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Statistics on Signed Permutations Groups (Extended Abstract)

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Abstract. A classical result of MacMahon shows that the length function and the major index are equidistributed over the symmetric groups. Through the years this result was generalized in various ways to signed permutation groups. In this paper we present several new generalizations, in particular, we study the effect of different linear orders on the letters $[-n, n]$ and generalize a classical result of Foata and Zeilberger.

Résumé. MacMahon a demontré que la fonction de longueur et l'indice majeur sont équi-distribué dans les groupes symétriques. Depuis, ce résultat a été generalisé aux groupes de permutations signées de plusieurs façons. Dans ce travail, nous présentons plusieurs généralisations, et en particulier, nous étudions l'effet d'imposer un ordre linéaire sur $[-n, n]$ et nous généralisons un résultat de Foata et Zeilberger.

1. Introduction

The *signed permutation groups*, also known as the Weyl groups of type B or as the hyperoctahedral groups, are fundamental objects in today's mathematics. A better understanding of these groups may help to advance research in many fields. One method of studying these groups is by using numerical statistics and finding their generating functions. This method was successfully applied in the case of the *symmetric groups*. MacMahon [**13**] considered four different statistics for a permutation π in the *symmetric group*: the number of *descents* ($des(\pi)$), the number of *excedances* ($exc(\pi)$), the *length* statistic ($\ell(\pi)$), and the *major index* ($maj(\pi)$). MacMahon showed that the *excedance* number is equidistributed with the *descent* number, and that the *length* is equidistributed with the *major index* over the *symmetric groups*.

We will discuss three types of statistics: *Eulerian* statistics, which are equidistributed with the *descent number*; *Mahonian* statistics, which are equidistributed with *length*; *Euler-Mahonian* pairs of statistics, which are equidistributed with the pair consisting of the *descent number* and the *major index*. Through the years many generalizations to MacMahon's results were found. In particular, Foata and Zeilberger found that the *Denert statistic* and the *excedance number* are Euler-Mahonian [**10**]. Recently, Adin and Roichman [**3**] generalized MacMahon's result on the *major index* to the *signed permutations groups*, by introducing a new Mahonian statistic, the *flag major index*. See also [**1**]. The associated signed Mahonian statistic was studied in [**2**]. In this extended abstract we will generalize the Foata-Zeilberger result to signed permutation groups, and will investigate the effect of different linear orders on the letters $[-n, n] \setminus \{0\}$ on the resulting generating functions.

The full background, proofs and extensions for *colored permutations groups* to this work can be found in [**8**].

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2. Background

2.1. Statistics on the Symmetric Group. In this subsection we present the main definitions, notation, and theorems on the *symmetric groups* (i.e., the Weyl groups of type A), denoted S_n.

DEFINITION 2.1. Let $\mathbf{N}$ the set of all the *natural numbers*, a *permutation* of order $n \in \mathbf{N}$ is a bijection $\pi : \{1, 2, 3, \dots, n\} \to \{1, 2, 3, \dots, n\}$.

REMARK 2.2. Permutations are traditionally written in a two-line notation as:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \pi(1) & \pi(2) & \pi(3) & \dots & \pi(n) \end{pmatrix}.$$

However for convenience we will use the shorter notation:

$$\pi = [\pi(1), \pi(2), \pi(3), \dots, \pi(n)].$$

For example: $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$ will be written as $\pi = [2, 4, 3, 1, 5]$.

DEFINITION 2.3. The *symmetric group* of degree $n \in \mathbf{N}$ (denoted S_n) is the group consisting of all the permutations of order n, with composition as the group operation.

DEFINITION 2.4. The Coxeter generators of S_n are $s_1, s_2, \dots, s_{n-1}$ where $s_i := [1, 2, \dots, i + 1, i, \dots, n]$.

It is a well-known fact that the *symmetric group* is a *Coxeter group* with respect to the above generating set $\{s_i \mid 1 \le i \le n-1\}$. This fact gives rise to the following natural statistic of permutations in the symmetric group:

DEFINITION 2.5. The *length* of a permutation $\pi \in S_n$ is defined to be:

$$\ell(\pi) := \min\{\ r \ge 0 \mid \pi = s_{i_1} \dots s_{i_r} \text{ for some } i_1, \dots, i_r \in [1, n]\ \}.$$

Here are other useful statistics on S_n that we are going to work with:

DEFINITION 2.6. Let $\pi \in S_n$. Define the following:

(1) The *inversion number* of π:

$$inv(\pi) := |\{(i, j) \mid 1 \le i < j \le n,\ \pi(i) > \pi(j)\}|.$$

Note that $inv(\pi) = \ell(\pi)$.

(2) The *descent set* of π: $Des(\pi) := \{1 \le i \le n - 1 \mid \pi(i) > \pi(i + 1)\}$.

(3) The *decent number* of π: $des(\pi) = |Des(\pi)|$.

(4) The *major-index* of π: $maj(\pi) := \sum\limits_{i \in Des(\pi)} i$.

(5) The *sign* of π: $sign(\pi) := (-1)^{\ell(\pi)}$.

(6) The *excedance number* of π: $exc(\pi) := |\{1 \le i \le n \mid \pi(i) > i\}|$.

EXAMPLE 2.7. Let $\pi = [2, 3, 1, 5, 4] \in S_5$. We can compute the above statistics on π, namely:

$$inv(\pi) = \ell(\pi) = 3,\ Des(\pi) = \{2, 4\},\ des(\pi) = 2,\ maj(\pi) = 6,$$
$$sign(\pi) = (-1)^3 = -1, \text{ and } exc(\pi) = 3.$$

REMARK 2.8. Throughout the paper we use the following notations for a nonnegative integer n:

$$[n]_q := \frac{1 - q^n}{1 - q},\ [n]_q! = [1]_q[2]_q \dots [n]_q,$$
$$[n]_{\pm q}! := [1]_q[2]_{-q}[3]_q[4]_{-q} \dots [n]_{(-1)^{n-1}q}, \text{ and also}$$
$$(a; q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1 - a)(1 - aq) \dots (1 - aq^{n-1}), & \text{otherwise.} \end{cases}$$

MacMahon [**13**] was the first to find a connection between these statistics. He discovered that the *excedance* number is equidistributed with the *descent* number, and that the *inversion* number is equidistributed with the *major index*:

THEOREM 2.9. [**13**]

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [1]_q[2]_q[3]_q \dots [n]_q = [n]_q!.$$

THEOREM 2.10. [**13**]

$$\sum_{\pi \in S_n} q^{exc(\pi)} = \sum_{\pi \in S_n} q^{des(\pi)}.$$

Gessel and Simion gave a similar factorial type product formula for the *signed Mahonian*:

THEOREM 2.11. [**14**, Cor. 2]

$$\sum_{\pi \in S_n} sign(\pi) q^{maj(\pi)} = [n]_{\pm q}!.$$

A bivariate generalization of MacMahon's Theorem 2.9 was achieved during the 1970's by Foata and Schützenberger :

THEOREM 2.12. [**9**]

$$\sum_{\pi \in S_n} q^{maj(\pi)} t^{des(\pi^{-1})} = \sum_{\pi \in S_n} q^{inv(\pi)} t^{des(\pi^{-1})}.$$

In the same article Foata and Schützenberger also proved another bivariate connection between the different statistics:

THEOREM 2.13. [**9**]

$$\sum_{\pi \in S_n} q^{maj(\pi^{-1})} t^{maj(\pi)} = \sum_{\pi \in S_n} q^{\ell(\pi)} t^{maj(\pi)}.$$

In 1990 during her research of the genus zeta function, Denert found a new statistic which was also Mahonian:

DEFINITION 2.14. [**6**] Let be $\pi \in S_n$, define the *Denert's statistic* to be:

$$\begin{aligned} den(\pi) \quad &:= \quad |\{1 \le l < k \le n \mid \pi(k) < \pi(l) < k\}| \\ &+ \quad |\{1 \le l < k \le n \mid \pi(l) < k < \pi(k)\}| \\ &+ \quad |\{1 \le l < k \le n \mid k < \pi(k) < \pi(l)\}|. \end{aligned}$$

Later in the same year Foata and Zeilberger proved that the pair of statistics (exc, den) is equidistributed with the pair (des, maj):

THEOREM 2.15. [**10**]

$$\sum_{\pi \in S_n} q^{exc(\pi)} t^{den(\pi)} = \sum_{\pi \in S_n} q^{des(\pi)} t^{maj(\pi)}.$$

2.2. Signed Permutations Groups. In this subsection we present the main definitions, notation and theorems for the classical Weyl groups of type B, also known as the *hyperoctahedral groups* or the *signed permutations groups*, and denoted B_n.

DEFINITION 2.16. The *hyperoctahedral group* of order $n \in \mathbf{N}$ (denoted B_n) is the group consisting of all the bijections σ of the set $[-n, n] \backslash \{0\}$ onto itself such that $\sigma(-a) = -\sigma(a)$ for all $a \in [-n, n] \backslash \{0\}$, with composition as the group operation.

REMARK 2.17. There are different notations for a permutation $\sigma \in B_n$. We will use the notation $\sigma = [\sigma(1), ..., \sigma(n)]$.

We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}. As in S_n we also have many different statistics; we will describe the main ones:

THEOREM 2.18. Let $\sigma \in B_n$, define the following statistics on σ:

(1) The *inversion number* of σ: $inv(\sigma) := |\{(i, j) \mid 1 \le i < j \le n,\ \sigma(i) > \sigma(j)\}|$.

(2) The *descent set* of σ:

$$Des(\sigma) := \{1 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}.$$

(3) The *type A descent number* of σ: $des_A(\sigma) := |Des(\sigma)|$.
(4) The *type B descent number* of σ:

$$des_B(\sigma) := |\{0 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}|, \text{ where here } \sigma(0) := 0.$$

(5) The *major index* of σ: $maj(\sigma) := \sum_{i \in Des(\sigma)} i$.
(6) The *negative set* of σ: $Neg(\sigma) := \{i \in [1, \ldots, n] \mid \sigma(i) < 0\ \}$.
(7) The *negative number* of σ: $neg(\sigma) := |Neg(\sigma)|$.
(8) The *negative number sum* of σ: $nsum(\sigma) := -\sum_{i \in Neg(\sigma)} \sigma(i)$.

It is well known (see, e.g. [**5**, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, \ldots, s_{n-1}\}$, where $s_i,\ 1 \leq i \leq n-1$, are defined as in S_n (see 2.4), and s_0 is defined as:

$$s_0 := [-1, 2, 3, \ldots, n].$$

This gives rise to another natural statistic on B_n, the *length* statistic:

DEFINITION 2.19. For all $\sigma \in B_n$ the *length* of σ is:

$$\ell(\sigma) := \min\{r \geq 0 \mid \sigma = s_{i_1} s_{i_2} \ldots s_{i_n} \text{ for some } i_1, \ldots, i_r \in [0, n-1]\}.$$

There is a well-known direct combinatorial way to compute this statistic:

THEOREM 2.20. ([**5**, Propositions 8.1.1 and 8.1.2]) For all $\sigma \in B_n$ the length of σ can be computed as:

$$\ell(\sigma) = inv(\sigma) - \sum_{i \in Neg(\sigma)} \sigma(i).$$

Using the last definition we can define another natural statistic on B_n, the *sign* statistic:

DEFINITION 2.21. For all $\sigma \in B_n$ the *sign* of σ is:

$$sign(\sigma) := (-1)^{\ell(\sigma)}.$$

The generating function of *length* is also called the Poincaré polynomial and can be presented in the following manner:

THEOREM 2.22. [**12**, §3.15]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = [2]_q [4]_q \ldots [2n]_q = \prod_{i=1}^{n} [2i]_q.$$

Recently, Adin and Roichman generalized MacMahon's result Theorem 2.9 to B_n, by introducing a new Mahonian statistic, the *flag major index*:

DEFINITION 2.23. [**3**] The *flag major index* of $\sigma \in B_n$ is defined as:

$$\text{flag-major}(\sigma) := 2maj(\sigma) + neg(\sigma),$$

where $maj(\sigma)$ is calculated with respect to the linear order

$$-1 < -2 < \ldots < -n < 1 < 2 < \ldots < n.$$

THEOREM 2.24. [**3**, §2]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} q^{flag-major(\sigma)} = [2]_q [4]_q \ldots [2n]_q.$$

REMARK 2.25. The previous result still holds if $maj(\sigma)$ is calculated with respect to the *natural order* $-n < -(n-1) < ... < -2 < -1 < 1 < 2 < ... < n-1 < n$, see also [**3**].

Adin, Brenti and Roichman introduced another statistic which was also Mahonian, the *nmaj* statistic:

DEFINITION 2.26. [**1**, §3.2] Let $\sigma \in B_n$ then the *negative major index* is defined as:

$$nmaj(\sigma) := maj(\sigma) - \sum_{i \in Neg(\sigma)} \sigma(i) = maj(\sigma) + nsum(\sigma).$$

THEOREM 2.27. [**1**]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} q^{nmaj(\sigma)}.$$

In the same article [**1**] they also defined a new descent multiset and new descent statistics, and found a new Euler-Mahonian bivariate distribution for these statistics:

DEFINITION 2.28. [**1**, §3.1 and §4.2] Let $\sigma \in B_n$ define:

(1) The *negative descent multiset* of σ:

$$NDes(\sigma) := Des(\sigma) \bigcup \{-\sigma(i) \mid i \in Neg(\sigma)\},$$

where $\bigcup$ stands for multiset union.

(2) The *negative descent* statistic of σ: $ndes(\sigma) := |NDes(\sigma)|$.

(3) The *flag-descent* number of σ: $fdes(\sigma) := des_A(\sigma) + des_B(\sigma) = 2des_A(\sigma) + \varepsilon(\sigma)$, where

$$\varepsilon(\sigma) := \begin{cases} 1, & \text{if } \sigma(1) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 2.29. [**1**, §4.3]

$$\sum_{\sigma \in B_n} t^{ndes(\sigma)} q^{nmaj(\sigma)} = \sum_{\sigma \in B_n} t^{fdes(\sigma)} q^{flag-major(\sigma)}.$$

In their article from 2005 Adin, Gessel, and Roichman gave a type B analogue to the Gessel-Simion Theorem(e.g. [**14**, Cor. 2]):

THEOREM 2.30. [**2**, §5.1]

$$\sum_{\sigma \in B_n} sign(\sigma) q^{flag-major(\sigma)} = [2]_{-q}[4]_q \ldots [2n]_{(-1)^n q}.$$

Where *flag major index* computed with respect to the linear order:

$$-1 < -2 < \ldots < -n < 1 < 2 < \ldots < n.$$

3. Main Results

3.1. Signed-Mahonian and Mahonian-Mahonian Statistics.

DEFINITION 3.1. A *linear order of length n, denoted K_n, is a bijection*

$$K_n : [-n, n] \backslash \{0\} \to [1, 2n].$$

We can calculate permutation statistics according to a linear order K_n, we use the following notation: $maj_{K_n}(\sigma)$, $des_{K_n}(\sigma)$, $flag - major_{K_n}(\sigma)$, $nmaj_{K_n}(\sigma)$ etc, to indicate that the corresponding statistic is calculated with respect to the linear order K_n. We also use the notation: $m >_{K_n} l$, to indicate, that according to the linear order K_n 'm' is larger than 'l', i.e. that $s = K_n(m)$, $r = K_n(l)$, and $s > r$.

EXAMPLE 3.2. Let K_n be a linear order and let $\sigma \in B_n$. Then:

$$maj_{K_n}(\sigma) := \sum_{\sigma(i) >_{K_n} \sigma(i+1)} i.$$

NOTE 3.3. Notice that for any linear order K_n, and for any $\sigma \in B_n$, $neg(\sigma) = neg_{K_n}(\sigma)$. This also applies to the *length* statistic, because it is defined with respect to the Coxeter generators, which do not depend on the choice of linear order.

The following proposition is a more general version of Remark 2.25:

PROPOSITION 3.4. Let K_n be a linear order then:

$$\sum_{\sigma\in B_n} q^{flag-major(\sigma)} = \sum_{\sigma\in B_n} q^{flag-major_{K_n}(\sigma)}.$$

In the following theorems we give simple factorial-type product formulas for the generating function for the signed-Mahonian and Mahonian-Mahonian statistics over B_n.

Let be N the natural order, $N: \ -n < -(n-1) < \ldots < -1 < 1 < \ldots < n-1 < n$, then:

THEOREM 3.5.

$$\sum_{\sigma\in B_n} sign(\sigma) q^{flag-major_N(\sigma)} = (q;-1)_n [n]_{\pm q^2}!.$$

The next theorem presents *signed-Mahonian* calculation using the new Mahonian statistic *nmaj*:

THEOREM 3.6.

$$\sum_{\sigma\in B_n} sign(\sigma) q^{nmaj_N(\sigma)} = (q;-q)_n [n]_{\pm q}!.$$

DEFINITION 3.7. Define the following set:

$$U_n := \{\tau \in B_n \mid \tau(1) < \tau(2) < \ldots < \tau(n-1) < \tau(n)\}.$$

There are several facts (see also [**1**],[**2**]) about the set U_n that can be directly concluded from the definition of U_n, namely: each $\sigma \in B_n$ has a unique representation as:

$$\sigma = \tau\pi \ (\tau \in U_n \text{ , and } \pi \in S_n).$$

DEFINITION 3.8. Define the following subsets of U_n:

(1) $U_{n1} := \{\tau \in U_n \mid \tau(1) = -n\}$.
(2) $U_{n2} := \{\tau \in U_n \mid \tau(n) = n\}$.

NOTE 3.9. $U_n = U_{n1} \uplus U_{n2}$, where $\uplus$ stands for disjoint union.

We also define two bijections from U_{n-1} one onto U_{n1}, and one onto U_{n2}:

DEFINITION 3.10. For $i \in 1,2$, define $\varphi_{ni} : U_{n-1} \to U_{ni}$ by:

(1) $\varphi_{n1}(\tau)(i) = \begin{cases} -n, & \text{i=1;} \\ \tau(i-1), & 2 \le i \le n. \end{cases}$

(2) $\varphi_{n2}(\tau)(i) = \begin{cases} \tau(i), & 1 \le i \le n-1; \\ n, & i = n. \end{cases}$

THEOREM 3.11.

$$\sum_{\sigma\in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} = \prod_{i=1}^{n} (1+qt^i)[n]_{q^2t}!.$$

PROOF. (Sketch, more detailed proof can be found at [**8**]) We will prove this theorem by reducing the problem to U_n:

$$\begin{aligned} \sum_{\sigma\in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} &= \sum_{\pi\in S_n,\ \tau\in U_n} q^{2maj(\pi)+neg(\tau)} t^{maj(\pi)+nsum(\tau)} \\ &= \sum_{\tau\in U_n} q^{neg(\tau)} t^{nsum(\tau)} \sum_{\pi\in S_n} q^{2maj(\pi)} t^{maj(\pi)} \\ &= \sum_{\tau\in U_n} q^{neg(\tau)} t^{nsum(\tau)} \sum_{\pi\in S_n} (q^2t)^{maj(\pi)}. \end{aligned}$$

We know according to Theorem 2.9 that: $\sum_{\pi\in S_n} (q^2t)^{maj(\pi)} = [n]_{q^2t}!$, and by calculation we get:

$$\begin{aligned} a_n = \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)} &= \sum_{\tau \in U_{n1}} q^{neg(\tau)} t^{nsum(\tau)} + \sum_{\tau \in U_{n2}} q^{neg(\tau)} t^{nsum(\tau)} \\ &= \sum_{\tau' \in U_{n-1}} q^{neg(\varphi_{n1}(\tau'))} t^{nsum(\varphi_{n1}(\tau'))} \\ &+ \sum_{\tau' \in U_{n-1}} q^{neg(\varphi_{n2}(\tau'))} t^{nsum(\varphi_{n2}(\tau'))} \\ &= \sum_{\tau' \in U_{n-1}} q^{neg(\tau')+1} t^{nsum(\tau')+n} + \sum_{\tau' \in U_{n-1}} q^{neg(\tau')} t^{nsum(\tau')} \\ &= qt^n a_{n-1} + a_{n-1} = (1+qt^n) a_{n-1}. \end{aligned}$$

We got the recurrence equation: $a_n = (1+qt^n)a_{n-1},\ a_1 = 1+qt$, and the solution to this equation is: $a_n = \prod_{i=1}^{n}(1+qt^i)$, and therefore; the general solution is:

$$\sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} = [n]_{q^2t}! \prod_{i=1}^{n}(1+qt^i)$$

$\square$

NOTE 3.12. Notice that substituting $t = 1$ in Theorem 3.11, we get Theorem 2.24 and the equation: $[n]_{q^2}!(1+q)^n = \prod_{i=1}^{n}[2i]_q$.

We can also calculate the generating function of *length* and *flag major index* by using a similar method:

THEOREM 3.13.

$$\sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{\ell(\sigma)} = A_n(q^2, t) \prod_{i=1}^{n}(1+qt^i),$$

where $A_n(q,t) = \sum_{\pi \in S_n} q^{maj(\pi)} t^{\ell(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} t^{inv(\pi)}$.

3.2. Flag-Excedance and Flag-Denerts Statistic. In this subsection we present the *flag-Denert's* statistic (denoted $fden$) and the *flag-excedance* (denoted $fexc$) statistic. We prove that the pair of statistics $(fden, fexc)$ are equidistributed with $(flag-major, fdes)$ over B_n and, therefore, the *flag-Denert* and *flag-excedance* statistics gives a type B generalization to the Foata-Zeilberger Theorem 2.15.

DEFINITION 3.14. Define the type b *excedance* number of $\sigma \in B_n$ to be:

$$exc_B(\sigma) := |\{\ 1 \le i \le n \mid i < |\sigma(i)|\ \}|.$$

DEFINITION 3.15. Define the *flag-excedance* of $\sigma \in B_n$ to be:

$$fexc(\sigma) := 2exc_B(\sigma) + \varepsilon(\sigma).$$

DEFINITION 3.16. Let n be a nonnegative integer. Define the following subset of B_n:

$$Color_2^n := \{\sigma \in B_n |\ \sigma(i) = \pm i,\ \forall i \in [1,n]\}.$$

NOTE 3.17. Notice that each $\sigma \in B_n$ has a unique representation as:

$$\sigma = \pi\tau,\ where\ \pi \in S_n, \tau \in Color_2^n.$$

DEFINITION 3.18. We define the friends order to be:

$$F : -1 < 1 < -2 < 2 < ... < -n < n.$$

We prove that the *flag-excedance* statistics is Eulerian:

THEOREM 3.19.

$$\sum_{\sigma \in B_n} q^{fexc(\sigma)} = \sum_{\sigma \in B_n} q^{fdes_F(\sigma)}.$$

We define the type B Denert's statistic (denoted den_B):

DEFINITION 3.20. Let $\sigma \in B_n$. Define the type B Denert's statistic to be:

$$\begin{aligned} den_B(\sigma) &= |\{1 \le l < k \le n \mid |\sigma(k)| < |\sigma(l)| < k\}| \\ &+ |\{1 \le l < k \le n \mid |\sigma(l)| < k < |\sigma(k)|\}| \\ &+ |\{1 \le l < k \le n \mid k < |\sigma(k)| < |\sigma(l)|\}|. \end{aligned}$$

REMARK 3.21. According to the definition of den_B we can see that:

$$den_B(\sigma) = den_B(\tau\pi) = den_B(\pi),\ \forall\sigma \in B_n,\ \tau \in Color_2^n,\ \pi \in S_n.$$

We define the *flag-Denert's* statistic (denoted $fden_B$), and prove that it is equidistributed with the *flag major index* over the *signed permutations groups*:

DEFINITION 3.22. Let $\sigma \in B_n$. Define the *flag-Denert's* statistic to be:

$$fden(\sigma) := 2den_B(\sigma) + neg(\sigma).$$

THEOREM 3.23.

$$\sum_{\sigma \in B_n} q^{fden(\sigma)} = \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)}.$$

We prove that the pair of statistics (*fden*,*fexc*) is equidistributed with (*flag-major*,*fdes*).

THEOREM 3.24.

$$\sum_{\sigma \in B_n} q^{fden(\sigma)} t^{fexc(\sigma)} = \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)} t^{fdes_F(\sigma)}.$$

PROOF. (Sketch, more detailed proof can be found at [**8**]) We use the Definitions 3.15, 3.22, [**8**, Lemma 6.4], and Theorem 2.15 and conclude the following equality:

$$\begin{aligned} \sum_{\sigma \in B_n} q^{fden(\sigma)} t^{fexc(\sigma)} &= \sum_{\sigma \in B_n} q^{2den_B(\sigma)+neg(\sigma)} t^{2exc_B(\sigma)+\varepsilon(\sigma)} \\ &= \sum_{\tau \in Color_2^n} q^{neg(\tau)} t^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2den(\pi)} t^{2exc(\pi)} \\ &= \sum_{\tau \in Color_2^n} q^{neg(\tau)} t^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2maj(\pi)} t^{2des(\pi)} \\ &= \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)} t^{fdes_F(\sigma)} \end{aligned}$$

□

Remark. Extensions to wreath products and more results may be found in [**8**].

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