Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Kempf collapsing and quiver loci

Allen Knutson and Mark Shimozono

Abstract. Let Q be a Dynkin quiver, that is, a directed graph whose underlying undirected graph has connected components given by Dynkin diagrams of root systems of types A, D, or E. Assign a fixed vector space to each vertex. Consider the set Rep of representations of the quiver Q with these fixed vector spaces. A product G of general linear groups acts on Rep by change of basis at each vertex. A quiver locus Ω is the closure of a G-orbit in Rep. The equivariant cohomology class (resp. K-class) of Ω is known as a quiver polynomial (resp. K-quiver polynomial).

Reineke proved that Ω is the image of a Kempf collapsing, which is a G-equivariant map from a vector bundle over a partial flag manifold. From this we deduce a formula for the quiver polynomial of Ω.

We extend Kempf's construction. On the numerical side, we give a formula for the equivariant cohomology class of the image of a Kempf collapsing. On the geometric side, we give sufficient conditions under which we can compute the equivariant K-class of the image. We observe that these conditions hold for Reineke's Kempf collapsings in types A and D, yielding a formula for the K-quiver polynomials for these loci.

The formulae are BGG/Demazure divided difference operators applied to a product of linear forms.

Résumé. Soit Q un carquois de Dynkin, c'est-à-dire un graphe orienté dont le graphe non-orienté sous-jacent est formé de composantes connexes de diagrammes de Dynkin de type A,D et E. Fixons un espace vectoriel à chaque sommet du graphe. Considérons l'ensemble Rep des représentations du carquois Q avec ces espaces vectoriels. Un produit G de groupes générals linéaires agit sur Rep en effectuant un changement de base à chacun des sommets. Le locus du carquois Ω est la fermeture d'une G-orbite dans Rep. La classe équivariante de la cohomologie (resp. K-classe) de Ω est un polynôme carquois (resp. K-polynôme carquois). Reineke a prouvé que Ω est l'image d'une application de Kempf, qui est une application G-équivariante d'un fibré vectoriel sur une variété de drapeau partielle. De ceci, nous pouvons en déduire une formule pour le polynôme carquois de Ω. Nous étendons la construction de Kempf. Du côté numérique, nous donnons une formule pour la classe équivariante de cohomologie de l'image d'une application de Kempf. Du côté géométrique, nous donnons des conditions suffisantes avec lesquelles nous pouvons calculer la classe K-invariante de l'image. Nous observons que ces conditions sont les mêmes pour l'application de Kempf pour les types A et D, générant une formule pour le K-polynôme carquois pour ces loci. Les formules sont des opérateurs BGG/Demazure de différence divisée appliqués à un produit de formes linéaires.

1. Introduction

Given a quiver representation one may define a torus-stable affine variety called a quiver locus. The universal torus-equivariant cohomology class of a quiver locus is called a **quiver polynomial**. The polynomials associated with the type A quiver admit many beautiful combinatorial formulae involving tableaux [**8**], rc-graphs [**3**] [**17**], lacing diagrams [**23**], factor sequences [**4**], etc. These quiver polynomials have been studied extensively due to their connection with Thom's theory of degeneracy loci [**29**], intersection theory, and Schubert calculus. We list some cases of quiver polynomials in order of increasing generality.

(1) Double Schur polynomials via the Giambelli-Thom-Porteous formula [**26**].
(2) Double Schubert polynomials [**18**].

2000 *Mathematics Subject Classification.* Primary 14N15; Secondary 05E05.
Key words and phrases. quiver polynomials, divided differences.
A. K. was partially supported by the NSF.
M. S. was partially supported by NSF grant DMS-0401012.

(3) Universal Schubert polynomials [**19**]. These specialize to quantum Schubert polynomials [**16**] among others.
(4) Quiver polynomials for the equioriented type A quiver [**4**] [**5**] [**23**].
(5) Quiver polynomials for the type A quiver with arbitrary orientation [**10**].

We present a divided difference formula for the quiver polynomial of any quiver locus belonging to a Dynkin quiver. We also give a divided difference formula for the more refined information given by the K-quiver polynomial, which is a Laurent polynomial associated with a quiver locus. The literature on the K-theoretic classes of degeneracy loci include [**11**] for Grassmannians, [**22**] for matrix Schubert varieties, [**9**] for the K-analogue of universal Schubert polynomials, [**7**] [**12**] [**13**] [**23**] [**24**] for the equioriented type A quiver, and [**10**] for a conjecture for type A with arbitrary orientation.

Our divided difference formulae are obtained through Kempf collapsings. A Kempf collapsing is a suitable map from a fiber bundle over a partial flag variety, to a vector space. This extends a construction of Kempf [**21**], who used it to derive geometric properties of the image of the collapsing map. The instance of this construction as applied to quiver loci has already been given by Reineke [**28**]; we found it independently.

We expect that our method applies to a suitable nontrivial family of quiver loci for quivers that are not necessarily of type ADE.

Since quiver loci are equivariant classes of subvarieties it follows that their multidegrees (the quiver polynomials) satisfy a certain kind of positivity: it is always possible to equivariantly and flatly degenerate a quiver locus Ω to a union $\Omega(0)$ of coordinate subspaces with multiplicities. This leaves the multidegree invariant. The multidegree is additive on maximum degree components, so the quiver polynomial is the positive sum of products of linear forms. Moreover the forms correspond to vectors that lie in an open half space (assuming the torus action was positive, as it is for quiver loci of Dynkin quivers), so positivity is well-defined. A similar formulation of positivity holds for the K-quiver polynomials.

Our divided difference formulae for the quiver and K-quiver polynomials are not obviously positive in the above sense. It would be desirable to obtain manifestly positive combinatorial formulae.

We give some recent examples of positive formulae for quiver polynomials. In the paper [**23**] (which circulated as a preprint in 2003) four positive formulae (pipe, tableau/Schur, component/Schubert, and ratio) were given for the quiver polynomials for the equioriented type A quiver. The pipe formula is positive in the above sense. The Schur formula was previously conjectured to be positive in [**4**]. The component formula was proved independently in [**5**] after its authors were shown the formula in the form of a conjecture. In [**10**] the component formula was generalized to the type A quiver with arbitrary orientation; this formula can also be obtained via Gröbner degeneration as in [**23**]. Previously in [**8**] a Schur-type formula was proved for Fulton's universal Schubert polynomials. In [**13**] [**24**] positive formulae were given for the K-quiver polynomials for the type A equioriented quiver.

2. Vague statement of "numerical" results

Theorem 2.1. (1) *Let Q be a quiver whose underlying undirected graph is a Dynkin diagram of type ADE, d any dimension vector and $\Omega \subset \mathrm{Rep} = \mathrm{Rep}(Q,d)$ a quiver locus. Then the quiver polynomial $H_{\mathrm{Rep}}(\Omega)$ is obtained by applying a divided difference operator to an explicit product of linear forms.*
(2) *For quivers of types A and D, the K-quiver polynomial $K_{\mathrm{Rep}}(\Omega)$ is obtained by applying a divided difference operator to an explicit product of linear forms.*

Conjecturally the formula for $K_{\mathrm{Rep}}(\Omega)$ also holds for quivers of type E. This kind of formula for quiver polynomials, is reminiscent of those for double Schubert and Grothendieck polynomials. However it is different in that for each quiver locus, one starts with a different product of linear forms, whereas all the Schubert and Grothendieck polynomials indexed by a permutation in a given symmetric group, are obtained by applying divided difference operators to a single product of linear forms. These formulae are new even in the equioriented type A case, where the quiver and K-quiver polynomials are known to be certain double Schubert and Grothendieck polynomials respectively, with the y variables set equal to the reverse of the x variables [**23**]. The most important ingredient is the product of linear forms, which depends in a subtle way on cohomological data that is calculated from the quiver locus.

Example 2.2. Let Q consist of two vertices connected by a single arrow, $d = (3,4)$ and $\Omega \subset \mathrm{Rep} = M_{3\times 4}$ the determinantal variety of 3×4 matrices of rank at most two. It is well-known that (in suitable coordinates)

$H_{\mathrm{Rep}}(\Omega)$ is the double Schubert polynomial $\mathfrak{S}_{1,2,5,3,4}(x;y)$, which coincides with the double Schur polynomial $s_2[x-y]$ where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3,y_4)$. Let $\partial_i^x f = (f - s_i f)/(x_i - x_{i+1})$ where s_i exchanges x_i and x_{i+1}, and define ∂_i^y for the similar operator in the y variables. Then our formula reads

$$\begin{aligned} H_{\mathrm{Rep}}(\Omega) = {} & \partial_2^y\partial_1^y\partial_3^y\partial_2^y\partial_1^x\partial_2^x\partial_1^x \circ \\ & (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_1-y_4)(x_2-y_2)(x_2-y_3)(x_2-y_4)(x_3-y_3)(x_3-y_4) \end{aligned}$$

3. Hilbert numerators and multidegrees

Quiver polynomials and K-quiver polynomials are instances of the constructions of the multidegree and Hilbert numerator. We recall these notions, following [**25**].

Let $T=(\mathbb{C}^*)^r$ be an algebraic torus and $X(T)\cong\mathbb{Z}^r$ be the group of algebraic group homomorphisms $T\to\mathbb{C}^*$. We write the group operation on $X(T)$ additively. Let $x_1,\dots,x_r$ be the standard basis of $X(T)$.

Let M be a T-module, that is, a vector space over $\mathbb{C}$ endowed with a rational T-action. For $\lambda\in X(T)$ a vector of weight λ is a nonzero vector $v\in M$ such that $t\cdot v=\lambda(t)v$ for all $t\in T$. Let $M_\lambda\subset M$ be the subspace of vectors of weight λ. Then $M\bigoplus_{\lambda\in X(T)}M_\lambda$. If $\dim M_\lambda<\infty$ for all λ then one may define

$$\mathrm{ch}_T\ M = \sum_{\lambda\in X(T)} \dim M_\lambda\, e^\lambda,$$

which is a formal Laurent series in the variables e^{x_i}.

Let Y be a finite-dimensional T-module. Suppose Y is **positive**, that is, all the weights of Y ($\lambda\in X(T)$ such that $Y_\lambda\neq 0$) lie on one side of a hyperplane in $\mathbb{R}^r$ through the origin. Consider the coordinate ring $\mathbb{C}[Y]$ of Y; it is a polynomial ring in a set $\mathcal{B}$ of coordinate functions on Y, which can be taken to be weight vectors. A basis of weight vectors in $\mathbb{C}[Y]$ is given by the set of monomials with variables in $\mathcal{B}$. Therefore the weight spaces of $\mathbb{C}[Y]$ are finite-dimensional, and using geometric series one obtains

$$\mathrm{ch}_T\ \mathbb{C}[Y] = \prod_{v\in\mathcal{B}}(1-e^{\mathrm{wt}(v)})^{-1} \tag{3.1}$$

where $\mathrm{wt}(v)\in X(T)$ is the weight of v.

Let $Z\subset Y$ be a T-stable closed subscheme, with defining ideal $I(Z)\subset\mathbb{C}[Y]$. Its coordinate ring is $\mathbb{C}[Z]\cong\mathbb{C}[Y]/I(Z)$. Since $\mathbb{C}[Z]$ is a quotient of $\mathbb{C}[Y]$ by a T-stable ideal, it has a basis of weight vectors given by a subset of that of $\mathbb{C}[Y]$. Thus $\mathbb{C}[Z]$ has finite-dimensional weight spaces and $\mathrm{ch}_T\ \mathbb{C}[Z]$ is a well-defined formal Laurent series. The **T-equivariant Hilbert numerator** of Z in the positive T-module Y is the formal Laurent series in the variables e^{x_i} defined by

$$K_Y(Z) = \frac{\mathrm{ch}_T\ \mathbb{C}[Z]}{\mathrm{ch}_T\ \mathbb{C}[Y]}. \tag{3.2}$$

Using a T-equivariant version of the Hilbert Syzygy Theorem it follows that $K_Y(Z)$ is in fact a Laurent polynomial: the formal series $\mathrm{ch}_T\ \mathbb{C}[Z]$ can always be expressed as a Laurent polynomial (namely, $K_Y(Z)$) divided by the denominator of $\mathrm{ch}_T\ \mathbb{C}[Y]$.

There are natural isomorphisms $K_T^*(Y)\cong K_T^*(\mathrm{pt})\cong R(T)=\mathbb{Z}[X(T)]=\mathbb{Z}[e^{\pm x_1},\dots,e^{\pm x_r}]$ where $R(T)$ is the ring of rational representations of T. The Hilbert numerator $K_Y(Z)$ may be regarded as an element in the T-equivariant K-theory $K_T^*(Y)$ of Y; it is the equivariant K-class of the structure sheaf $\mathcal{O}_Z$ of Z.

There is a surjective ring homomorphism $\mathbb{Z}[e^{\pm x_1},\dots,e^{\pm x_r}]\to\mathbb{Z}[x_1,\dots,x_r]$ that sends a Laurent polynomial to its lowest degree nonvanishing homogeneous term, where e^λ is formally expressed as $e^\lambda=\sum_{i\geq 0}\lambda^i/i!$. The **multidegree** of Z is the polynomial $H_Y(Z)$ given by the image of the Hilbert numerator $K_Y(Z)$ under this map. It can be shown that $H_Y(Z)$ is a polynomial with integer coefficients: $H_Y(Z)\in\mathbb{Z}[x_1,\dots,x_r]$. More canonically, there are isomorphisms $H_T^*(Y)\cong H_T^*(\mathrm{pt})\cong\mathrm{Sym}_{\mathbb{Z}}(X(T))\cong\mathbb{Z}[x_1,\dots,x_r]$ where $\mathrm{Sym}_{\mathbb{Z}}$ is the symmetric algebra with integer coefficients. Then $H_Y(Z)$ is identified with the element of the equivariant cohomology ring $H_T^*(Y)$ given by the T-equivariant fundamental class of the T-stable subvariety Z of Y, and the above ring homomorphism is the T-equivariant Chern map $K_T^*(Y)\to H_T^*(Y)$.

Suppose Z is a coordinate subspace, that is, it is defined by the vanishing of some subset $\mathcal{B}'\subset\mathcal{B}$ of the set of coordinates $\mathcal{B}$ of Y. Then directly from the definitions one may easily compute the Hilbert numerator and multidegree:

$$K_Y(Z) = \prod_{v\in\mathcal{B}'}(1-e^{\mathrm{wt}(v)}) \qquad H_Y(Z) = \prod_{v\in\mathcal{B}'}(-\mathrm{wt}(v)). \tag{3.3}$$

The Hilbert numerator $K_Y(Z)$ is a more subtle geometric invariant than the multidegree $H_Y(Z)$ since the latter is only the leading term of the former.

4. Quiver polynomials

To each quiver representation we define its quiver polynomial and K-quiver polynomial as the multidegree and Hilbert numerator of its associated quiver locus.

A **quiver** is a finite directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 is the set of directed edges. Each directed edge $a \in Q_1$ has a head $ha \in Q_0$ and a tail $ta \in Q_0$. A **dimension vector** is a function $d : Q_0 \to \mathbb{Z}_{\geq 0}$: it assigns to each vertex $i \in Q_0$ a nonnegative integer $d(i)$. A **representation** V of the quiver Q of dimension d, is a collection of linear maps V_a, one for each arrow $a \in Q_1$, with $V_a : \mathbb{C}^{d(ta)} \to \mathbb{C}^{d(ha)}$. Equivalently, V is a list of matrices where $V_a \in M_{d(ta)\times d(ha)}$; here matrices act on row vectors. Let $\mathrm{Rep} = \mathrm{Rep}(Q, d) = \prod_{a\in Q_1} M_{d(ta)\times d(ha)}$ be the set of representations of Q of dimension d. Say that $V, W \in \mathrm{Rep}$ are **equivalent** if V is taken to W by a change of basis in the vector spaces at the vertices, that is, there is an element $g = (g_i)_{i\in Q_0} \in G = G(Q, d) = \prod_{i\in Q_0} GL(d(i))$ such that $W_a = g_{ta} V_a g_{ha}^{-1}$ for all $a \in Q_1$. Thus an equivalence class of quiver representations of Q of dimension d is a G-orbit in $\mathrm{Rep}(Q, d)$.

A **quiver locus** in Rep is a subvariety of the form $\Omega = \overline{G \cdot V}$ for some $V \in \mathrm{Rep}$. Let $T \subset G$ be the maximal torus consisting of tuples of diagonal matrices. Since Ω is G-stable and closed it is also T-stable and therefore defines T-equivariant classes $K_{\mathrm{Rep}}(\Omega) \in K^*_T(\mathrm{Rep})$ and $H_{\mathrm{Rep}}(\Omega) \in H^*_T(\mathrm{Rep})$. These are by definition the **K-quiver polynomial** and **quiver polynomial** of the quiver locus Ω.

More specifically, let $T^i \subset GL(d(i))$ be the subgroup of diagonal matrices in the i-th component of G for $i \in Q_0$ and let $T = \prod_{i\in Q_0} T^i \subset G$. Let $X^i = \{x^i_1, x^i_2, \dots, x^i_{d(i)}\}$ be a basis of $X(T^i)$. Then $K_T(\mathrm{Rep}) \cong \mathbb{Z}[e^{\pm x^i_j}]$. Since Ω is G-stable it defines a G-equivariant class in $K^*_G(\mathrm{Rep})$. But there are natural isomorphisms $K^*_G(\mathrm{Rep}) \cong K^*_T(\mathrm{Rep})^W \cong \mathbb{Z}[e^{\pm x^i_j}]^W$ where $W = \prod_{i\in Q_0} S_{d(i)}$ is the Weyl group of G, the product of symmetric groups where $S_{d(i)}$ permutes the i-th set of variables X^i. So $K_{\mathrm{Rep}}(\Omega)$ is a W-symmetric Laurent polynomial. Similarly $H^*_G(\mathrm{Rep}) \cong H^*_T(\mathrm{Rep})^W \cong \mathbb{Z}[x^i_j]^W$, and the quiver polynomial $H_{\mathrm{Rep}}(\Omega)$ is W-symmetric.

Remark 4.1. The above action of T on $\mathrm{Rep}(Q, d)$ is *not* positive if and only if there is some directed cycle C in Q such that for every vertex i on C, $d(i) > 0$. In this situation the Hilbert numerator of some quiver loci in $\mathrm{Rep}(Q, d)$ are not well-defined. However we may consider the action of a bigger group $G \times T'$ on Rep where $T' = (\mathbb{C}^*)^{|Q_1|}$ is a torus with a copy of $\mathbb{C}^*$ for each arrow $a \in Q_1$, where the a-th copy of $\mathbb{C}^*$ acts on the a-th component of $\mathrm{Rep}(Q, d)$ by scaling. The torus $T^+ = T \times T'$ in G^+ acts positively on Rep. In particular if $\Omega \subset \mathrm{Rep}$ is a quiver locus that is also stable under G^+ then its quiver and K-quiver polynomial with respect to the T^+-module Rep, are well-defined. If Q has no directed cycles then the T^+ polynomials specialize to the usual quiver polynomials by setting to zero the basis elements of $X(T')$.

More generally one may consider the Hilbert numerators and multidegrees of G^+-orbit closures or other G^+-stable subvarieties of Rep with respect to the T^+-module Rep.

Example 4.2. Let Q consist of a single vertex and a single loop and fix the dimension n. Then $\mathrm{Rep} = M_n$ is the $n \times n$ matrices and $G = GL(n)$ acts by conjugation. The indecomposables of $\mathbb{C}Q$ are Jordan blocks. With the notation of the previous Remark, the $G^+ = G \times \mathbb{C}^*$-stable quiver loci are the closures of conjugacy classes of nilpotent matrices.

More generally if Q is a directed cycle then the quiver loci given by G-orbits of nilpotent elements of Rep, are also G^+-stable.

5. Representations of Q

We recall some of the representation theory of Dynkin quivers. This provides an indexing set for the quiver loci and other key ingredients for our divided difference formula for the quiver polynomials. See [**14**] [**15**] for excellent survey information.

5.1. Path Algebra. The **path algebra** $\mathbb{C}Q$ is the associative algebra over $\mathbb{C}$ with generating set $Q_0 \cup Q_1$ and relations (for all $i, j \in Q_0$ and $a \in Q_1$)

$$\begin{aligned} i \cdot j &= \delta_{i,j} i \\ i \cdot a &= \delta_{i,ta} a \\ a \cdot j &= \delta_{ha,j} a. \end{aligned} \tag{5.1}$$

Using these relations it follows that for $a, b \in Q_1$, the product ab is zero unless $ha = tb$. Hence $\mathbb{C}Q$ has a basis given by paths, where a path of length zero is an element of Q_0, and a path of length $m > 0$ is a sequence $a_1 a_2 \cdots a_m$ with $a_i \in Q_1$ where $ha_k = ta_{k+1}$ for $1 \le k \le m-1$. Since a path has a unique starting vertex and unique ending vertex, it follows that the elements $i \in Q_0$ are a complete set of orthogonal idempotents in $\mathbb{C}Q$. Let $\mathbb{C}Q$-Mod be the category of finite-dimensional right $\mathbb{C}Q$-modules. Let $V \in \mathbb{C}Q$-Mod. We have $V = \bigoplus_{i \in Q_0} V_i$ where $V_i = V \cdot i$. One easily checks that the linear map V_a given by the action of a on V, is zero on V_j for $j \ne ta$ and its image lies in V_{ha}. So without loss we may consider V_a as a linear map from $V_{ta} \to V_{ha}$. Thus we see that a $\mathbb{C}Q$-module is just a quiver representation and vice versa. Let $g : V \to W$ be a $\mathbb{C}Q$-module isomorphism. Firstly g is a linear isomorphism. Since g intertwines the action of $i \in Q_0$, g restricts to an isomorphism $g_i : V_i \to W_i$ for all i. In particular V and W have the same dimension vector d. So we may regard V and W as being elements of $\mathrm{Rep}(Q, d)$. Since g intertwines the action of $a \in Q_1$, it must satisfy $g_{ta} V_a = W_a g_{ha}$ or equivalently $g_{ta} V_a g_{ha}^{-1} = W_a$. Therefore V and W are isomorphic if and only if the corresponding elements of Rep are in the same G-orbit.

So the problem of classifying G-orbits on Rep is the same as that of classifying finite-dimensional $\mathbb{C}Q$-modules up to isomorphism.

5.2. An index set for quiver loci. An **indecomposable** module is one that is not the direct sum of two nonzero submodules. By definition every module is the direct sum of indecomposables. So the isomorphism class of a $\mathbb{C}Q$-module is determined by the multiplicities of its indecomposable summands. Let Ind_Q be the set of isomorphism classes of indecomposable $\mathbb{C}Q$-modules. One special kind of indecomposable module is a **simple** module, one that has no proper submodule. For each vertex $i \in Q_0$ there is a corresponding simple $\mathbb{C}Q$-module S_i: it has $\mathbb{C}^1$ at vertex i and zero vector spaces at the other vertices, and all maps are zero.

Gabriel's Theorem characterizes the quivers Q with finitely many indecomposables.

THEOREM 5.1. [**20**] *The following are equivalent for a quiver Q.*

1. *$G(Q, d)$ has finitely many orbits on $\mathrm{Rep}(Q, d)$ for all d.*
2. *Ind_Q is finite.*
3. *The undirected graph X underlying Q is the Dynkin diagram of a simply-laced root system Φ (that is, its connected components are Dynkin diagrams of type ADE).*

Suppose this holds. Then there is a bijection $\mathrm{Ind}_Q \to \Phi^+$ of the indecomposables with the positive roots Φ^+ of Φ. This bijection sends the simple $\mathbb{C}Q$-module S_i to the simple root α_i and in general sends $I \in \mathrm{Ind}_Q$ to its dimension vector, where a function $d : Q_0 \to \mathbb{Z}$ is identified with the element $\sum_{i \in Q_0} d(i)\alpha_i$ of the root lattice of Φ.

For $\beta \in \Phi^+$ let I_β be the indecomposable with dimension vector β. The modules in Ind_Q may be constructed explicitly using reflection functors [**6**] but we don't require this construction.

EXAMPLE 5.2. The equioriented type A quiver (A_{n+1}) is depicted below; we use the vertex set $Q_0 = \{0, 1, 2, \ldots, n\}$ and directed edges going from $a-1$ to a for $1 \le a \le n$.

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}$$

For $0 \le i \le j \le n$, let I_{ij} be the indecomposable corresponding to the root $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. It can be realized by placing $\mathbb{C}^1$ at vertices i through j with identity maps connecting them and zero maps elsewhere.

REMARK 5.3. Let $\beta \in Q^+$. Say that a map $m : \Phi^+ \to \mathbb{Z}_{\ge 0}$ is a **Kostant partition** of β if $\beta = \sum_{\alpha \in \Phi^+} m(\alpha)\alpha$. By Gabriel's Theorem, the isomorphism classes of $\mathbb{C}Q$-modules of dimension d, are parametrized by the Kostant partitions of d. Write $\Omega_m = \overline{G \cdot \phi}$ for any $\phi \in \mathrm{Rep}$ such that $\phi \cong \bigoplus_{\alpha\in\Phi^+} I_\alpha^{\oplus m(\alpha)}$.

5.3. Hom, Ext, and the Euler form. For $M, N \in \mathbb{C}Q$-Mod let $\mathrm{Hom}_Q(M,N)$ be the vector space of $\mathbb{C}Q$-module homomorphisms from M to N. Let $\mathrm{Ext}^i_Q(-,N)$ be the i-th cohomology group of the functor $\mathrm{Hom}_Q(-,N)$ applied to a projective resolution of M. The homological form on $\mathbb{C}Q$-Mod is defined by

$$\langle M\, ,\, N\rangle = \sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i_Q(M,N).$$

It is *not* symmetric. The category $\mathbb{C}Q$-Mod is hereditary. In particular $\mathrm{Ext}^i_Q(M,N) = 0$ for all $i \geq 2$, so that

$$\langle M\, ,\, N\rangle = \dim \mathrm{Hom}_Q(M,N) - \dim \mathrm{Ext}^1_Q(M,N). \tag{5.2}$$

Ringel [**27**] observed that the homological form $\langle M\, ,\, N\rangle$ depends only on the dimension vectors d_M and d_N of M and N. Define the **Euler form** on functions $Q_0 \to \mathbb{Z}$ by

$$\langle d\, ,\, e\rangle = \sum_{i\in Q_0} d(i)e(i) - \sum_{a\in Q_1} d(ta)e(ha).$$

Then the homological form on M and N is the Euler form on their dimension vectors:

$$\langle M\, ,\, N\rangle = \langle d_M\, ,\, d_N\rangle.$$

Remark 5.4. Dynkin quivers are precisely those with positive definite Euler form. For Q a Dynkin quiver, a vector $d : Q_0 \to \mathbb{Z}_{\geq 0}$ is a positive root if and only if $\langle d\, ,\, d\rangle = 1$.

5.4. Auslander-Reiten quiver. This material comes from [**1**]. Say that a $\mathbb{C}Q$-module homomorphism f is **irreducible** if it is nonzero, and for every factorization $f = h \circ g$ as a composition of $\mathbb{C}Q$-module homomorphisms, either g is split injective or h is split surjective. The **Auslander-Reiten quiver** Γ_Q of Q is the directed graph with vertex set given by Ind_Q and with a directed edge from M to N if there is an irreducible map $M \to N$.

Proposition 5.1. *Let Q be a Dynkin quiver. Then for $\beta, \gamma \in \Phi^+$, then there is an arrow from I_β to I_γ in Γ_Q if and only if $\beta \neq \gamma$ and $\langle \beta\, ,\, \gamma\rangle > 0$.*

Remark 5.5. For Dynkin quivers Q the Auslander-Reiten quiver Q has no cycles. Therefore there is a partial order $\preccurlyeq_Q$ on Ind_Q or Φ^+ given by $\beta \preccurlyeq_Q \gamma$ if there is a directed path from I_β to I_γ in Γ_Q.

Proposition 5.2. *For Q Dynkin and $\alpha, \beta \in \Phi^+$:*

1. $\mathrm{Hom}_Q(I_\alpha, I_\beta) = 0$ *if* $\alpha > \beta$.
2. $\mathrm{Ext}^1_Q(I_\alpha, I_\beta) = 0$ *if* $\alpha \leq \beta$.
3. $\langle \alpha\, ,\, \beta\rangle = \dim \mathrm{Hom}_Q(\alpha,\beta)$ *for* $\alpha \leq \beta$.
4. $\langle \alpha\, ,\, \beta\rangle = -\dim \mathrm{Ext}^1_Q(\alpha,\beta)$ *for* $\alpha > \beta$.

Proposition 5.2 says that the matrix $\langle \alpha\, ,\, \beta\rangle$ for $\alpha, \beta \in \Phi^+$, written with respect to any linear extension of $\preccurlyeq_Q$, agrees with $\dim \mathrm{Hom}_Q(I_\alpha, I_\beta)$ on or above the diagonal and with $-\dim \mathrm{Ext}^1_Q(I_\alpha, I_\beta)$ below the diagonal. Thus one can read off all the important homological information about $\mathbb{C}Q$-Mod just from the Euler form under an appropriate ordering of positive roots.

5.5. Reduced expressions. We recall from [**2**] [**31**] a combinatorial way to construct the Auslander-Reiten quiver Γ_Q when Q is Dynkin. Suppose X is an undirected graph that is the Dynkin diagram of a simply-laced root system Φ, with Weyl group W and distinguished set $\{s_i \in W \mid i \in Q_0\}$ of simple reflections. An **orientation** of X is a directed graph Q that yields X if the directions on edges are forgotten. The Weyl group acts on the set $\mathrm{Or}(X)$ of orientations of X: for $Q \in \mathrm{Or}(X)$ and $i \in Q_0$, $s_iQ \in \mathrm{Or}(X)$ is obtained from Q by reversing all the directed edges that touch the vertex i.

Let $w_0 \in W$ be the longest element. Let Red be the set of reduced words for w_0, that is, the set of sequences $i_\bullet = (i_N, \dots, i_2, i_1)$ such that $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1}$ with N minimal. Say that $i_\bullet \in \mathrm{Red}$ is adapted to $Q \in \mathrm{Or}(X)$ and write $i_\bullet \in \mathrm{Red}_Q$, if for every j the vertex i_j is a sink in the directed graph $s_{i_{j-1}} \cdots s_{i_2} s_{i_1} Q$. For every orientation Q of X, $\mathrm{Red}_Q \neq \emptyset$. Moreover Red_Q is a commutation class (two reduced words are in the same commutation class, if they are reachable from each other by commuting Coxeter relations $s_i s_j = s_j s_i$ where i and j are nonadjacent vertices in X). However $\bigcup_{Q \in \mathrm{Or}(X)} \mathrm{Red}_Q \subsetneq \mathrm{Red}$.

Fix $i_\bullet \in \mathrm{Red}_Q$. It defines a total ordering $\leq_{i_\bullet}$ on the set of positive roots Φ^+ by the sequence $\gamma_1 < \gamma_2 < \ldots$ where $\gamma_j = s_{i_j} \cdots s_{i_2} \alpha_{i_1}$. Then it is a theorem of [**2**] that the total orders $\leq_{i_\bullet}$ for $i_\bullet \in \mathrm{Red}_Q$, are the set of linear extensions of the partial order $\preccurlyeq_Q$.

The Auslander-Reiten quiver Γ_Q of $Q \in \mathrm{Or}(X)$ is traditionally drawn with arrows going from right to left and smaller elements pointing towards bigger ones. It turns out that there is a nice planar embedding of Γ_Q such that the poset element γ_j is placed in the i_j-th row for all j. Even better, this graph is the 1-skeleton of a topological complex [**31**].

Example 5.6. Let Q be the A_3 quiver with both arrows pointing to the middle:

$$0 \longrightarrow 1 \longleftarrow 2$$

We use $i_\bullet = (2, 0, 1, 2, 0, 1) \in \mathrm{Red}_Q$. The induced total ordering $\leq_{i_\bullet}$ on Ind_Q is given below, with labeling of Φ^+ as in Example 5.2.

$$\alpha_{11} < \alpha_{01} < \alpha_{12} < \alpha_{02} < \alpha_{22} < \alpha_{00}. \tag{5.3}$$

The Auslander-Reiten quiver Γ_Q is depicted below.

$$\begin{tikzcd}
0 & & I_{01} \arrow[dr] & & I_{22} \\
1 & I_{11} \arrow[ur] \arrow[dr] & & I_{02} \arrow[ur] \arrow[dr] & \\
2 & & I_{12} \arrow[ur] & & I_{00}
\end{tikzcd}$$

The matrix for the Euler form on pairs of elements of Φ^+ with respect to the total order (5.3) is given by

$$(\langle \alpha\, ,\, \beta \rangle)_{\alpha, \beta \in \Phi^+} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Example 5.7. Let Q be the D_4 quiver with the following orientation:

$$\begin{tikzcd}
 & 1 \arrow[d] & \\
4 & 2 \arrow[l] \arrow[r] & 3
\end{tikzcd}$$

We use $w_0 = s_1 s_2 s_4 s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4 s_3$. We label the indecomposables by their dimension vectors. For example, 1211 means $(1, 2, 1, 1)$ or $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The total ordering for the above reduced word is given by the list

$$0010, 0001, 0111, 1111, 0101, 0110, 1211, 0100, 1110, 1101, 1100, 1000 \tag{5.4}$$

and the AR quiver is given by

$$\begin{tikzcd}
1 & & & 1111 \arrow[dr] & & 0100 \arrow[dr] & & 1000 \\
2 & & 0111 \arrow[ur] \arrow[dr] \arrow[ddr] & & 1211 \arrow[ur, leftrightarrow] \arrow[dr] \arrow[ddr] & & 1100 \arrow[ur] & \\
3 & 0010 \arrow[ur] & & 0101 \arrow[ur] & & 1110 \arrow[ur] & & \\
4 & 0001 \arrow[uur] & & 0110 \arrow[uur] & & 1101 \arrow[uur] & &
\end{tikzcd}$$

5.6. Orbit representatives. Let Q be a Dynkin quiver, $d : Q_0 \to \mathbb{Z}_{\geq 0}$ a dimension vector and m a Kostant partition of d. For our divided difference formula for quiver polynomials we define a representative element $\phi_m \in \mathrm{Rep}(Q,d)$ in the $G = G(Q,d)$-orbit indexed by m, that is, $\phi_m \cong \bigoplus_{\alpha\in\Phi^+} I_\alpha^{\oplus m(\alpha)}$.

Pick any particular matrix representation for each indecomposable I_α and by abuse of notation denote it by I_α. Consider an *ordered* direct sum $I_\bullet = I_1 \oplus I_2 \oplus \cdots \oplus I_M$ that has $m(\alpha)$ summands I_α for all $\alpha \in \Phi^+$, with the property that

$$\mathrm{Ext}^1_Q(I_j, I_i) = 0 \qquad \text{if } i < j. \tag{5.5}$$

This condition holds if we list the indecomposables in the *reverse* of the total order on Ind_Q given by $\leq_{i_\bullet}$ for any $i_\bullet \in \mathrm{Red}_Q$.

We view $I_\bullet$ as a point in $\mathrm{Rep}(Q,d)$. As such $I_\bullet$ is "block diagonal": for each $a \in Q_1$ the a-th component of $I_\bullet$ is "block diagonal" with "diagonal" blocks given by the a-th components of $I_1, I_2, \ldots, I_M$ in that order.

EXAMPLE 5.8. Take Q to be the equioriented A_2 quiver, $d(0) = e$ and $d(1) = f$. Take the quiver locus X_r given by the determinantal variety of $e \times f$ matrices of rank at most r. Then the $G = GL(e) \times GL(f)$-orbit associated to X_r has Kostant partition m with $m(\alpha_0) = e - r$, $m(\alpha_0 + \alpha_1) = r$, and $m(\alpha_1) = f - r$. So if $e = 3$, $f = 4$, and $r = 2$ then an appropriate ordering of the indecomposables in $I_\bullet$ is given by $I_\bullet = I_{\alpha_0} \oplus I_{\alpha_0+\alpha_1} \oplus I_{\alpha_0+\alpha_1} \oplus I_{\alpha_1} \oplus I_{\alpha_1}$. The element $I_\bullet \in M_{3\times 4}$ is the matrix

$$I_\bullet = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where each I_{α_0} is a 1×0 matrix, each $I_{\alpha_0+\alpha_1}$ is a 1×1 identity matrix, and each I_{α_1} is a 0×1 matrix.

Fix $I_\bullet$ as above. Define the Levi subgroup $L(I_\bullet) \subset G(Q,d)$ by

$$L(I_\bullet) = \prod_{k=1}^{M} G(Q, d(I_k)).$$

We regard $L(I_\bullet)$ as a block diagonal subgroup of $G = G(Q,d)$: for each $i \in Q_0$, the i-th component of $L(I_\bullet)$ are the block diagonal matrices in the i-th component of $G(Q,d)$ with block sizes coming from the i-th components of $G(Q, d(I_k))$. It acts on the direct product

$$\mathrm{Rep}(I_\bullet) = \prod_{k=1}^{M} \mathrm{Rep}(Q, d(I_k)). \tag{5.6}$$

We regard $\mathrm{Rep}(I_\bullet) \subset \mathrm{Rep}$ similarly as the "block diagonal" elements of Rep.

If I is an indecomposable $\mathbb{C}Q$-module with dimension vector d_I, then it is easy to show for our situation that $\overline{G(Q,d_I)\cdot I} = \mathrm{Rep}(Q,d_I)$. It follows that

$$\mathrm{Rep}(I_\bullet) = \overline{L(I_\bullet)\cdot I_\bullet}. \tag{5.7}$$

Let $P(I_\bullet) \subset G$ be the parabolic subgroup given by the lower block triangular subgroup of G with Levi factor $L(I_\bullet)$. For $i \in Q_0$ its i-th component is lower block triangular with diagonal blocks given by those of the i-th component of $L(I_\bullet)$.

Finally, let $Z(I_\bullet) \subset \mathrm{Rep}$ be the "lower block triangular" coordinate subspace of Rep, such that for $a \in Q_1$ the a-th component of $Z(I_\bullet)$ consists of the matrices with zeroes in the entries strictly above the "block diagonal" given by the a-th component of $\mathrm{Rep}(I_\bullet)$ and arbitrary entries allowed elsewhere.

LEMMA 5.9.

$$Z(I_\bullet) = \overline{P(I_\bullet)\cdot I_\bullet}. \tag{5.8}$$

The proof of this fact, which is equivalent to the condition (5.5), follows easily from the definition of Ext. This is precisely the point where the careful ordering of the indecomposables in $I_\bullet$ is used.

EXAMPLE 5.10. For $I_\bullet$ as in Example 5.8,

$$L(I_\bullet) = T_3 \times T_4 \subset GL(3) \times GL(4)$$
$$P(I_\bullet) = B_- \times B_- \subset GL(3) \times GL(4).$$

So $L(I_\bullet)$ is the maximal torus and $P(I_\bullet)$ is the product of lower triangular Borels. (This always holds in type A: each positive root α_{ij} contains at most one copy of each simple root). $\mathrm{Rep}(I_\bullet)$ and $Z(I_\bullet)$ are the coordinate subspaces of Rep given by

$$\mathrm{Rep}(I_\bullet) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} \qquad Z(I_\bullet) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}. \tag{5.9}$$

6. Kempf collapsings

The geometric construction of a Kempf collapsing κ leads to divided difference formulae for the equivariant cohomology class and K-theory class of the image of κ. We recall Reineke's construction, which realizes a Dynkin quiver locus as the image of a Kempf collapsing. This yields divided difference formula for quiver and K-quiver polynomials.

Let G be a reductive algebraic group over $\mathbb{C}$ and P a parabolic subgroup. Let Y be a finite-dimensional G-module and $Z \subset Y$ a P-stable closed subscheme. In our application G is the product of general linear groups of the form $G(Q, d)$ and Z is a linear subspace of $\mathrm{Rep}(Q, d)$. Consider the G-equivariant fiber bundle $G \times^P Z$ over the partial flag variety G/P with fiber Z over the identity:

$$G \times^P Z = (G \times Z)/P.$$

Here P acts diagonally on the right by $(g, z)p = (gp, p^{-1} \cdot z)$. Consider the map

$$\begin{aligned} \kappa : G \times^P Z &\to Y \\ (g, z)P &\mapsto gz. \end{aligned}$$

We call κ a **Kempf collapsing**. The map κ is proper so its image is closed.

THEOREM 6.1. [**21**] *Suppose that*
- *Z has rational singularities.*
- *$\mathcal{O}_Y \to \kappa_* \mathcal{O}_{G\times^P Z}$ is surjective.*
- *$R^j \kappa_* \mathcal{O}_{G\times^P Z} = 0$ for $j > 0$.*

Then Im κ *is normal and Cohen-Macaulay. If in addition κ is birational to its image, then* Im κ *has rational singularities.*

Kempf suggests a condition to guarantee these criteria: that Z is a linear subspace and a completely reducible P-module. In our application the latter condition doesn't hold so we don't assume it. Here is our extension of Kempf's result.

THEOREM 6.2. *Suppose Z has rational singularities and $R^j \kappa_* \mathcal{O}_{G\times^P Z} = 0$ for $j > 0$. Let $\widetilde{\mathrm{Im}\ \kappa}$ be the normalization of the image of κ.*
- *If the general fiber of κ is connected, then $\widetilde{\mathrm{Im}\ \kappa}$ has rational singularities.*
- *If the general fiber of κ is connected and* Im κ *is normal (hence has rational singularities), then $\kappa_* \mathcal{O}_{G\times^P Z} = \mathcal{O}_{\mathrm{Im}\ \kappa}$.*
- *Conversely, if $\kappa_* \mathcal{O}_{G\times^P Z} = \mathcal{O}_{\mathrm{Im}\ \kappa}$, then all fibers of κ are connected, and* Im κ *is normal (hence has rational singularities).*

Even without the last two conditions, the Kempf collapsing still determines the multidegree of Im κ.

THEOREM 6.3. *Suppose Z has rational singularities. Let $m_0 = H_Z(Y)$. Construct a sequence of polynomials $m_1, m_2, \ldots$ where each polynomial is obtained from the previous one by a divided difference operator $\partial_\alpha = \frac{1}{\alpha}(1 - s_\alpha)$, where α varies over the set of simple roots of G (taken in the Borel opposite to one contained in P), and the action of s_α on $\mathrm{Sym}_{\mathbb{Z}}(X(T))$ is induced from the reflection action on $X(T)$. Don't apply a divided difference operator if the result is 0, and only stop when all ∂_α give the result 0. This process always terminates after the same number of steps, and the last polynomial in this sequence is c times $H_Y(\mathrm{Im}\ \kappa)$, where c is the number of components in a general fiber of κ.*

When we have both connected fibers and the vanishing of higher direct images of κ, then we can compute the Hilbert numerator $K_Y(\widetilde{\mathrm{Im}\ \kappa})$.

Theorem 6.4. *Suppose Z has at worst rational singularities, the general fiber of κ is connected, and $R^j\kappa_*\mathcal{O}_{G\times^P Z} = 0$ for $j > 0$.*

Let $m_0 = K_Y(Z)$, and construct a sequence of Laurent polynomials $m_1, m_2, \dots$ by applying Demazure operators $\pi_\alpha := (1-\exp(-\alpha))^{-1}(1-\exp(\alpha)s_\alpha)$ to m_0, where α varies over the set of simple roots of G. Stop when the application of any π_α leaves the result unchanged. This process terminates after finitely many steps. The last Laurent polynomial in this sequence is $K_Y(X)$ where X is the pushforward of $\mathcal{O}_{\widetilde{\mathrm{Im}\ \kappa}}$ under the normalization map $\widetilde{\mathrm{Im}\ \kappa} \to \mathrm{Im}\ \kappa \to Y$.

Explicitly,

$$K_Y(X) = \sum_{w\in W} w \cdot \frac{K_Y(Z)}{\prod_{\beta\in\Phi^+}(1-\exp(-\beta))}$$

where Φ^+ is the set of positive roots relative to the opposite of some Borel subgroup between T and P.

Remark 6.5. In Theorems 6.3 and 6.4, let w_0 be the longest element in the Weyl group W. One may take a reduced word for w_0 and apply the sequence of divided differences indicated by the reduced word. In cohomology one should skip an operator if its result is zero.

The general machine of Kempf collapsings and divided differences may be applied to quiver loci via Reineke's construction [**28**].

Theorem 6.6. *Suppose Q is a Dynkin quiver and d is any dimension vector. Then each orbit closure $\Omega \subseteq \mathrm{Rep}(Q,d)$ is the image of a linear Kempf collapsing, i.e. there exists a parabolic subgroup $P \subset G$ and a P-invariant linear subspace $Z \subset \mathrm{Rep}$ such that $\Omega = G\cdot Z$.*

By Lemma 5.9 a suitable choice for P and Z is given by $P(I_\bullet)$ and $Z(I_\bullet)$ where $I_\bullet$ is chosen as in section 5.6. Then one may use the product formulae (3.3) for the starting element of the divided difference formulae and apply divided differences to get the desired quiver or K-quiver polynomial.

Example 6.7. Continuing Examples 5.8 and 5.10, let $x_1, x_2, x_3, y_1, y_2, y_3, y_4$ be the standard basis of $X(T)$ where $T \subset GL(3)\times GL(4)$ is the maximal torus. $[Z(I_\bullet)]_T$ is the product of linear forms $(x_i - y_j)$ where (i,j) runs over the positions in $M_{3\times 4}$ where $Z(I_\bullet)$ contains a zero entry. We recover Example 2.2:

$$\begin{aligned}
[Z(I_\bullet)]_T &= (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_1-y_4)(x_2-y_2)(x_2-y_3)(x_2-y_4) \\
&\quad \times (x_3-y_3)(x_3-y_4) \\
[X_2]_G &= \partial_2^y\partial_1^y\partial_3^y\partial_2^y\partial_1^x\partial_2^x\partial_1^x[Z(I_\bullet)]_T \\
&= s_2[x-y]
\end{aligned}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3, y_4)$. Note that two divided difference operators must be omitted from a reduced decomposition of the longest element of $W(G(Q,d)) = S_3 \times S_4$.

Example 6.8. Let Q be the type D4 quiver in Example 5.7, $d = (2,3,2,2)$, T the maximal torus in $G(Q,d)$, and let $X(T)$ have basis a_1, a_2, b_1, b_2, b_3, c_1, c_2, d_1, d_2. Consider

$$I_\bullet = I_{(1,1,0,1)} \bigoplus I_{(1,1,1,0)} \bigoplus I_{(0,1,1,1)},$$

ordering terms as in the reverse of the total order (5.4) on Ind_Q. Then $P(I_\bullet) = B$ consists of the product $\prod_{i\in Q_0} B^i$ of lower triangular subgroups $B^i \subset GL(d(i))$. Let $z^1 \in M_{2\times 3}$, $z^2 \in M_{3\times 2}$, and $z^3 \in M_{3\times 2}$ be the matrices corresponding to the arrows $(1,2)$, $(2,3)$, and $(2,4)$ in Q_1 respectively. Then the point $I_\bullet$ and the subspace $Z = Z(I_\bullet)$ are given by

$$I_\bullet^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad I_\bullet^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_\bullet^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$z^1 = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \end{pmatrix} \qquad z^2 = \begin{pmatrix} 0 & 0 \\ * & 0 \\ * & * \end{pmatrix} \qquad z^3 = \begin{pmatrix} * & 0 \\ * & 0 \\ * & * \end{pmatrix}$$

$Z(I_\bullet)$ has the equations $z^1_{12} = z^1_{13} = z^1_{23} = 0$, $z^2_{11} = z^2_{12} = z^2_{22} = 0$, and $z^3_{12} = z^3_{22} = 0$. To compute the multidegree of the corresponding quiver locus Ω, we start with the multidegree $H_{\mathrm{Rep}}(Z) = (a_1 -$

$b_2)(a_1-b_3)(a_2-b_3)(b_1-c_1)(b_1-c_2)(b_2-c_2)(b_1-d_2)(b_2-d_2)$. Applying ∂_1^a, ∂_1^c, and ∂_1^d we obtain $(a_1-b_3)(a_2-b_3)(b_1-c_1)(b_1-c_2)(b_1+b_2-d_1-d_2)$. Applying $\partial_1^b\partial_2^b\partial_1^b$ we obtain the answer

$$H_{\mathrm{Rep}}(\Omega) = s_{11}[a-b] + s_1[a-b]s_1[b-c] + s_1[a-b]s_1[b-d] + s_1[b-c]s_1[b-d]$$

where $s_\lambda[X-Y]$ is the double Schur polynomial. Note how the answer can be expressed as a positive sum of products of double Schur polynomials in differences of sets of variables, where the differences correspond to arrows in Q_1. This seems to be an instance of a Schur or component formula (a la [**23**]) in type D.

7. Future directions

We believe that the method of Kempf collapsing yields divided difference formulae for a nontrivial family of quiver loci for any quiver Q. In Remark 4.1 it was explained how one may define multidegrees and Hilbert numerators for an arbitrary Q but with a condition on the quiver locus. Under those conditions, consider Example 4.2 consisting of $M_{n\times n}$ under the adjoint action of $G = GL(n)$ and in particular the closure X of a nilpotent conjugacy class. Then X has a Kempf collapsing [**30**], but the best choice of the space Z is not the direct sum of the indecomposables as in the Dynkin case. One may choose Z to be the set of matrices that are strictly lower block triangular with diagonal block of sizes given by the transpose of the partition coming from the nilpotent Jordan blocks, and P to be the lower block triangular parabolic for the same set of diagonal blocks.

As indicated in the introduction, it would also be nice to obtain "positive" formulae for the (K-)quiver polynomials.

References

[1] M. Auslander, I. Reiten, and S. O. Smalø, Representation theory of Artin algebras, Cambridge Stud. Adv. Math. 36, Cambridge University Press, 1995.

[2] R. Bedard, On commutation classes of reduced words in Weyl groups, European J. Combin. **20** (1999), no. 6, 483–505.

[3] N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. **2** (1993), no. 4, 257–269.

[4] A. S. Buch and W. Fulton, Chern class formulas for quiver varieties, Invent. Math. **135** (1999), no. 3, 665–687.

[5] A. S. Buch, L. M. Fehér, and R. Rimányi, Positivity of quiver coefficients through Thom polynomials, Adv. Math. 197 (2005), no. 1, 306–320.

[6] I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev, Coxeter functors and Gabriel's theorem, Uspehi Mat. Nauk **28** (1973), no. 2 (170), 19–33.

[7] A. S. Buch, A. Kresch, H. Tamvakis, M. Shimozono, and A. Yong, Stable Grothendieck polynomials and K-theoretic factor sequences, preprint.

[8] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong, Schubert polynomials and quiver formulas, Duke Math. J. **122** (2004), no. 1, 125–143.

[9] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong, Grothendieck polynomials and quiver formulas, Amer. J. Math. **127** (2005), no. 3, 551–567.

[10] A. S. Buch and R. Rimanyi, A formula for non-equioriented quiver orbits of type A, arXiv:math.AG/0412073.

[11] A. S. Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians. Acta Math. **189** (2002), no. 1, 37–78.

[12] A. S. Buch, Grothendieck classes of quiver varieties. Duke Math. J. 115 (2002), no. 1, 75–103.

[13] A. S. Buch, Alternating signs of quiver coefficients, J. Amer. Math. Soc. **18** (2005), no. 1, 217–237.

[14] H. Derksen, Lecture notes on quiver representations, http://www.math.lsa.umich.edu/%7Ehderksen/math711.w01/math711.html.

[15] H. Derksen and J. Weyman, Quiver representations, Notices Amer. Math. Soc. **52** (2005), no. 2, 200–206.

[16] S. V. Fomin, S. Gelfand, and A. Postnikov, Quantum Schubert polynomials, JAMS **10** (1997) 565–596.

[17] S. V. Fomin and A. N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993). Discrete Math. **153** (1996), no. 1-3, 123–143.

[18] W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, Duke Math. J. **65** (1992) 381–420.

[19] W. Fulton, Universal Schubert polynomials, Duke Math. J. **96** (1999) 575–594.

[20] P. Gabriel, Unzerlegbare Darstellungen, I, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309.

[21] G. Kempf, On the collapsing of homogeneous bundles. Invent. Math. **37** (1976) 229–239.

[22] A. Knutson and E. Miller, Gröbner geometry of Schubert polynomials, to appear in Ann. of Math.

[23] A. Knutson, E. Miller, and M. Shimozono, Four positive formulae for type A quiver polynomials, to appear in Invent. Math.

[24] E. Miller, Alternating formulas for K-theoretic quiver polynomials, Duke Math. J. **128** (2005), no. 1, 1–17.

[25] E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

[26] I. R. Porteous, Simple singularities of maps, Proceedings of Liverpool Singularities Symposium, I (1969/70), 286–307, Lecture Notes in Math., Vol. 192, Springer, Berlin, 1971.
[27] C. M. Ringel, Representations of K-species and bimodules, J. Algebra **41** (1976) 269–302.
[28] M. Reineke, Quivers, desingularizations and canonical bases, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), 325–344, Progr. Math., 210, Birkhäuser Boston, Boston, MA, 2003.
[29] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.
[30] J. Weyman, Cohomology of vector bundles and syzygies, Cambridge Tracts in Mathematics, 149, Cambridge University Press, Cambridge, 2003.
[31] S. Zelikson, Auslander-Reiten quivers and the Coxeter complex, Algebr. Represent. Theory **8** (2005), no. 1, 35–55.

Department of Mathematics, University of California, San Diego
E-mail address: allenk@math.ucsd.edu

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123 USA
E-mail address: mshimo@vt.edu