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The Chromatic Symmetric Function of Symmetric Caterpillars and Near-Symmetric Caterpillars

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Abstract. For every proper coloring κ of a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, one obtains a monomial of degree n defined by $\mathbf{x}^\kappa = x_{\kappa(v_1)}x_{\kappa(v_2)}\ldots x_{\kappa(v_n)}$. Summing these monomial terms over all proper colorings of a given graph G gives the chromatic symmetric function $X_G(\mathbf{x})$. Using Stanley's expansion of the chromatic symmetric function in the power sum basis $\{p_\lambda(\mathbf{x})\}_{\lambda\vdash n}$ of the space Λ^n of homogeneous symmetric functions of degree n, we identify properties of our graph as various coefficients of the $p_\lambda(\mathbf{x})$ in this expansion for $X_G(\mathbf{x})$.

We focus on caterpillars, that is, those trees which becomes a path when all of its vertices of degree one, are deleted. This path is known as the spine of the caterpillar. A caterpillar C is said to be symmetric if there is an isomorphism that exchanges the endpoints of the spine, and is called near-symmetric if the caterpillar becomes symmetric upon shifting a single edge of C into the spine.

We use the coefficients of $p_\lambda(\mathbf{x})$ in the expansion of $X_C(\mathbf{x})$, for λ being a partition with two parts, to show that the chromatic symmetric function distinguishes symmetric and near-symmetric caterpillars from all other caterpillars. We also show that if two trees have a different number of leaves, then they also have different chromatic symmetric functions.

Résumé. Nous sommes intéressés dans le problème de si la fonction symétrique chromatique $X_G(\mathbf{x})$ distingue les arbres nonisomorphe. En utilisant l'expansion de Stanley de la fonction symétrique chromatique dans la base $\{p_\lambda(\mathbf{x})\}_{\lambda\vdash n}$ de l'espace Λ^n des fonctions symétriques homogènes de degré n, nous identifions des propriétés de notre graphique comme divers coefficients de $p_\lambda(\mathbf{x})$ dans cette expansion pour $X_G(\mathbf{x})$.

Nous concentrons sur chenilles, c'est-à-dire, ces arbres qui devient un chemin quand tous ses sommets du degré un sont supprimés. Ce chemin est connu comme épine de la chenille. Une chenille C s'appelle symétrique s'il y a un isomorphisme qui échange les sommets finaux de l'épine, et s'appelle proche-symétrique si la chenille devient symétrique par l'insertion d'un arc de C en l'épine.

Nous employons les coefficients de $p_\lambda(\mathbf{x})$ dans l'expansion de $X_C(\mathbf{x})$, pour λ étant une cloison avec deux parts, pour prouver que la fonction symétrique chromatique distingue les chenilles symétriques et chenilles proche-symétriques de tous autres chenilles. Aussi, nous prouvons que la fonction symétrique chromatique distingue les arbres qui ont un nombre différent de sommets du degré un.

1. Introduction

In this paper we shall only consider the case of *simple* graphs, that is, those with no loops or multiple edges. Let $\mathbf{x} = x_1, x_2, \ldots$ be a countable sequence of commutative inderterminates and G be a graph with vertex set V and edge set E. Given a coloring κ of G, that is a map $\kappa : V \to \mathbb{N}$, we write $\mathbf{x}^\kappa$ for the monomial term of degree $n = |V|$ defined by

$$\mathbf{x}^\kappa = \prod_{v\in V} x_{\kappa(v)}.$$

The *chromatic symmetric function* $X_G(\mathbf{x})$ is then defined by taking

$$X_G(\mathbf{x}) = \sum_\kappa \mathbf{x}^\kappa, \tag{1}$$

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where the sum is over all *proper* colorings κ, i.e. those colorings for which $\kappa(u) \neq \kappa(v)$ for every edge uv of the graph G.

Any coloring of a graph partitions the vertex set into a finite number of color classes, and given a proper coloring of G, permuting these color classes yields another proper coloring of G. Thus $X_G(\mathbf{x})$ is symmetric in the inderterminates $x_1, x_2, x_3, \ldots$. For convenience, we shall drop reference to the variables and write X_G in place of $X_G(\mathbf{x})$.

The *chromatic polynomial* $\chi(G,k)$ of a graph G gives the number of proper colorings using only k colors. We note that $X_G(1^k) = \chi(G,k)$, where 1^k denotes setting $x_1 = x_2 = \ldots = x_k = 1$ and $x_{k+1} = x_{k+2} = \ldots = 0$, since then a monomial survives if, and only if, it comes from a proper coloring using the colors $\{1, 2, \ldots, k\}$, in which case the contribution to the sum is 1. It is easy to see that the chromatic polynomial of any n-vertex tree T is given by $\chi(T,k) = k(k-1)^{n-1}$.

We are interested in the following question of Stanley [**Stanley, 1995**].

Problem 1.1. *Does the chromatic symmetric function distinguish every pair of nonisomorphic trees? That is, given trees T_1 and T_2, do we have $X_{T_1} = X_{T_2}$ if, and only if, $T_1 \cong T_2$?*

The rest of the paper is structured as follows. In the next section we derive some straightforward results for graphs. In Section 3 we look at a labelling procedure for caterpillars, and discuss its relation to symmetric caterpillars. Section 4 uses this labelling procedure to solve Problem 1.1 in the case of symmetric and near-symmetric caterpillars. In Section 5 we turn to counting the number of n-vertex symmetric caterpillars. Finally, in Section 6, we conclude by collecting our results, showing the existence of certain families of graphs.

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2. Definitions and General Results

If $\{p_\lambda(\mathbf{x})\}_{\lambda \vdash n} = \{p_\lambda\}_{\lambda \vdash n}$ is the power sum basis of Λ^n, the space of homogeneous symmetric functions of degree n, then we have the following.

Theorem 2.1. [**Stanley, 1995**, Theorem 2.5] *For an n-vertex graph G*

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(F)},$$

where $\lambda(F)$ is the partition of n whose parts correspond to the sizes of the connected components in the spanning subgraph of G with edge set F.

From its definition, it is clear that X_G is homogeneous of degree n. Hence graphs with a different number of vertices have different chromatic symmetric functions.

We shall use the notation $[p_\lambda]X_G$ to denote the coefficient of p_λ in the expansion of X_G in terms of the basis $\{p_\lambda\}_{\lambda \vdash n}$ of Λ^n. From Theorem 2.1 we have

$$[p_\lambda]X_G = \sum_{\substack{F \subseteq E \\ \lambda(F)=\lambda}} (-1)^{|F|}. \tag{2}$$

The only way to obtain the partition $\lambda(F) = (1^n)$ is for F to include no edges of G, so the only contribution to the coefficient of $p_{(1^n)}$ comes from $F = \emptyset$. Hence, for each graph G,

$$[p_{(1^n)}]X_G = 1. \tag{3}$$

The only way to obtain the partition $\lambda(F) = (2, 1^{n-2})$ is for F to include a single edge of G. Hence the only contributions to the coefficient of $p_{(2,1^{n-2})}$ is from the sets F with $|F| = 1$. Thus Equation 2 gives

$$[p_{(2,1^{n-2})}]X_G = -|E|, \tag{4}$$

for every graph with edge set E. Similarly

$$[p_{(2^k,1^{n-2k})}]X_G = (-1)^k \mu_k(G), \tag{5}$$

where $\mu_k(G)$ is the number of ways of selecting k vertex-disjoint edges in G, that is, the *number of matchings in G of size k.*

In the interest of Problem 1.1, we turn to the case where our graph G is a tree T. We restrict to the case of $n \geq 3$ vertices. Then the partition $\lambda(F) = (k, 1^{n-k})$ arises precisely when the edge set F determines a k-vertex subtree of T, requiring exactly $k-1$ edges. Thus Equation 2 gives

$$[p_{(k,1^{n-k})}]X_G = (-1)^{k-1} T_k, \tag{6}$$

where T_k is the number of k-vertex subtrees of T. More generally, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$, we can show that

$$[p_\lambda]X_T = (-1)^{n-j} T_\lambda, \tag{7}$$

where T_λ is the number of partitions of T into disjoint subtrees of size $\lambda_1, \lambda_2, \ldots, \lambda_j$.

Within the context of trees, it is common to refer to a vertex of degree one as a *leaf.* Vertices of degree larger than one are called *internal* vertices. We say that an edge of a graph is *internal* if both of its endpoints are internal. Otherwise at least one endpoint of the edge is a leaf, and we call the edge *external.*

Every edge that is removed from a tree T increases the number of connected components by one, so to obtain a partition $\lambda(F)$ with two parts requires F to be of the form $E - \{e\}$, where e is an edge of T. In the next few sections we inspect the partitions obtained by removing internal edges, and show how the coefficients of these partitions in X_T help attack Problem 1.1 in the case of caterpillars. Before moving in that direction, we inspect the simpler case of partitions obtained by removing an external edge from a tree.

PROPOSITION 2.2. *If T is an n-vertex tree with $n \geq 3$, then*

$$[p_{(n-1,1)}]X_T = (-1)^n L(T),$$

where $L(T)$ is the number of leaves of T.

Thus the chromatic symmetric function distinguishes trees with a different number of leaves.

PROOF. Every leaf is the endpoint of some external edge of T, and since there are at least three vertices in T, no edge of T has a leaf as both of its endpoints. Thus the number of leaves in T is the same as the number of external edges in T.

To obtain the partition $\lambda(F) = (n-1, 1)$ in Equation 2, the edge subset F must isolate a single vertex of T. This can be accomplished when the set $F \subseteq E$ excludes a single external edge of T, and this is the only way this partition can arise. Since there are $n-1$ edges in T, these F have $|F| = n-2$, and hence

$$[p_{(n-1,1)}]X_T = (-1)^{n-2} L(T), \tag{8}$$

where $L(T)$ is the number of leaves of T.

□

3. Caterpillars, Spine Sets, and Symmetry

A *caterpillar* C is a tree which contains a path consisting of internal vertices of C such that every vertex of C that is not on the path is adjacent to a vertex on the path. This path is called the *spine* of the caterpillar. With our definitions, the spine of a caterpillar is the unique subgraph induced by the set of internal vertices of the caterpillar. If we do *not* make the requirement that the vertices of the spine be internal vertices of C, which may prove convenient in some instances, then the spine is no longer unique.

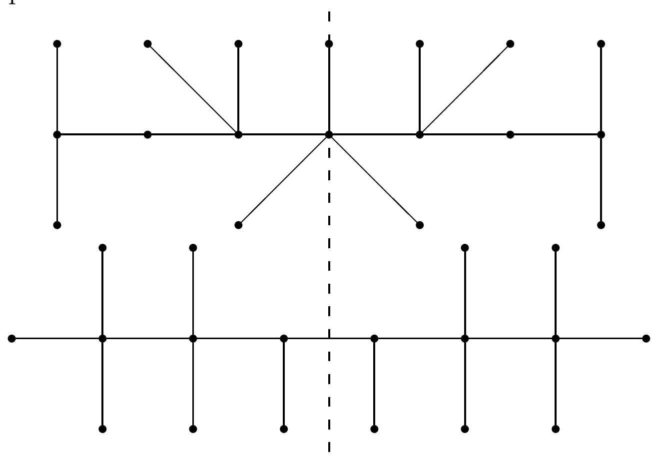
If the spine of a caterpillar consists of the path of vertices $x_1, x_2, \ldots, x_k$, then we call $\delta = (deg(x_1), deg(x_2), \ldots, deg(x_k))$ a *degree sequence* of the spine; the other degree sequence being $(deg(x_k), deg(x_{k-1}), \ldots, deg(x_1))$. We call the caterpillar *symmetric* if the degree sequence δ is palindromic, that is, when the two possible degree sequences of the spine are equal. Equivalently, a caterpillar C is symmetric if there is an automorphism of C that switches endpoints of the spine. Visually, suitably drawn, one half of the caterpillar is the mirror image of the other.

Example:

Here we see two symmetric caterpillars.

$\delta = (3, 2, 4, 5, 4, 2, 3)$

$\delta = (4, 4, 3, 3, 4, 4)$



Given an n-vertex caterpillar, we shall create a labelling of its edges with the numbers $1, 2, 3, \ldots, n-1$ as follows.

First take an endpoint of the spine and mark it. Now starting at the marked vertex, we iterate:

(1) Let u be the vertex of the spine which has just been marked.
(2) Label the unlabelled external edges incident to u with the smallest unused labels among $1, 2, 3, \ldots n-1$.
(3) If there is an unlabelled internal edge incident with u, say $e = uv$, then label e with the smallest unused label among $1, 2, 3, \ldots n-1$, mark vertex v, and proceed back to 1. If there is no unlabelled internal edge incident to u, then the labelling is complete.

Collecting the labels of the internal edges of a caterpillar C gives rise to a set $\mathscr{S}_C \subseteq \{2, 3, \ldots, n-2\}$ called a *spine set* of C. Note that two spine sets of a given caterpillar are possible, since either endpoint of the spine could have been chosen to be initially marked in the labelling procedure. If a degree sequence of the spine is $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ with δ_1 corresponding to the degree of vertex v, then the spine set of C one obtains by initially marking v is

$$\mathscr{S}_C = \{\delta_1, \delta_1 + \delta_2 - 1, \delta_1 + \delta_2 + \delta_3 - 2, \ldots, \delta_1 + \delta_2 + \ldots + \delta_{k-1} - k + 2\}.$$

Conversely, given any set $S \subseteq \{2, 3, \ldots, n-2\}$, say $S = \{x_1, x_2, \ldots, x_k\}$ where $x_1 < x_2 < \ldots < x_k$, we can associate to S the n-vertex caterpillar $\mathscr{C}_S$ that has spine set S by using the caterpillar whose spine has the degree sequence given by

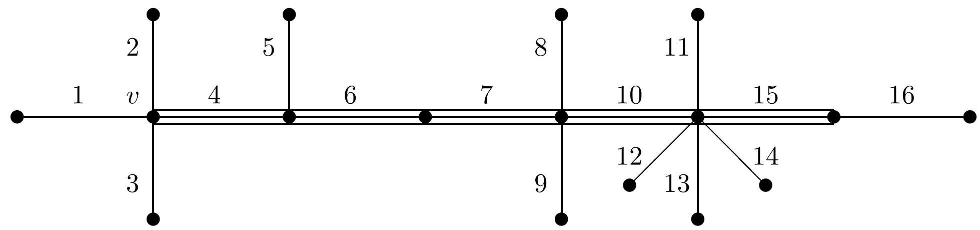
$$\delta = (x_1, x_2 - x_1 + 1, x_3 - x_2 + 1, \ldots, x_k - x_{k-1} + 1, n - x_k).$$

Then for each n-vertex caterpillar C with k internal edges, we have

$$\mathscr{C}_{\mathscr{S}_C} \cong C. \tag{9}$$

Example:

Below we see a caterpillar that has had its edges labelled as described by the above procedure, where the vertex labelled v is the one that was initially marked. The spine of the caterpillar has been highlighted.



From this we obtain the spine set $\mathscr{S}_C = \{4, 6, 7, 10, 15\}$.

Conversely, given the set $S = \{4, 6, 7, 10, 15\}$, we can construct the 17 vertex caterpillar $\mathscr{C}_S$ with $|S| = 5$ internal edges by taking a caterpillar with spine degree sequence given by

$$\begin{aligned}\delta &= (4, 6-4+1, 7-6+1, 10-7+1, 15-10+1, 17-15)\\ &= (4, 3, 2, 4, 6, 2).\end{aligned}$$

This is exactly the caterpillar we began with.

Given a set $S \subseteq \{2, 3, \ldots, n-2\}$, we call the set $S' = \{n - i|\ i \in A\}$ the *reflection* of S. If the set S satisfies $S = S'$ we shall call S a *symmetric* subset. The following result shows that the two spine sets one can obtain from a caterpillar are reflections of one another.

LEMMA 3.1. *For each $S \subseteq \{2, 3, \ldots, n-2\}$ we have $\mathscr{S}_{\mathscr{C}_S} \in \{S, S'\}$.*

PROOF. If $S \subseteq \{2, 3, \ldots, n-2\}$, then there are n vertices in $C = \mathscr{C}_S$ and $n-1$ edges. Suppose an endpoint of the spine of C is chosen to be initially marked, and the labelling procedure has been completed, producing a spine set T.

When the internal edge e of C was labelled k there must have been $n-1-k$ edges left to label. Further, starting the labelling procedure from the opposite endpoint of the spine, when we reach the point of labelling e, these $n-k-1$ edges are exactly the edges that have been labelled. Thus e will be labelled $n-k$, as required. □

Since this result shows $\mathscr{S}'_C$ is also a spine set for the caterpillar C, Equation 9 yields

$$\mathscr{C}_{\mathscr{S}'_C} \cong C. \tag{10}$$

COROLLARY 3.2. *If C_1 and C_2 are n-vertex caterpillars, then $C_1 \cong C_2$ if, and only if, either $\mathscr{S}_{C_1} = \mathscr{S}_{C_2}$ or $\mathscr{S}_{C_1} = \mathscr{S}'_{C_2}$.*

PROOF. If $C_1 \cong C_2$, then there is an isomorphism between the two which takes the spine of one caterpillar onto the spine of the other. If we perform the labelling procedure on each caterpillar by starting at the ends of the spine which correspond through the isomorphism, we will produce the same spine set for each caterpillar; that is, $\mathscr{S}_{C_1} = \mathscr{S}_{C_2}$. If we had started the labelling procedure from the opposite end of one of the spines, then the proof of Lemma 3.1 shows that we obtain the reflected spine set. In which case $\mathscr{S}_{C_1} = \mathscr{S}'_{C_2}$.

Conversely, if either $\mathscr{S}_{C_1} = \mathscr{S}_{C_2}$ or $\mathscr{S}_{C_1} = \mathscr{S}'_{C_2}$, then by using either Equation 9 or Equation 10 we obtain $C_1 \cong C_2$, as desired. □

PROPOSITION 3.3. *An n-vertex caterpillar C is symmetric if, and only if, its spine set $\mathscr{S}_C$ is a symmetric subset of $\{2, 3, \ldots, n-2\}$.*

PROOF. If a caterpillar is symmetric, that is, the degree sequence of the spine is palindromic, then the labelling procedure would produce the *same* spine set $\mathscr{S}_C$ from either end. Since we know from the proof of Lemma 3.1 that labelling from the opposite end should give the *reflected* spine set, this shows that if the caterpillar C is symmetric, then so is its spine set $\mathscr{S}_C$.

Conversely, if the degree sequence of the spine is not palindromic, we can easily check that the corresponding spine set is not symmetric. Suppose the degree sequence for C is $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ where

$$\delta_1 = \delta_k,\ \delta_2 = \delta_{k-1},\ \ldots,\ \text{and}\ \delta_t = \delta_{k-t+1}, \tag{11}$$

but

$$\delta_{t+1} \neq \delta_{k-t}. \tag{12}$$

Then, labelling the caterpillar from the one end of the spine gives the spine set

$$\{\delta_1, \delta_1 + \delta_2 - 1, \ldots, \delta_1 + \delta_2 + \ldots + \delta_{k-1} - k + 2\}$$

while labelling the caterpillar from the opposite end of the spine gives the spine set

$$\{\delta_k, \delta_k + \delta_{k-1} - 1, \ldots, \delta_k + \delta_{k-1} + \ldots + \delta_2 - k + 2\}.$$

The two spine sets above are written with elements shown in increasing order of size. Thus to check if they are the same sets, we need only check that, in the order shown, the j-th element of one matches the j-th element of the other, for each j. The first t elements of these sets (in the order shown) are the same by Equation 11, but by using both Equations 11 and 12, we find that the $t+1$-th elements, $\delta_1 + \delta_2 + \ldots + \delta_{t+1} - k + 2$ and $\delta_k + \delta_{k-1} + \ldots + \delta_{k-t} - k + 2$ respectively, differ. □

Corollary 3.4. *If C_1 and C_2 are n-vertex caterpillars and at least one of them is symmetric, then $C_1 \cong C_2$ if, and only if, $\mathscr{S}_{C_1} = \mathscr{S}_{C_2}$.*

Proof. Without loss of generality, let C_2 be symmetric. Proposition 3.3 gives $\mathscr{S}_{C_2} = \mathscr{S}'_{C_2}$. Now Corollary 3.2 gives the desired result. □

4. Results on X_C

4.1. A Bound on Coefficients. For each $i \in \mathscr{S}_C$, i corresponds to some internal edge e_i of C, and the graph obtained by removing the edge e_i from C consists of two disjoint caterpillars with $i-1$ and $n-i-1$ edges respectively. Hence the set $F = E - \{e_i\}$ induces the partition

$$\lambda(F) = (i, n-i). \tag{13}$$

Whenever λ is a partition with two parts and C is a caterpillar there is a straightforward bound on the coefficient of p_λ, namely

Proposition 4.1. *Let C be an n-vertex caterpillar and λ have two parts. Then either*

(1) $(-1)^n[p_\lambda]X_C = L(C)$, *if* $\lambda = (n-1, 1)$, *or*
(2) $0 \leq (-1)^n[p_\lambda]X_C \leq 2$ *otherwise.*

Proof. From Proposition 2.2, we have $[p_\lambda]X_T = (-1)^n L(T)$ in the case of $\lambda = (n-1, 1)$. Any other partition λ with two parts can only arise as $\lambda(E - \{e_i\})$ for some $i \in \mathscr{S}_C$. We show that any such λ can arise at most twice.

We are looking for occurrences of $\lambda = (j, n-j)$, and λ can only arise from the edges, if there are any, which would correspond to the potential elements j and $n-j$ of $\mathscr{S}_C$. Thus the magnitude of the coefficient of p_λ could be at most 2, if both j, $n-j \in \mathscr{S}_C$. □

From the proof of Proposition 4.1, we have the following fact.

Corollary 4.2. *If $\lambda = (j, n-j)$, $1 < j < n$, is a partition of n into two parts and C is a n-vertex caterpillar, then $[p_\lambda]X_C = (-1)^n|\{j, n-j\} \cap \mathscr{S}_C|$.*

4.2. Symmetric and Near-Symmetric Caterpillars.

THEOREM 4.3. *The chromatic symmetric function distinguishes the symmetric caterpillars from the nonsymmetric caterpillars. Further, it distinguishes among the symmetric caterpillars.*

PROOF. Let C be a symmetric caterpillar. We saw in Proposition 3.3 that a given n-vertex caterpillar C was symmetric if, and only if, $\mathscr{S}_C$ is symmetric. That is, $n-j \in \mathscr{S}_C$ if, and only if, $j \in \mathscr{S}_C$.

In the case when n is even and $j = \frac{n}{2}$ we have $j = n-j$, but otherwise we have $j \neq n-j$. Thus, for a symmetric caterpillar, Corollary 4.2 gives

$$(-1)^n[p_{(j,n-j)}]X_C = 0 \text{ if } j \notin \mathscr{S}_C, \tag{14}$$

$$(-1)^n[p_{(j,n-j)}]X_C = 2 \text{ if } j \in \mathscr{S}_C \text{ and } j \neq \frac{n}{2}, \tag{15}$$

and if n is even, then

$$(-1)^n[p_{(\frac{n}{2},\frac{n}{2})}]X_C = 1 \text{ if } \frac{n}{2} \in \mathscr{S}_C. \tag{16}$$

We have shown all symmetric caterpillars satisfy Equations 14, 15, and 16. Conversely, if a caterpillar satisfies Equations 14, 15, and 16, we shall show it is symmetric. Let a caterpillar C satisfy Equations 14, 15, and 16 and let j be a member of $\mathscr{S}_C$. To show C is symmetric, we need only show that $n-j \in \mathscr{S}_C$. If $j = \frac{n}{2}$, then $n-j = j$, so immediately $n-j \in \mathscr{S}_C$. If $j \neq \frac{n}{2}$, then by Equation 15 and Corollary 4.2 we find $n-j \in \mathscr{S}_C$, as required.

Hence we can use the chromatic symmetric function to distinguish the symmetric caterpillars from those that are nonsymmetric. Further, by Equations 14, 15, 16, and Corollary 4.2, the spine set of the caterpillar can be determined from its chromatic symmetric function. From Equation 9 we know that the spine set of a caterpillar determines the caterpillar. Thus chromatic symmetric function distinguishes the symmetric caterpillars from one another.

□

We can now make a slight perturbation of Theorem 4.3. Towards this end, we shall say that a nonsymmetric caterpillar C is *near-symmetric* if $\mathscr{S}_C \cup \{i\}$ is a symmetric subset for some number $i \in \{2, 3, \dots n-2\}$.

Example: The caterpillar C with 11 vertices whose spine set is $\mathscr{S}_C = \{3, 4, 8\}$ is near-symmetric, as $\{3, 4, 7, 8\}$ is a symmetric subset of $\{2, 3, \dots, 9\}$.

THEOREM 4.4. *The chromatic symmetric function distinguishes the near-symmetric caterpillars from those caterpillars which are not near-symmetric. Further, it distinguishes among the near-symmetric caterpillars.*

PROOF. Let C be a near-symmertric caterpillar, say with $\mathscr{S}_C \cup \{n-i\}$ being a symmetric subset. Then necessarily $i \in \mathscr{S}_C$. Looking at the coefficients of p_λ in X_C for partitions into two parts gives

$$(-1)^n[p_{(j,n-j)}]X_C = 0 \text{ if } j \notin \mathscr{S}_C, \tag{17}$$

$$(-1)^n[p_{(j,n-j)}]X_C = 2 \text{ if } j \in \mathscr{S}_C \text{ and } j \neq \frac{n}{2},\ i, \tag{18}$$

$$(-1)^n[p_{(i,n-i)}]X_C = 1, \tag{19}$$

and if n is even, then

$$(-1)^n[p_{(j,n-j)}]X_C = 1 \text{ if } j = \frac{n}{2} \in \mathscr{S}_C. \tag{20}$$

Conversely, any caterpillar C which satisfies Equations 17, 18, 19, and 20 for some value i is found to be near-symmetric upon considering Corollary 4.2, as adding $n-i$ to $\mathscr{S}_C$ creates a symmetric subset.

As before, from Equations 17, 18, 19, and 20 and Corollary 4.2 we see that the chromatic symmetric function of a near-symmetric caterpillar determines the spine set $\mathscr{S}_C$ of the caterpillar. Then by Equation 9, we can recover C from $\mathscr{S}_C$. □

Combining Theorems 4.3 and 4.4 we obtain the following result.

Theorem 4.5. *Let $\mathcal{C}$ be the set of caterpillars and $\mathcal{S}$ be the set of caterpillars that are either symmetric or near-symmetric. Then if $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{S}$, we have $X_{C_1} = X_{C_2}$ if, and only if, $C_1 \cong C_2$.*

5. Counting Symmetric Caterpillars

Proposition 5.1. *Let $\mathcal{S}(n,k)$ denote the number of nonisomorphic n-vertex symmetric caterpillars with k internal edges.*

(1) *If k is even, $\mathcal{S}(n,k) = \binom{\lfloor \frac{n-3}{2} \rfloor}{\frac{k}{2}}$.*

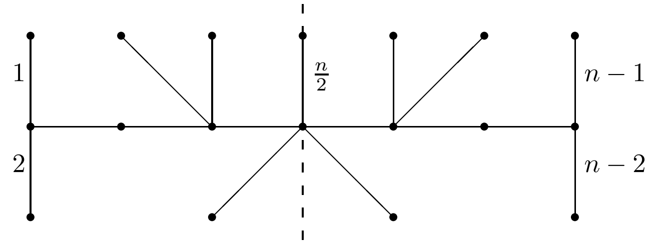
(2) *If k is odd, then*

   (a) $\mathcal{S}(n,k) = \binom{\frac{n}{2}-2}{\frac{k-1}{2}}$ *when n is even, and*

   (b) $\mathcal{S}(n,k) = 0$ *when n is odd.*

Proof. Suppose k, the number of edges in the spine, is even. Then visually the line of symmetry of C crosses the spine at the vertex in the center of the spine.

If n is even, then $n-1$, the number of edges, is odd. Hence one of the edges is forced to be along the line of symmetry of C. Under our labelling procedure, and by redrawing if necessary, we can assume the edge along the line of symmetry is labelled $\frac{n}{2}$. For example:

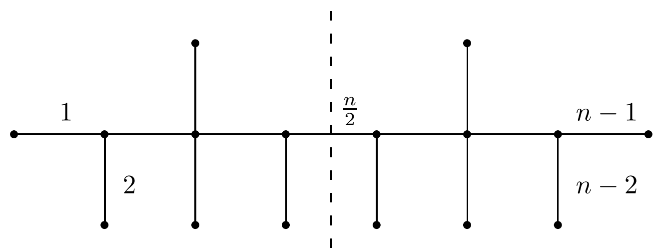


From the symmetry of the caterpillar, the rest of the caterpillar is determined once we know which $\frac{k}{2}$ of the first $\frac{n}{2}-1$ edges are internal edges. Thus we count the number of sets $S \subseteq \{2,3,\ldots,\frac{n}{2}-1\}$ with $\frac{k}{2}$ elements, giving $\binom{\frac{n}{2}-2}{\frac{k}{2}} = \binom{\frac{n-4}{2}}{\frac{k}{2}} = \binom{\lfloor\frac{n-3}{2}\rfloor}{\frac{k}{2}}$ symmetric caterpillars.

If n is odd, then the number of edges is even and, by redrawing if necessary, $\frac{n-1}{2}$ of the $n-1$ edges lie on each side of the line of symmetry. Further, knowing which of the first $\frac{n-1}{2}$ edges are internal edges determines the caterpillar. Thus we count sets of the form $S \subseteq \{2,3,\ldots,\frac{n-1}{2}\}$ containing $\frac{k}{2}$ elements, obtaining $\binom{\frac{n-1}{2}-1}{\frac{k}{2}} = \binom{\frac{n-3}{2}}{\frac{k}{2}} = \binom{\lfloor\frac{n-3}{2}\rfloor}{\frac{k}{2}}$ symmetric caterpillars. This completes the proof of 1.

Now if k, the number of edges in the spine, is odd, then visually the line of symmetry of C bisects the central edge of the spine. Apart from this edge, every other edge is paired with its reflection across the line of symmetry. Thus the total number of edges is odd, forcing n to be even. This gives $\mathcal{S}(n,k)=0$ for odd n.

If we assume n is even and C is symmetric, then the central edge of the spine of C is labelled $\frac{n}{2}$ by our labelling procedure, and, as before, knowing the internal edges of one side of the caterpillar determines the other.



Thus we seek to count all sets of the form $S \subseteq \{2,3,\ldots,\frac{n}{2}-1\}$ containing $\frac{k-1}{2}$ elements. This gives $\binom{\frac{n}{2}-2}{\frac{k-1}{2}}$ symmetric caterpillars in this final case. □

From this result, one can check that the total number of symmetric caterpillars with n vertices is $2^{\lfloor\frac{n-2}{2}\rfloor}$. A more direct approach can be found in [**Harary/ Schwenk, 1973**], where it is also shown that the total number of caterpillars with n vertices is $2^{n-4}+2^{\lfloor\frac{n-4}{2}\rfloor}$.

6. Conclusions

In the majority of this paper we have remained within the context of caterpillars as opposed to trees in general. As previously noted, there are $2^{n-4}+2^{\lfloor\frac{n-4}{2}\rfloor}$ n-vertex caterpillars. We have proved the result in the case of symmetric caterpillars, of which there are $2^{\lfloor\frac{n-2}{2}\rfloor}$, and also in the case of near-symmetric caterpillars [**Morin, 2005**].

By collecting various results, we find that we have proved the following.

PROPOSITION 6.1. *There are collections $\mathcal{Q}_n$ of n-vertex graphs such that:*

(1) $\lim_{n\to\infty} |\mathcal{Q}_n| = \infty$,
(2) $\chi(G_1,k)=\chi(G_2,k)$ *for every pair of graphs* G_1, $G_2 \in \mathcal{Q}_n$, *and*
(3) *If* G_1, $G_2 \in \mathcal{Q}_n$ *and* $X_{G_1}=X_{G_2}$, *then* $G_1=G_2$.

Proof. We look at the collection $\mathcal{Q}_n$ of symmetric caterpillars with n vertices. We have Property 1 by Proposition 5.1. Since all the caterpillars in $\mathcal{Q}_n$ has n vertices, we have $\chi(G,k) = k(k-1)^{n-1}$ for each $G \in \mathcal{Q}_n$. Finally Theorem 4.3 gives Property 3. $\square$

We note that $\mathcal{Q}_n$ could have also been chosen to be the set of near-symmetric n-vertex caterpillars in the above proof.

References

[Harary/ Schwenk, 1973] Frank Harary and Allen J. Schwenk, "The Number of Caterpillars", *Discrete Mathematics* **6** (1973), 359-365.

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