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The Horn recursion for Schur P- and Q- functions
Extended Abstract

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Abstract. A consequence of work of Klyachko and of Knutson-Tao is the Horn recursion to determine when a Littlewood-Richardson coefficient is non-zero. Briefly, a Littlewood-Richardson coefficient is non-zero if and only if it satisfies a collection of Horn inequalities which are indexed by smaller non-zero Littlewood-Richardson coefficients. There are similar Littlewood-Richardson numbers for Schur P- and Q- functions. Using a mixture of combinatorics of root systems, combinatorial linear algebra in Lie algebras, and the geometry of certain cominuscule flag varieties, we give Horn recursions to determine when these other Littlewood-Richardson numbers are non-zero. Our inequalities come from the usual Littlewood-Richardson numbers, and while we give two very different Horn recursions, they have the same sets of solutions. Another combinatorial by-product of this work is a new Horn-type recursion for the usual Littlewood-Richardson coefficients.

Résumé. Une des conséquences du travail de Klyachko et de Knutson-Tao est un système de récurrences de Horn pour déterminer quand un coefficient de Littlewood-Richardson est non nul. En bref, un tel coefficient est non nul si et seulement si il satisfait une collection d'inégalités de type Horn, dont les indices sont des coefficients de Littlewood-Richardson plus petits et non nuls. Il existe des nombres de Littlewood-Richardson comparables pour les P- et Q- fonctions de Schur. En utilisant des outils provenant combinatoire des systèmes de racines, d'algèbre linéaire dans le contexte des algébre de Lie, et de la géométrie des variétés de drapeaux cominiscules, nous obtenons un système de récurrences de type Horn pour déterminer quand cette famille de nombres de Littlewood-Richardson sont non nuls. Ces inégalités sont basées sur les nombres de Littlewood-Richardson habituels, et même si les deux systèmes sont très différents, ils ont la même solution. Une autre conséquence de ce travail est une nouvelle récurrence de type Horn pour les coefficients Littlewood-Richardson habituels.

Introduction

The Littlewood-Richardson numbers $a^{\lambda}_{\mu,\nu}$ for partitions λ, μ, ν are important in many areas of mathematics. For example, they are the structure constants of several related rings with distinguished bases: the ring of symmetric functions with its basis of Schur functions, the representation ring of $\mathfrak{sl}_n$ with its basis of irreducible highest weight modules, the external representation ring of the tower of symmetric groups with its basis of irreducible modules, and the cohomology ring of the Grassmannian with its basis of Schubert classes [**Ful97, Mac95, Sta99**]. The combinatorics of Littlewood-Richardson numbers are extremely interesting and now we have many formulas for them, including the original Littlewood-Richardson formula [**LR34**]. Despite this deep and prolonged interest in Littlewood-Richardson numbers, one of the most fundamental questions about them was not asked until about a decade ago:

$$\text{When is } a^{\lambda}_{\mu,\nu} \text{ non-zero?}$$

This question came from (of all places) a problem in linear algebra concerning the possible eigenvalues of a sum of hermitian matrices. The answer to this problem is given by the Horn inequalities: the eigenvalues

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which can and do occur are the solutions to a set of linear inequalities, and the inequalities themselves come from non-negative integral eigenvalues solving this problem for smaller matrices.

The same inequalities answer our question about Littlewood-Richardson numbers. A Littlewood-Richardson number $a^{\lambda}_{\mu,\nu}$ is non-zero if and only if the triple of partitions (λ,μ,ν) satisfy certain linear inequalities, and the inequalities themselves come from triples indexing smaller non-zero Littlewood-Richardson coefficients. This description is a consequence of work of Klyachko [**Kly98**] which linked eigenvalues of sums of hermitian matrices, highest weight modules of $\mathfrak{sl}_n$, and the Schubert calculus for Grassmannians, and then Knutson and Tao's proof [**KT99**] of Zelevinsky's Saturation Conjecture [**Zel99**]. This work implies Horn's Conjecture [**Hor62**] about the eigenvalues of sums of Hermitian matrices. These results have wide implications in mathematics (see the surveys [**Ful98, Ful00**]) and have raised many new and evocative questions. For example, the Horn inequalities give the answer to questions in several different realms of mathematics (representation theory, combinatorics, Schubert calculus, eigenvalues), but it was initially mysterious why any one of these questions should have a recursive answer, as the proofs travelled through so many other parts of mathematics.

Another question, which was the *point de départ* for the results we describe here, is the following: are there related numbers whose non-vanishing has a similar recursive answer? Our main result is a recursive answer for the non-vanishing of the analogs of Littlewood-Richardson coefficients for Schur P-functions, and the same for Schur Q-functions. We give one set of inequalities which determine non-vanishing for the P-functions and a different set of inequalities for the Q-functions. Because each Schur P-function is a non-zero multiple of a corresponding Schur Q-function, a Littlewood-Richardson number for P-functions is non-zero if and only if the corresponding number for Q-functions is non-zero, and thus our two sets of inequalities have the same sets of solutions.

Another consequence of our work is a new set of recursive Horn-type inequalities for the ordinary Littlewood-Richardson numbers $a^{\lambda}_{\mu,\nu}$. While these new inequalities are clearly related to the ordinary Horn inequalities, they are definitely quite different. (We explain this below.)

Before we define some of these objects and give the different recursions, we remark that our results were proved using a mixture of the combinatorics of root systems, combinatorial linear algebra in Lie algebras, and the geometry of certain *cominuscule* flag varieties G/P. Cominuscule flag varieties are (almost all of) the flag varieties whose geometrically defined Bruhat order (which is the Bruhat order on the cosets W/W_P of the Weyl groups) is a distributive lattice.

The alert reader will notice that these inequalities for Schur P- and Q-functions are not strictly recursive because they are indexed by ordinary Littlewood-Richardson numbers which are non-zero. The reason for the term recursive is that the inequalities stem from a geometric recursion among all cominuscule flag varieties which is not evident from viewing only the subclass corresponding to, say the Schur Q-functions.

This abstract does not dwell on the geometry, but rather on the fascinating combinatorial consequences of these recursions. The last section of this extended abstract does however give a broad view of some of the key geometric ideas which underly our recursion. The results here are proved in the forthcoming preprint by the authors, "The recursive nature of the cominuscule Schubert calculus".

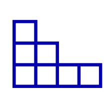
1. The classical Horn recursion

For more details and definitions concerning the various flavors of Schur functions that arise here, we recommend the book of Macdonald [**Mac95**]. *Schur functions S_λ* are symmetric functions indexed by *partitions λ*, which are weakly decreasing sequences of nonnegative integers, $\lambda\colon \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The Schur function S_λ is homogeneous of degree $|\lambda| := \lambda_1 + \cdots + \lambda_n$. Schur functions form a basis for the $\mathbb{Z}$-algebra of symmetric functions. Thus there are integral *Littlewood-Richardson* numbers $a^{\lambda}_{\mu,\nu}$ defined by the identity

$$S_\mu \cdot S_\nu \;=\; \sum_{\lambda} a^{\lambda}_{\mu,\nu}\, S_\lambda\,.$$

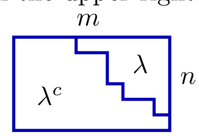
Homogeneity gives the necessary relation $|\lambda| = |\mu| + |\nu|$ for $a^{\lambda}_{\mu,\nu} \neq 0$.

A partition λ is represented by its *diagram*, which is a left-justified array of boxes in the positive quadrant with λ_i boxes in row i. Thus

$(4,2,1) \;\leftrightarrow$ 

Partitions are partially ordered by the inclusion of their diagrams. Let $n \times m$ be the rectangular partition with n parts, each of size m.

For $\lambda \subset n \times m$ ($\lambda_1 \leq m$ and $\lambda_{n+1} = 0$), define λ^c to be the partition obtained from the set-theoretic difference $n \times m - \lambda$ of diagrams (placing λ in the upper right corner of $n \times m$). Thus we have



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The Horn-type inequalities we give are naturally stated in terms of *symmetric Littlewood-Richardson numbers*. For $\lambda, \mu, \nu \subset n \times m$, define

$$\begin{aligned} a_{\lambda,\mu,\nu} &:= \text{Coefficient of } S_{n\times m} \text{ in } S_\lambda S_\mu S_\nu \\ &= \text{Coefficient of } S_{\lambda^c} \text{ in } S_\mu S_\nu \ = \ a^{\lambda^c}_{\mu,\nu}\,. \end{aligned}$$

We say that a triple of partitions $\lambda, \mu, \nu \subset n \times m$ is *feasible* if $a_{\lambda,\mu,\nu} \neq 0$. This convenient terminology comes from geometry.

Definition 1.1. Suppose that $\lambda \subset n \times m$ and $\alpha \subset r \times (n-r)$, where $0 < r < n$. Define

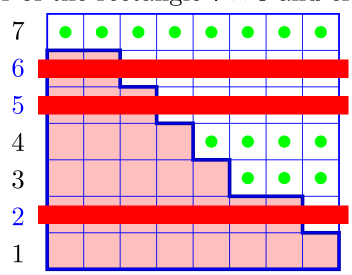
$$I_n(\alpha) \ := \ \{n-r+1-\alpha_1,\ n-r+2-\alpha_2,\ \ldots,\ n-\alpha_r\}\,.$$

Draw λ in the upper right corner of the $n \times m$ rectangle, and number the rows Cartesian-style. Define $|\lambda|_\alpha$ to be the number of boxes that remain in λ after crossing out the rows indexed by $I_n(\alpha)$.

Example 1.2. Suppose that $n = 7$, $m = 8$, and $r = 3$, and we have $\lambda = 8654310$ and $\alpha = 311$. Then the set $I_7(\alpha)$ is

$$\{7-3+1-3,\ 7-3+2-1,\ 7-3+3-1\} \ = \ \{2,5,6\}\,.$$

If we place λ in the upper-right corner of the rectangle 7×8 and cross out the rows indexed by $I_7(\alpha)$,



we count the dots • to see that $|\lambda|_\alpha = 15$.

Theorem 1.3 (Classical Horn Recursion: Klyachko [**Kly98**], Knutson-Tao [**KT99**]).
A triple $\lambda, \mu, \nu \subset n \times m$ is feasible if and only if $|\lambda| + |\mu| + |\nu| = nm$, and

$$|\lambda|_\alpha + |\mu|_\beta + |\nu|_\gamma \ \leq \ (n-r)m$$

for all feasible triples $\alpha, \beta, \gamma \subset r \times (n-r)$ and for all $0 < r < n$.

The first condition, $|\lambda| + |\mu| + |\nu| = nm$, is due to homogeneity.

2. Symmetric Horn recursion

Since replacing a partition λ by its conjugate λ^t (interchanging rows with columns) induces an involution on the algebra of symmetric functions, there is a version of the Horn recursion where one crosses out columns instead of rows. It turns out that there are necessary inequalities obtained by crossing out both rows and columns, including possibly a different number of each. The cominuscule recursion reveals a sufficient subset of these.

Definition 2.1. Let $0 < r < \min\{n,m\}$ and suppose that $\lambda \subset n \times m$, $\alpha \subset r \times (n-r)$, and we have another partition $\alpha' \subset r \times (m-r)$. Define $I_n(\alpha)$ as before, and set

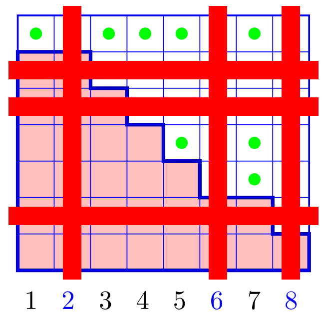
$$I_m(\alpha') \ := \ \{m-r+1-\alpha'_1,\ m-r+2-\alpha'_2,\ \ldots,\ m-\alpha'_r\}\,.$$

Draw λ in the upper right corner of the $n \times m$ rectangle and cross out the rows indexed by $I_n(\alpha)$ and the columns indexed by $I_m(\alpha')$. Define $|\lambda|_{\alpha,\alpha'}$ to be the number of boxes that remain in λ.

Example 2.2. We use the same data as in Example 1.2, and set $\alpha' = 410$. Then

$$I_8(\alpha') \;=\; \{8-3+1-4,\; 8-3+2-1,\; 8-3+3-0\} \;=\; \{2,6,8\}\,.$$

If we now cross out the rows indexed by $I_7(\alpha)$ and the columns indexed by $I_8(\alpha')$,



we count the dots • to see that $|\lambda|_{\alpha,\alpha'} = 8$.

Theorem 2.3 (Symmetric Horn Recursion).
A triple $\lambda, \mu, \nu \subset n \times m$ is feasible if and only if $|\lambda| + |\mu| + |\nu| = nm$, and

$$|\lambda|_{\alpha,\alpha'} + |\mu|_{\beta,\beta'} + |\nu|_{\gamma,\gamma'} \;\leq\; (m-r)(n-r)$$

for all pairs of feasible triples $\alpha, \beta, \gamma \subset r \times (n-r)$ and $\alpha', \beta', \gamma' \subset r \times (m-r)$ and for all $0 < r < \min\{m, n\}$.

3. Schur P- and Q- functions

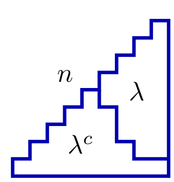
The algebra of symmetric functions has a natural odd subalgebra which comes from its structure as a combinatorial Hopf algebra [**ABS06**]. This algebra was first studied by Schur in the context of the theory of projective representations of the symmetric group. This odd subalgebra has a pair of distinguished bases, the *Schur P-functions* and the *Schur Q-functions*. They are indexed by strict partitions, which are strictly decreasing sequences of positive integers λ: $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. They are proportional: $Q_\lambda = 2^k P_\lambda$, where λ has k parts.

We have Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$ and $d^\lambda_{\mu,\nu}$ indexed by triples of strict partitions and defined by the identities

$$Q_\mu \cdot Q_\nu \;=\; \sum_\lambda c^\lambda_{\mu,\nu}\, Q_\lambda \qquad \text{and} \qquad P_\mu \cdot P_\nu \;=\; \sum_\lambda d^\lambda_{\mu,\nu}\, P_\lambda\ .$$

Combinatorial formulas for these numbers were given in work of Worley [**Wor84**], Sagan [**Sag87**], and Stembridge [**Ste89**].

Let $\triangle_n \colon n > n{-}1 > \cdots > 2 > 1$ be the strict partition of staircase shape. Then $\lambda \subset \triangle_n$ if $\lambda_1 \leq n$. If $\lambda \subset \triangle_n$, define λ^c to be the partition obtained from the set-theoretic difference $\triangle_n - \lambda$ of diagrams (placing λ in the upper right corner of $\triangle_n$). Thus we have



As before, the Horn-type inequalities are naturally stated in terms of symmetric Littlewood-Richardson numbers. For $\lambda, \mu, \nu \subset \triangle_n$, define

$$\begin{aligned} c_{\lambda,\mu,\nu} \;&:=\; \text{Coefficient of } Q_{\triangle_n} \text{ in } Q_\lambda Q_\mu Q_\nu \\ &=\; \text{Coefficient of } Q_{\lambda^c} \text{ in } Q_\mu Q_\nu \;=\; c^{\lambda^c}_{\mu,\nu}\,. \end{aligned}$$

A triple of strict partitions $\lambda, \mu, \nu \subset n \times m$ is *feasible* if $c_{\lambda,\mu,\nu} \neq 0$.

We similarly define symmetric Littlewood-Richardson numbers $d_{\lambda,\mu,\nu}$ for the Schur P-functions. Since the two bases are proportional, the corresponding coefficients are as well. In particular the sets of triples λ, μ, ν for which the corresponding coefficients are feasible are the same. Nevertheless, we give two very different sets of inequalities which determine the feasibility of these numbers, arising from the different geometric origins of Schur Q-functions and Schur P-functions.

Definition 3.1. Let $0 < r < n$ and suppose that $\lambda \subset \triangle_n$ is a strict partition and $\alpha \subset r \times (n-r)$ is an ordinary partition. Draw λ in the upper right corner of the staircase $\triangle_n$. Number the inner corners $1, 2, \ldots, n$ and, for each number in $I_n(\alpha)$, cross out the row and column emanating from that inner corner. Then let $[\lambda]_\alpha$ be the number of boxes that remain in λ.

Definition 3.2. Let $0 < r < n+1$ and suppose that $\lambda \subset \triangle_n$ is a strict partition and $\alpha \subset r \times (n+1-r)$ is an ordinary partition. Draw λ in the upper right corner of the staircase $\triangle_n$. Number the outer corners $1, 2, \ldots, n, n+1$ and for each number in $I_{n+1}(\alpha)$, cross out the row and column emanating from the corresponding outer corner. Then let $\{\lambda\}_\alpha$ be the number of boxes that remain in λ.

Example 3.3. For example, suppose that $n = 8$ and $r = 4$, we have $\lambda = 8643$ and $\alpha = 4220$. Then

$$\begin{aligned} I_8(\alpha) &= \{8-4+1-4,\ 8-4+2-2,\ 8-4+3-2,\ 8-4+4-0\} = \{1,4,5,8\}, \\ I_9(\alpha) &= \{9-4+1-4,\ 9-4+2-2,\ 9-4+3-2,\ 9-4+4-0\} = \{2,5,6,9\}. \end{aligned}$$

and if we place λ in the upper-right corner of the rectangle $\triangle_8$ and cross out the rows and columns emanating from the inner corners indexed by $I_8(\alpha)$, we see that $[\lambda]_\alpha = 6$. If we instead cross out the rows and columns emanating from the outer corners indexed by $I_9(\alpha)$, we see that $\{\lambda\}_\alpha = 5$. The two diagrams are shown in Figure 1, on the left and right, respectively.

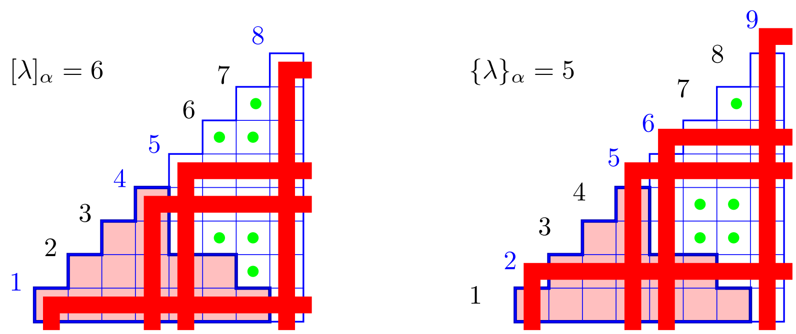


Figure 1. Computation of $[\lambda]_\alpha = 6$ and of $\{\lambda\}_\alpha = 5$

Note that the homogeneity of the multiplication of Schur P-functions and Schur Q-functions implies that

$$|\lambda| + |\mu| + |\nu| \;=\; |\triangle_n| \;=\; \binom{n+1}{2}, \tag{3.1}$$

is necessary for a triple $\lambda, \mu, \nu \subset \triangle_n$ to be feasible.

Theorem 3.4 (Horn recursion for Schur P- and Q-functions).
A triple $\lambda, \mu, \nu \subset \triangle_n$ of strict partitions is feasible if and only if one of the following two equivalent conditions hold:
(1) *The homogeneity condition* (3.1) *holds, and for all feasible $\alpha, \beta, \gamma \subset r \times (n-r)$ and all $0 < r < n$, we have*

$$[\lambda]_\alpha + [\mu]_\beta + [\nu]_\gamma \;\leq\; \binom{n+1-r}{2},$$

or else

(2) *The homogeneity condition* (3.1) *holds, and for all feasible* $\alpha, \beta, \gamma \subset r \times (n+1-r)$ *and all* $0 < r < n+1$ *with* r *even, we have*

$$\{\lambda\}_\alpha + \{\mu\}_\beta + \{\nu\}_\gamma \ \leq \ \binom{n+1-r}{2}.$$

4. Remarks on the geometry of the proof

We first give some general idea of the ingredients in our proof, and then explain a little bit of the relation of this geometry to the combinatorics given here.

A flag manifold G/P (G is a reductive algebraic group and P is a parabolic subgroup) has a Bruhat decomposition into Schubert cells indexed by cosets W/W_P, where W is the Weyl group of G and W_P that of P. The closures of the Schubert cells are the Schubert varieties, and cohomology classes associated to them (Schubert classes) form bases for the cohomology ring of G/P. Standard results in geometry show that the structure constants (generalized Littlewood-Richardson numbers) are the number of points in triple intersections of general translates of Schubert varieties (and hence are non-negative).

If a structure constant is non-zero, then any triple intersection of corresponding Schubert varieties (not just a general intersection) is non-empty, and general intersections are transverse. Conversely, if a structure constant is zero, then any corresponding general intersection is empty, and a non-empty intersection is never transverse. The key idea is to replace the difficult question on non-emptiness of a general intersection of Schubert varieties by the easier question of the transversality of a (not completely general) intersection. This was used by one of us to show transversality of intersections in the Grassmannian of lines [**Sot97**], but its use to study the Horn problem is due to Belkale [**Bel02**], who first gave a geometric proof of the Horn recursion for the Grassmannian.

This idea transfers the analysis from the flag manifold G/P to its tangent space T_pG/P at a given point p. In fact, all of our diagrams are just pictures of the root-space decompositions of T_pG/P for the corresponding varieties. In our proof, we consider three Schubert varieties which contain the point p, and then move them independently by the stabilizer P of p so that they are otherwise general. If it is possible to move the three tangent spaces inside of T_pG/P so that they meet transversally, then the triple is feasible, and if not, then it is not.

This explains where cominuscule flag manifolds come in. The maximal reductive, or Levi, subgroup L of the parabolic group P acts on the tangent space T_pG/P to G/P at that point. Our arguments (moving the tangent spaces to Schubert varieties around by elements of L) require that L act on T_pG/P with finitely many orbits, and this is one characterization of cominuscule flag manifolds G/P.

It also explains why there is a recursion. The argument requires us to consider the stabilizer Q in L of a certain linear subspace of T_pG/P—the tangent space to an orbit of L through a general point of the intersection of general translates of the tangents to the three Schubert varieties. Then the Schubert calculus inside of the smaller flag manifold L/Q is used to analyze the transversality of that triple intersection. Fortunately, the flag manifold L/Q is itself cominuscule, which is the source of our recursion.

We briefly illustrate these comments on the three flag manifolds that arose in this extended abstract.

4.1. The classical Grassmannian. Let $Gr(n, m+n)$ be the Grassmannian of n-planes in $m+n$ space. The general linear group $GL(m+n, \mathbb{C})$ acts on $Gr(n, m+n)$. If $H \in Gr(n, m+n)$ then $T_H Gr(n, m+n)$ may be identified with the set of n by m matrices (more precisely with $\mathrm{Hom}(H, \mathbb{C}^{m+n}/H)$). The Levi subgroup is the group $GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ which acts linearly on the rows and columns of n by m matrices. The orbits of this group are simply matrices of a fixed rank, r, and the subgroup Q is the stabilizer of a pair (K, K'), where $K \subset H$ and $K' \subset \mathbb{C}^{m+n}/H$ both have dimension r. This explains why in Definition 2.1, the number of rows crossed out equals the number of columns crossed out. The smaller cominuscule flag variety L/Q is the product of two Grassmannians, $Gr(r, n) \times Gr(r, m)$.

The Schubert varieties of $Gr(n, m+n)$ are indexed by partitions λ which fit in the $n \times m$ rectangle, and its cohomology ring is the algebra of Schur functions with these restricted indices.

4.2. The Lagrangian Grassmannian. Fix a non-degenerate alternating bilinear (symplectic) form on $\mathbb{C}^{2n}$. Let $LG(n)$ be the set of maximal isotropic (Lagrangian) subspaces in $\mathbb{C}^{2n}$, each of which has dimension n. This is the quotient of the symplectic group by the parabolic subgroup corresponding to the long root, $LG(n) = Sp(2n, \mathbb{C})/P_0$.

Since $H \in LG(n)$ is isotropic the symplectic form identifies $\mathbb{C}^{2n}/H$ with the dual of H, and $T_H LG(n)$ is identified with the space of quadratic forms on H. The Levi subgroup is the general linear group $GL(H)$. Identifying H with $\mathbb{C}^n$, the Levi becomes $GL(n, \mathbb{C})$ and $T_H LG(n)$ is the set of $n \times n$ symmetric matrices. (Symmetric matrices are parametrized by their weakly lower triangular parts, which correspond to the staircase shape $\triangle_n$ where the order of the columns has been reversed.) The general linear group acts simultaneously on the rows and columns of symmetric matrices. The orbits are simply symmetric matrices of a fixed rank, r, and the subgroup Q is the stabilizer of the null space of such a matrix. The smaller cominuscule flag variety L/Q is the Grassmannian $G(r, n)$.

The Schubert varieties of $LG(n)$ are indexed by strict partitions λ which fit inside the staircase $\triangle_n$, and its cohomology ring is the algebra of Schur Q-functions with these restricted indices.

4.3. The Orthogonal Grassmannian. Fix a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n+2}$. The set of maximal isotropic subspaces (each of which has dimension $n+1$) of $\mathbb{C}^{2n+2}$ has two isomorphic components. Let $OG(n+1)$ be one of these components. This is the quotient of the even orthogonal group by a parabolic subgroup P corresponding to one of the roots at the fork in the Dynkin diagram, $OG(n+1) = SO(2n+2, \mathbb{C})/P$.

If $H \in OG(n+1)$ is isotropic, then $T_H OG(n+1)$ is identified with the space of alternating forms on H. The Levi subgroup is the general linear group $GL(H)$. Identifying H with $\mathbb{C}^{n+1}$, then the Levi becomes $GL(n+1, \mathbb{C})$ and $T_H OG(n+1)$ is the set of $(n+1) \times (n+1)$ anti-symmetric matrices. (Anti-symmetric matrices are parametrized by their lower triangular parts, and these strictly lower triangular matrices correspond to the staircase shape where the order of the columns has been reversed.) The general linear group acts simultaneously on the rows and columns of anti-symmetric matrices. The orbits are simply anti-symmetric matrices of a fixed rank. However, and this comes from the details of the proof and the roots of $SO(2n+2, \mathbb{C})$, the subgroup Q is the stabilizer of an even-dimensional subspace of H. The smaller cominuscule flag variety L/Q is the Grassmannian $G(r, n+1)$, where r is even.

The Schubert varieties of $OG(n+1)$ are indexed by strict partitions λ which fit inside the staircase $\triangle_n$, and its cohomology ring is the algebra of Schur P-functions with these restricted indices.

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