - |S|$. Hence it follows that

$$\left| \bigcap_{i \in S} A_i \right| = \left(\!\!\binom{k-|S|}{r_1}\!\!\right) \cdots \left(\!\!\binom{k-|S|}{r_m}\!\!\right) .$$

Since the result depends again only on the size of S it follows that

$$c_k^{(m)}(r_1, \ldots, r_m) = \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{k-i}{r_1}\!\!\right) \cdots \left(\!\!\binom{k-i}{r_m}\!\!\right) (-1)^i . \tag{2.7}$$

The Table in Fig. 1 reports the first terms of the sequences $c_n^{(m)}$, with $m = 0, 1, \ldots, 6$. We remark that for $m \geq 3$ the sequence $c_n^{(m)}$ is not present in [**27**].

3. Enumeration of m-compositions trough formal languages

A large amount of combinatorial properties of m-compositions can simply be derived by encoding them as words on an infinite alphabet. In fact, an m-composition can be viewed as the concatenation of its columns. This implies that the set $\mathcal{C}^{(m)}$ of all m-composition is equivalent to the free language $\mathcal{A}^*$ on the infinite alphabet $\mathcal{A}^{(m)} = \{a_\mu \, : \, \mu \in \mathcal{M}^{(m)}_{\neq 0}\}$, where $\mathcal{M}^{(m)}_{\neq 0}$ is the set of all multisets $\mu : [m] \to \mathbb{N}$ with positive order and the letter a_μ corresponds to the column $\begin{bmatrix}\mu(1) & \ldots & \mu(m)\end{bmatrix}^T$. Substituting each letter a_μ with an indeterminate x_μ, it follows immediately that the generating series of $\mathcal{C}^{(m)}$ is

$$c(X) = \frac{1}{1 - \sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} x_\mu} \tag{3.1}$$

where $X = \{x_\mu \, : \, \mu \in \mathcal{M}^{(m)}_{\neq 0}\}$. In particular, for $x_\mu = x^{\mathrm{ord}(\mu)}$ we get the generating series

$$c^{(m)}(x) = \sum_{n \geq 0} c^{(m)}_n \, x^n = \frac{1}{1 - h(x)} \quad \text{where} \quad h(x) = \sum_{k \geq 1} \left(\!\!\binom{m}{k}\!\!\right) x^k = \frac{1}{(1-x)^m} - 1 \, .$$

Hence

$$c^{(m)}(x) = \frac{(1-x)^m}{2(1-x)^m - 1} \, , \tag{3.2}$$

from which we can derive the recurrence (2.3) already obtained in the previous section. Similarly, for $x_\mu = x^{\mathrm{ord}(\mu)} y$ we get the generating series

$$c(x,y)^{(m)} = \sum_{n,k \geq 0} c^{(m)}_{n,k} \, x^n y^k = \frac{1}{1 - h(x) y}. \tag{3.3}$$

By the series (3.2), and making some easy computations, we obtain the following results:

(1) a recurrence relation for the numbers $c^{(m)}_{n+1}$,

$$c^{(m)}_{n+1} = -\delta_{n,0} + 2c^{(m)}_n + \sum_{k=0}^{n} \binom{m+k-1}{k+1} c^{(m)}_{n-k}$$

which generalizes the following identity satisfied by the number $c^{(2)}_n$ of 2-compositions [**12**]: $c^{(2)}_{n+2} = 3c^{(2)}_{n+1} + c^{(2)}_n + \ldots + c^{(2)}_0$.

(2) the following Binet-like formula:

$$c^{(m)}_n = \frac{1}{2} \left[\delta_{n,0} + \frac{1}{m \sqrt[m]{2}} \sum_{k=0}^{m-1} \frac{\omega_m^k}{(x_k)^{n+1}} \right]$$

where $x_k = 1 - \frac{1}{\sqrt[m]{2}} \omega_m^k$, $k = 0, 1, \ldots, m-1$, and $\omega_m = \mathrm{e}^{2\pi \mathrm{i}/m}$ is a primitive root of the unity. From this expression we obtain an asymptotic expansion for the coefficients $c^{(m)}_n$,

$$c^{(m)}_n \sim -\frac{A_0}{2x_0^{n+1}} = \frac{1}{2m(\sqrt[m]{2} - 1)} \left(\frac{\sqrt[m]{2}}{\sqrt[m]{2} - 1} \right)^n \qquad \text{as} \quad n \to \infty \, .$$

In particular we have

$$c^{(m)}_{n+1} \sim \frac{\sqrt[m]{2}}{\sqrt[m]{2} - 1} \, c^{(m)}_n \qquad \text{as} \quad n \to \infty \, .$$

A regular language for m-compositions. Extending the encoding used in [**7**] for the ordinary compositions, we are able to prove that m-compositions can be encoded as words on the alphabet $\mathcal{A}_m = \{a_1, \cdots, a_m, b_1, \ldots, b_m\}$. Let us define a map $\ell : \mathcal{C}^{(m)} \to \mathcal{A}_m^*$ setting

$$\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \overset{\ell}{\longmapsto} a_1 \, , \quad \ldots \quad \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \overset{\ell}{\longmapsto} a_m \, , \qquad + \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \overset{\ell}{\longmapsto} b_1 \, , \quad \ldots \quad + \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \overset{\ell}{\longmapsto} b_m \tag{3.4}$$

and proceeding as follows. First of all write an m-composition M as the formal sum (i.e. juxtaposition) of its columns (as in the previous case). Then write each column as juxtaposition of simple columns where a simple column is a column in which all the entries except one are zero. We stipulate to order the simple columns according to the position of the nonzero entry. At this point write each simple column as juxtaposition of elementary columns, where an elementary column is a column in which all the entries are zero except one equal to 1. Hence, if the nonzero entry of a simple column is k then it will be written as the juxtaposition of k elementary columns. Finally substitute each elementary column with the corresponding letter according to the encoding in (3.4). An example will explain better the correspondence. Consider the 3-composition

$$M = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Following the described procedure we have

$$M = \begin{matrix} 2 \\ 0 \\ 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} + \begin{matrix} 2 \\ 1 \\ 2 \end{matrix} = \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} + \begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{matrix}$$

and hence $\ell(M) = a_1a_1a_3b_2b_1a_3b_1a_1a_2a_3a_3$.

Let $\mathcal{L}_m = \ell(\mathcal{C}^{(m)})$ be the language on the alphabet $\mathcal{A}_m$ corresponding to the m-compositions. The words of $\mathcal{L}_m$ are characterized by the following conditions: i) each word begins with one letter $a_1, \ldots, a_m$; ii) each letter a_i or b_i can be followed by any b_j, while it can be followed by a letter a_j only when $i \leq j$. This implies that these words have a unique factorization of the form xy where:

(1) x is a non-empty word of the form $a_1^{i_1} \ldots a_m^{i_m}$, with $i_1, \ldots, i_m \geq 0$;
(2) y is a (possibly empty) word $y = y_1 \ \ldots \ y_k$, with $y_r = b_j \, a_j^{q_j} \ldots a_m^{q_m}$, with $q_j, \ldots, q_m \geq 0$.

According to such a characterization $\mathcal{L}_m$ is a regular language defined by the unambiguous regular expression:

$$\varepsilon + \left(a_1^+ a_2^* \ldots a_m^* + a_2^+ a_3^* \ldots a_m^* + \ldots + a_m^+ \right) \left(b_1 a_1^* a_2^* \ldots a_m^* + b_2 a_2^* \ldots a_m^* + \ldots + b_m a_m^* \right)^*$$

where, as usual, ε denotes the empty word.

4. Combinatorial interpretations

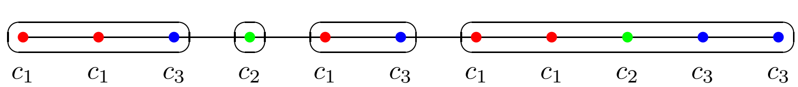
In this section we present three combinatorial interpretations for m-compositions. Here, for brevity's sake, we will give only the basic definitions, even though the relations between the structural properties of the different classes deserve a further investigation.

4.1. Colored linear partitions. m-compositions can be interpreted in terms of linear species [**5, 16**] as follows. Let $C = \{c_1, \ldots, c_m\}$ be a set of colors totally ordered in the natural way $c_1 < \cdots < c_m$. We say that the linearly ordered set $[n] = \{1, 2, \ldots, n\}$ is *m-colored* when each element is colored with one color in C respecting the following condition: if c_i and c_j are the respective colors of two elements x and y, with $x \leq y$, then $i \leq j$. In other words, an m-coloring of $[n]$ is an order preserving map $\gamma : [n] \to C$. We define an m-colored linear partition of $[n]$ as a linear partition in which each block is m-colored.

The m-compositions of length k of n are equivalent to the m-colored linear partitions of $[n]$ with k blocks. Indeed any $M \in \mathcal{C}_{n,k}^{(m)}$ corresponds to the m-colored linear partition π of $[n]$ obtained transforming the i-th column $(h_1, \ldots, h_m)$ of M into the i-th block of π of size $h_1 + \cdots + h_m$ with the first h_1 elements colored with c_1, ..., the last h_m elements colored with c_m, for every $1 \leq i \leq k$. For instance, the 3-composition

$$M = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

corresponds to the following 3-colored partition of the set $\{1, \ldots, 11\}$:



Let $\mathbf{Comp}_m$ be linear species of the m-compositions, i.e. the linear species of m-colored linear partitions. To give a structure of this species on a linearly ordered set L is equivalent to assign a linear partition π on L and then an m-coloring, that is an order preserving map in C, on each block of π. Then, if $\mathbf{G}$ denotes the uniform linear species and $\mathbf{Map}^{(m)}_{\neq\emptyset}$ denotes the linear species of the order preserving maps from a nonempty linear order to the set of colors C, we have that

$$\mathbf{Comp}_m = \mathbf{G} \circ \mathbf{Map}^{(m)}_{\neq\emptyset} .$$

An order preserving map $f : [k] \to [m]$ is equivalent to a multiset of order k on the set $[m]$. Hence it follows that

$$\mathbf{Card}(\mathbf{Map}^{(m)}_{\neq\emptyset}; x) = \sum_{k\geq 1} \left(\!\!\binom{m}{k}\!\!\right) x^k = \frac{1}{(1-x)^m} - 1$$

and consequently $\mathbf{Card}(\mathbf{Comp}_m; x) = \mathbf{Card}(\mathbf{G}; x) \circ \mathbf{Card}(\mathbf{Map}^{(m)}_{\neq\emptyset}; x) = c^{(m)}(x)$.

Using this interpretation we can obtain some useful identities. Let $\pi \in \mathbf{Comp}_m[L]$, where $L = \{x_1, \ldots, x_{i+1}, \ldots, x_{i+j+1}\}$ has size $i+j+1$. The element x_{i+1} belongs to a block of the form $\{x_{i-h+1}, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k+2}\}$ where $h, k \in \mathbb{N}$. Removing such a block, π splits into two m-colored linear partitions of a linear order of size $i-h$ and a linear order of size $j-k$, respectively. Then it follows that

$$c^{(m)}_{i+j+1} = \sum_{h,k\geq 0} \left(\!\!\binom{m}{h+k+1}\!\!\right) c^{(m)}_{i-h} c^{(m)}_{j-k} . \tag{4.1}$$

Recall that $\left(\!\binom{m}{i+j+1}\!\right)$ gives the number of all the order maps $f : [i+j+1] \to [m]$. Suppose that $f(i+1) = k$, with $k \in [m]$. Since f is order preserving, it follows that $f(x) \in [k]$ for every $x \in [i]$ and $f(x) \in \{k, \ldots, m\}$ for every $x \in \{i+2, \ldots, i+j+1\}$. Then

$$\left(\!\!\binom{m}{i+j+1}\!\!\right) = \sum_{k=1}^{m} \left(\!\!\binom{k}{i}\!\!\right) \left(\!\!\binom{m-k+1}{j}\!\!\right) = \sum_{k=0}^{m-1} \binom{i+k}{i} \left(\!\!\binom{m-k}{j}\!\!\right) . \tag{4.2}$$

4.2. Surjective families. Let $P_1, \ldots, P_m$ and Q be linearly ordered sets. Consider a family $\{f_i : P_i \to Q\}_{i=1}^m$ of order preserving maps with the following property: for every element $q \in Q$ there exists at least one index i and one element $p \in P_i$ such that $q = f_i(p)$. The single maps are not necessarily surjective but every element of the codomain admits at least one preimage along one of the maps of the family. Hence we call *surjective family* any family with such a property.

Now we can ask how many surjective families are there, when $|P_1| = r_1, \ldots, |P_m| = r_m$ and $|Q| = k$. The answer is: $c^{(m)}_k(r_1, \ldots, r_m)$. Indeed given a surjective family $\{f_i : P_i \to Q\}_{i=1}^m$ we can build up an m-composition M of length k as follows. The i-row of M is generated by the map $f_i : P_i \to Q$ taking as entries the numbers of the preimages of the elements of Q along f_i, that is defining it as $[\ |f_i^\bullet(1)| \ldots |f_i^\bullet(k)|\ ]$, where $f_i^\bullet(y)$ denotes the set of all preimages of y along f_i. Clearly the sum of this row is $|P_i| = r_i$. Moreover any column of M is different from the zero vector for the characterizing property of the surjective families. So, finally, we have that M is an m-composition of length k with row-sum vector $(r_1, \ldots, r_k)$.

4.3. Labelled bargraphs. A *bargraph* is a column-convex polyomino, such that the lower edge lies on the horizontal axis. It is uniquely defined by the heights of its columns, see Figure 2 (a). The enumeration of bargraphs according to perimeter, area, and site-perimeter has been treated in [**25, 26**], related to the study of percolation models, and more recently, by an analytical point of view, in [**8**]. For basic definitions on polyominoes we refer to [**6**].

Here we deal with *labelled bargraphs*, i.e. bargraphs whose cells are all labelled with positive integer numbers, and such that, for each column, the label of a cell is less then or equal to the label of the cell immediately above (if any), see Figure 2 (b). The degree is the maximal label of the bargraph. For any given $m \geq 1$, every m-composition of an integer n can be represented as a labelled bargraph of degree $j \leq m$ having n cells, as follows. Let M be an m-composition of $n \geq 0$, having length k, and let $\mathbf{c}_i^T = (a_{1j}, \ldots, a_{mj})$ be the j-th column of M. We build a bargraph made of k columns, of degree m at most, where the j-th column has exactly $a_{1j} + \ldots + a_{mj}$ cells, and a_{ij} is the number of cells with label i in the j-th column, which are placed, according to the definition of labelled bargraph, just above the cells

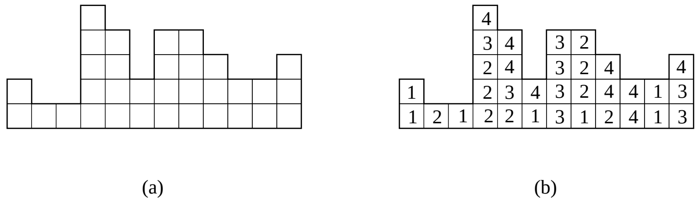


FIGURE 2. (a) a bargraph; (b) a labelled bargraph of degree 4.

with label $i-1$ (if any). For instance, the bargraph in Fig. 2 (b) is associated with the 4-composition of 33:

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}.$$

Of course, ordinary compositions (i.e. 1-compositions) are represented as bargraphs of degree 1, i.e. the usual bargraphs, as already pointed out in [**23**].

Some subclasses of m-compositions. The simple correspondence between m-compositions and labelled bargraphs can be applied to determine bijections between particular subclasses. So, for instance we can consider:

(1) the set of bargraphs having all the m labels in each column (Fig. 3 (a)); it corresponds to the set of m-compositions containing no 0s. The generating function of such objects is $1+\frac{\frac{x^m}{(1-x)^m}}{1-\frac{x^m}{(1-x)^m}} = \frac{1}{1-\left(\frac{x}{1-x}\right)^m}$.

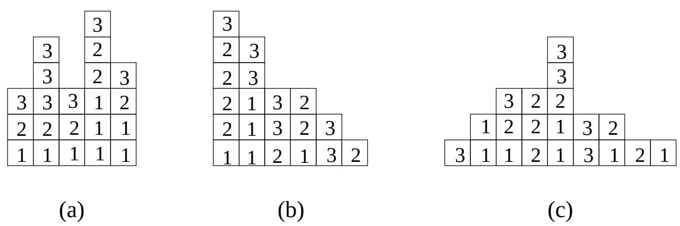


FIGURE 3. Labelled bargraphs of degree 3: (a) having all the labels in each of its columns; (b) a 3-partition; (c) a labelled stack of degree 3.

(2) the set of *labelled Ferrers diagrams*, i.e. those labelled bargraphs for which each column has height greater than or equal to the height of the column on its right, see Fig. 3 (b). A labelled Ferrers diagram of degree m corresponds to an m-composition such that the sum of the entries of each column is greater than or equal to the sum of the entries of column on its right. We call these objects m-partitions. This definition is motivated by the fact that the ordinary partitions correspond to Ferrers diagrams, i.e. labelled Ferrers diagrams of degree 1. For instance, the bargraph in Fig. 3 (b) corresponds to the 3-partition of 20:

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 0 & 2 & 0 \end{bmatrix}.$$

(3) the set of *labelled stacks*, i.e. of those labelled bargraphs for which each row is connected; these objects have indeed the shape of stack polyominoes, see Fig. 3 (c). A labelled stack of degree m corresponds to an m-composition such that the sequence $c_1, \ldots, c_k$ is unimodal, being c_i the sum of the entries of the i-th column.

The problem of enumerating labelled Ferrers diagrams and labelled stacks has been solved in [**24**] in a more general context.

5. Combinatorial properties of m-compositions

5.1. Cassini-like identities. In [**12**] it has been proved that the numbers $c_n^{(2)}$ of all 2-compositions of n satisfy the Cassini-like identity: $c_n^{(2)}c_{n+2}^{(2)} - (c_{n+1}^{(2)})^2 = -2^{n-1}$, for every $n \geq 1$. Here we prove that such an identity can be generalized to the numbers $c_n^{(m)}$. Specifically we prove that

$$\begin{vmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-1}^{(m)} & c_{n+m}^{(m)} & \cdots & c_{n+2m-2}^{(m)} \end{vmatrix} = (-1)^{\lfloor m/2 \rfloor}\, 2^{n-1} \tag{5.1}$$

for every $m, n \geq 1$. Let $C_n^{(m)} = [\ c_{n+i+j}^{(m)}\ ]_{i,j=0}^{m-1}$. Its i-th row is $\mathbf{r}_i = [\ c_{n+i+j}^{(m)}\ ]_{j=0}^{m-1}$. In particular, by recurrence (2.3), the last row is

$$\mathbf{r}_m = 2\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}\mathbf{r}_{m-k} = 2\sum_{k=1}^{m-1}\binom{m}{k}(-1)^{k-1}\mathbf{r}_{m-k} + (-1)^{m-1}2\,\mathbf{r}_0$$

where $\mathbf{r}_0 = [\ c_{n-1+j}^{(m)}\ ]_{j=0}^{m-1}$. Then subtracting to the last row the following linear combination of the previous rows

$$2\sum_{k=1}^{m-1}\binom{m}{k}(-1)^{k-1}\mathbf{r}_{m-k}$$

the last row of $\det C_n^{(m)}$ becomes $(-1)^{m-1}2\,\mathbf{r}_0$. Extracting $(-1)^{m-1}2$ from the last line and then shifting cyclically all rows downward we obtain that

$$\det C_n^{(m)} = 2\det C_{n-1}^{(m)}\,.$$

Then, for every $n \geq 1$, it follows that: $\det C_n^{(m)} = 2^{n-1}\det C_1^{(m)}$. So we have to compute only the determinant of the matrix $C_1^{(m)} = [\ c_{i+j+1}^{(m)}\ ]_{i,j=0}^{m-1}$. By identity (4.1) we have the decomposition $C_1^{(m)} = L_m M_m L_m^T$ where $L_m = [\ c_{i-j}^{(m)}\ ]_{i,j=0}^{m-1}$, $M_m = \left[\ \left(\!\binom{m}{i+j+1}\!\right)\ \right]_{i,j=0}^{m-1}$ and L_m^T is the transpose of L_m. Since L_m is a triangular matrix with unitary diagonal elements, it follows that $\det C_1^{(m)} = \det M_m$. Now identity (4.2) implies that $M_m = B_m\widetilde{B}_m$ where $B_m = [\ \binom{i+j}{i}\ ]_{i,j=0}^{m-1}$ and $\widetilde{B}_m = \left[\ \left(\!\binom{m-i}{j}\!\right)\ \right]_{i,j=0}^{m-1}$. Being $\widetilde{B}_m = J_m B_m$ where $J_m = [\delta_{i+j,m-1}]_{i,j=0}^{m-1}$, it is $M_m = B_m J_m B_m$ and $\det M_m = \det J_m(\det B_m)^2$. Since, as very well known, $\det J_m = (-1)^{\lfloor m/2 \rfloor}$ and $\det B_m = 1$, it follows that $\det M_m = (-1)^{\lfloor m/2 \rfloor}$ and consequently $\det C_1^{(m)} = (-1)^{\lfloor m/2 \rfloor}$. Finally we have: $\det C_n^{(m)} = (-1)^{\lfloor m/2 \rfloor}2^{n-1}$, for every $n \geq 1$.

5.2. m-compositions without zero rows. In this section we will study the m-compositions in which every row is different from the zero vector. We begin by determining an expression for the number $f_n^{(m)}$ of all such m-compositions of n. Let A_i be the set of all m-compositions $M \in \mathcal{C}_n^{(m)}$ where the i-th row is zero. Then

$$f_n^{(m)} = |A_1' \cap \cdots \cap A_m'| = \sum_{S\subseteq[m]}(-1)^{|S|}\left|\bigcap_{i\in S}A_i\right|.$$

Since $\bigcap_{i\in S}A_i$ is clearly in a bijective correspondence with the set of all $(m-|S|)$-compositions of n, it follows that

$$f_n^{(m)} = \sum_{k=0}^{m}\binom{m}{k}(-1)^k c_n^{(m-k)} = \sum_{k=0}^{m}\binom{m}{k}(-1)^{n-k}c_n^{(k)}\,. \tag{5.2}$$

On the other hand, the set $\mathcal{C}_n^{(m)}$ can be partitioned according to the number of zero rows and this yields the following identity:

$$c_n^{(m)} = \sum_{k=0}^{m}\binom{m}{k}f_n^{(k)}\,. \tag{5.3}$$

Clearly this formula can be also obtained formally by inverting (5.2). From (5.2) also follows that the generating series for the numbers $f_n^{(m)}$ is

$$f^{(m)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} c^{(k)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{(1-x)^k}{2(1-x)^k - 1} .$$

Then this series has the form

$$f^{(m)}(x) = \frac{x^m p_m(x)}{(1-2x)(1-4x+2x^2)\cdots(2(1-x)^m - 1)} \tag{5.4}$$

where $p_m(x)$ is a polynomial with degree (less than or) equal to $\binom{m}{2}$. This implies that, for $n \geq 1$, the numbers $f_n^{(m)}$ satisfy a homogeneous linear recurrence with constant coefficients of order $\binom{m+1}{2}$, which can be deduced from the denominator of the series (5.4).

Now we will establish an explicit formula for the numbers $f_n^{(m)}$. Since $f_n^{(m)}$ counts all m-compositions in which every row-sum is nonzero, it immediately follows that

$$f_n^{(m)} = \sum_{k \geq 0} \sum_{\substack{(r_1,\ldots,r_m) \in \mathbb{Z}_+{}^m \\ r_1+\cdots+r_m = n}} c_k^{(m)}(r_1,\ldots,r_m) = \sum_{k \geq 0} \sum_{\substack{\rho \in \mathbb{Z}_+{}^m \\ |\rho| = n}} c_k^{(m)}(\rho)$$

where $\rho = (r_1, \ldots, r_m)$ and $|\rho| = r_1 + \cdots + r_m$. Then, using (2.7), we have the formula

$$f_n^{(m)} = \sum_{\substack{\rho \in \mathbb{Z}_+{}^m \\ |\rho| = n}} \sum_{k \geq 0} \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{k-i}{r_1}\!\!\right) \cdots \left(\!\!\binom{k-i}{r_m}\!\!\right) (-1)^i . \tag{5.5}$$

Clearly $f_n^{(m)} = 0$ whenever $n < m$. Consider now the case $m = n$. In this case we have only the vector $\rho = (1, \ldots, 1)$ and the identity (5.5) becomes

$$f_n^{(n)} = \sum_{k \geq 0} \left[\sum_{i=0}^{k} \binom{k}{i} (k-i)^n (-1)^i \right] .$$

The sum in the brackets is very well known and gives the number of surjective functions from a set of size n to a set of size k. Moreover it can be expressed it terms of the Stirling numbers of the second kind, and precisely it is equal to $\left\{ {n \atop k} \right\} k!$. Then

$$f_n^{(n)} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! .$$

But also this sum is very well known, and gives the number t_n of all *preferential arrangements* on a set of size n (sequence A000670 in [**27**]). So, in conclusion, we have that $f_n^{(n)} = t_n$.

This result can be generalized. Indeed in the formula for $f_{n+1}^{(n)}$ we have only the n vectors $\rho = (1, \ldots, 1, 2, 1, \ldots, 1)$. Hence (5.5) becomes

$$f_{n+1}^{(n)} = \frac{n}{2} \sum_{k \geq 0} \sum_{i=0}^{k} \binom{k}{i} (k-i)^n (k-i+1)(-1)^i = \frac{n}{2} \left(\sum_{k \geq 0} \left\{ {n+1 \atop k} \right\} k! - \sum_{k \geq 0} \left\{ {n \atop k} \right\} k! \right)$$

that is

$$f_{n+1}^{(n)} = \frac{n}{2} \left(t_{n+1} + t_n \right) .$$

Similarly, when we consider $f_{n+2}^{(n)}$, we have only the n vectors $\rho = (1, \ldots, 1, 3, 1, \ldots, 1)$ and the $\binom{n}{2}$ vectors $\rho = (1, \ldots, 1, 2, 1, \ldots, 1, 2, 1, \ldots, 1)$. Hence (5.5), after simplification, becomes

$$f_{n+2}^{(n)} = \frac{n}{24} \left[(3n+1) t_{n+2} + 6(n+1) t_{n+1} + (3n+5) t_n \right] .$$

All these results suggest that there exist polynomials $p_i^{(k)}(x)$ such that

$$f_{n+k}^{(n)} = \sum_{i=0}^{k} p_i^{(k)}(n) \, t_{n+i} .$$

The nature of such polynomials needs some further investigations.

m/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	2	2	4	4	8	8	16	16	32	32
2	1	2	5	8	18	28	62	96	212	328	724	1120
3	1	3	9	19	48	96	236	468	1146	2270	5556	11004
4	1	4	14	36	101	240	648	1520	4082	9560	25660	60088
5	1	5	20	60	185	501	1470	3910	11390	30230	88002	233530
6	1	6	27	92	309	930	2939	8640	27048	79280	247968	726672

Figure 4. Table of the numbers $p_n^{(m)}$.

5.3. m-compositions with palindromic rows. An ordinary composition is *palindromic* when its elements are the same in the given or in the reverse order. In the literature palindromic compositions have been studied by various authors [**10, 11, 23**]. Here we generalize this definition to the m-compositions saying that an m-composition is palindromic when all its rows are palindromic. For instance the following is a palindromic 4-composition of length 5 of 24:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Clearly every m-compositions with palindromic rows has the form $[M|M_s]$ when its length is even and the form $[M|\mathbf{v}|M_s]$ when its length is odd, where M is an arbitrary m-composition, M_s is the specular m-composition obtained from M by reversing every row and $\mathbf{v}$ is an arbitrary column vector. Hence the generating series for the m-compositions with palindromic rows is given by

$$p^{(m)}(x) = \sum_{n\geq 0} p_n^{(m)}\, x^n = c^{(m)}(x^2) + \left(\frac{1}{(1-x)^m} - 1\right) c^{(m)}(x^2) = \frac{1}{(1-x)^m}\, c^{(m)}(x^2) = \frac{(1+x)^m}{2(1-x^2)^m - 1}.$$

From this identity it immediately follows that

$$p_n^{(m)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\!\!\binom{m}{n-2k}\!\!\right) c_k^{(m)}.$$

The first terms of $p_n^{(m)}$ are reported in Fig. 4. Let now $q_n^{(m)}$ be the number of all m-compositions of n with palindromic non zero rows. With arguments completely similar to the ones used in the case of ordinary m-compositions we have that

$$p_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} q_n^{(k)}, \qquad q_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} p_n^{(k)}.$$

Notice that when $n = m$ there is just one n-composition with palindromic rows, given by the column vector with all entries equal to 1. Hence $q_n^{(n)} = 1$.

5.4. m-compositions of Carlitz type. We say that an m-composition is of *Carlitz type* when no two adjacent columns are equal. When $m = 1$ we obtain the ordinary *Carlitz compositions* [**9**]. As in Section 2, also m-compositions of Carlitz type can be viewed as words on the infinite alphabet $\mathcal{A}^{(m)} = \{a_\mu : \mu \in \mathcal{M}_{\neq 0}^{(m)}\}$. Let $\mathcal{Z}$ be the set of all words corresponding to the m-composition of Carlitz type and let $\mathcal{Z}_\mu$ be the subset of $\mathcal{Z}$ formed exactly by the words ending with a_μ, for every $\mu \in \mathcal{M}_{\neq 0}^{(m)}$. It immediately follows that

$$\mathcal{Z} = 1 + \sum_{\mu \in \mathcal{M}_{\neq 0}^{(m)}} \mathcal{Z}_\mu \qquad \text{and} \qquad \mathcal{Z}_\mu = (\mathcal{Z} - \mathcal{Z}_\mu) a_\mu \qquad \forall\, \mu \in \mathcal{M}_{\neq 0}^{(m)}.$$

m/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	3	4	7	14	23	39	71	124	214
2	1	2	5	18	53	162	505	1548	4756	14650	45065	138622
3	1	3	12	58	255	1137	5095	22749	101625	454116	2028939	9065145
4	1	4	22	136	793	4660	27434	161308	948641	5579224	32811986	192971168
5	1	5	35	265	1925	14056	102720	750255	5480235	40031030	292408771	2135917405
6	1	6	51	458	3984	34788	303902	2654064	23179743	202445610	1768099107	15442052496

FIGURE 5. Table of the numbers $z_n^{(m)}$.

In order to obtain the generating series associated with the languages $\mathcal{Z}$ and $\mathcal{Z}_\mu$ it is sufficient to replace the letter a_μ with the indeterminate x_μ, thus obtaining the linear system

$$z(X) = 1 + \sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} z_\mu(X) \qquad \text{and} \qquad z_\mu(X) = (z(X) - z_\mu(X))x_\mu \qquad \forall\, \mu \in \mathcal{M}^{(m)}_{\neq 0}$$

from which

$$z_\mu(X) = \frac{x_\mu}{1 + x_\mu}\, z(X) \qquad \text{and then} \qquad z(X) = \frac{1}{1 - \displaystyle\sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} \frac{x_\mu}{1 + x_\mu}}\,.$$

Setting $x_\mu = x^{\operatorname{ord}(\mu)}$, we obtain the generating series for the coefficients $z_n^{(m)}$ giving the number of all m-compositions of Carlitz type of n. Specifically we have

$$z^{(m)}(x) = \sum_{n \geq 0} z_n^{(m)}\, x^n = \frac{1}{1 - \displaystyle\sum_{k \geq 1} \left(\!\!\binom{m}{k}\!\!\right) \frac{x^k}{1 + x^k}}\,. \tag{5.6}$$

For $m = 1$ we reobtain the generating series for the ordinary Carlitz compositions. The sequence $z_n^{(1)}$ appears in [**27**] as the sequence #A003242, while for $m \geq 2$ the corresponding sequences are absent. The first terms of $z_n^{(m)}$ are reported in Fig. 5.

From series (5.6) it is possible to obtain the following explicit formula for the numbers $z_n^{(m)}$. Indeed

$$\begin{aligned}
z^{(m)}(x) &= \sum_{k \geq 0} \left(\sum_{n \geq 1} \left(\!\!\binom{m}{n}\!\!\right) \frac{x^n}{1 + x^n} \right)^k = \sum_{k \geq 0} \sum_{a_1 \geq 1} \left(\!\!\binom{m}{a_1}\!\!\right) \frac{x^{a_1}}{1 + x^{a_1}} \cdots \sum_{a_k \geq 1} \left(\!\!\binom{m}{a_k}\!\!\right) \frac{x^{a_k}}{1 + x^{a_k}} \\
&= \sum_{k \geq 0} \sum_{a_1, \dots, a_k \geq 1} \left(\!\!\binom{m}{a_1}\!\!\right) \cdots \left(\!\!\binom{m}{a_k}\!\!\right) \frac{x^{a_1}}{1 + x^{a_1}} \cdots \frac{x^{a_k}}{1 + x^{a_k}} \\
&= \sum_{k \geq 0} \sum_{\substack{a_1, \dots, a_k \geq 1 \\ b_1, \dots, b_k \geq 1}} \left(\!\!\binom{m}{a_1}\!\!\right) \cdots \left(\!\!\binom{m}{a_k}\!\!\right) (-1)^{b_1 + \cdots + b_k - k} x^{a_1 b_1 + \cdots + a_k b_k}\,.
\end{aligned}$$

Then

$$z^{(m)}(x) = \sum_{n \geq 0} \left(\sum_{k \geq 0} \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^k \\ \alpha \cdot \beta = n}} \left(\!\!\binom{m}{\alpha}\!\!\right) (-1)^{|\beta| - k} \right) x^n$$

where if $\alpha = (a_1, \dots, a_k)$ and $\beta = (b_1, \dots, b_k)$ then $\alpha \cdot \beta = a_1 b_1 + \cdots + a_k b_k$, $|\beta| = b_1 + \cdots + b_k$ and $\left(\!\binom{m}{\alpha}\!\right) = \left(\!\binom{m}{a_1}\!\right) \dots \left(\!\binom{m}{a_k}\!\right)$. Finally, we have the following expression

$$z_n^{(m)} = \sum_{k \geq 0} \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^k \\ \alpha \cdot \beta = n}} \left(\!\!\binom{m}{\alpha}\!\!\right) (-1)^{|\beta| - k}\,.$$

With the same argument used in [**9**] by Carlitz it is possible to obtain the following expression for the series $z^{(m)}(x)$:

$$z^{(m)}(x) = \frac{1}{1 + \sum_{k\geq 1} (-1)^k \frac{1-(1-x^k)^m}{(1-x^k)^m}} . \tag{5.7}$$

Let now $g_n^{(m)}$ be the number of all m-compositions of Carlitz type of n without zero rows. With arguments completely similar to the ones used in the case of ordinary m-compositions we have that

$$z_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} g_n^{(k)}, \qquad g_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} z_n^{(k)} .$$

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