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# Algebraic structures on Grothendieck groups of a tower of algebras

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Abstract. The Grothendieck group of the tower of symmetric group algebras has a self-dual graded Hopf algebra structure. In this work, we define the general notation of a tower of algebras and study the two Grothendieck groups on this tower. Using representation theory, we prove that the two Grothendieck groups are graded Hopf algebras. Moreover, we define a paring and show that the two Grothendieck groups are dual to each other as Hopf algebras.

Résumé. Les configurations gréées sont des objets combinatoires inspirés par l'ansatz de Bethe, et qui sont en correspondence avec les éléments cristallins de plus haut poids. Dans cette note, nous introduisons le concept de "configurations gréées généralisées", en construisant une structure cristalline dans l'espace des configurations gréées.

## 1. Introduction

In 1977, L. Geissinger realized that *Sym* (symmetric functions in infinite variables) is a self-dual graded Hopf algebra [**6**], which can be interpreted as the self-dual Grothendieck Hopf algebra of the tower of symmetric groups $\bigoplus_{n\geq 0}\mathbb{C}S_n$ using the work of Frobenius and Schur. After this, mathematicians have encountered many instances of combinatorial Hopf algebras that can be realized as the Grothendieck Hopf algebras of a tower of algebras. In each instance, they study a pair of dual Hopf algebras, and it turns out that this duality can be interpreted as the duality of the Grothendieck groups of an appropriate tower of algebras. For example, C. Malvenuto and C. Reutenauer established the duality between the Hopf algebra of *NSym* (noncommutative symmetric functions) and the Hopf algebra of *QSym* (quasi-symmetric functions) when looking at the combinatorics of descents [**12**]. Later, D. Krob and J.-Y. Thibon showed that this duality can be interpreted as the duality of the Grothendieck groups associated to $\bigoplus_{n\geq 0} H_n(0)$ the tower of Hecke algebras at q = 0 [**10**]. More recently, N. Bergeron, F. Hivert, and J.-Y. Thibon showed that if one uses $\bigoplus_{n\geq 0} HCl_n(0)$ the tower of Hecke-Clifford algebras at q = 0, then one gets a similar interpretation for the duality between the $Peak$ algebra and its dual [**2**].

In this work, we study the algebraic structure on the Grothendieck groups $G_0(A)$ and $K_0(A)$ in the more general case where $(A = \bigoplus_{n\geq 0} A_n, \rho_{m,n})$ is a graded algebra and each component $A_n$ is an algebra. We will call $A$ a tower of algebras if it satisfies some conditions. No formal study of this kind has been done so far. Up to this point it was not clear what were the right conditions to impose on a tower of algebra to get the desired algebraic structure on their Grothendieck groups. Here, we find a list of axioms on a tower of algebras which will imply that their Grothendieck groups are graded Hopf algebras. Moreover, we define a paring and show that the corresponding Grothendieck groups are dual to each other as Hopf algebras if the tower of algebras satisfying an additional condition.

This paper is divided into 5 sections as follows. Section 1 is the introduction. In Section 2 we recall some definitions and propositions about bialgebras and Grothendieck groups. In Section 3 we discuss the axioms on a tower of algebras $(A = \bigoplus_{n\geq 0} A_n, \rho_{m,n})$ with $\rho$ preserving unities so that their Grothendieck

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groups are graded Hopf algebras. Moreover, we define a paring and show that the Grothendieck groups are dual to each other as Hopf algebras. In Section 4 we weaken the condition of $\rho$ and modify the definitions of inductions and restrictions to get the similar results as above. In Section 5 we will give some examples to indicate that the Grothendieck groups of a tower of algebras satisfying these axioms are Hopf algebras dual to each other, and these axioms are necessary.

## 2. Notations and Propositions

In this section there is a brief review of some ideas from the theory of bialgebras [**6**] and Grothendieck groups [**8**] which is useful for later discussion.

Definition 2.1. Let $K$ be a commutative ring. A $K$-*algebra* $B$ is a $K$-module with *multiplication* $\pi : B \otimes_K B \to B$ and *unit map* $\mu : K \to B$ satisfying *associativity* and *unitary property*, i.e., $\pi(\pi \otimes 1) = \pi(1 \otimes \pi)$ and $\pi(\mu \otimes 1) = \pi(1 \otimes \mu)$, where 1 is the identity map of module $B$. Denote this algebra by the triple $(B, \pi, \mu)$.

A $K$-*coalgebra* $C$ is a $K$-module with *comultiplication* $\Delta : C \to C \otimes C$ and *counit map* $\epsilon : C \to R$ satisfying *coassociativity* and *counitary property*, i.e., $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ and $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta$, where 1 is the identity map of module $C$. Denote this coalgebra by the triple $(C, \Delta, \epsilon)$.

If a $K$-module $B$ is simultaneously an algebra and a coalgebra it is called a *bialgebra* provided these structures are compatible in the sense that the comultiplication and counit are algebra homomorphisms. Explicitly this means that $\epsilon(\mu(1)) = 1$, $\epsilon(gh) = \epsilon(g)\epsilon(h)$, $\Delta\mu(1) = \mu(1) \otimes \mu(1)$, and $\Delta(gh) = \Sigma g_i h_p \otimes g'_i h'_p$ if $\Delta(g) = \Sigma g_i \otimes g'_i$ and $\Delta(h) = \Sigma h_p \otimes h'_p$, where 1 is the unity of $K$ and $gh = \pi(g \otimes h)$. This is equivalent to requiring that the multiplication and unit map are coalgebra homomorphisms. Denote this bialgebra by the 5-tuple $(B, \pi, \mu, \Delta, \epsilon)$.

A $K$-linear map $\gamma : H \to H$ on a bialgebra $H$ is an *antipode* if for all $h$ in $H$, $\Sigma h_i \gamma(h'_i) = \epsilon(h)1_H = \Sigma\gamma(h_i)h'_i$ when $\Delta h = \Sigma h_i \otimes h'_i$. A *Hopf algebra* is a bialgebra with antipode.

Definition 2.2. An algebra $B$ is a *graded algebra* if there is a direct sum decomposition $B = \bigoplus B_i$ $(i \geq 0)$ such that the product of homogeneous of degrees $p$ and $q$ is homogeneous of degree $p + q$, that is, $\pi(B_p \otimes B_q) \subseteq B_{p+q}$, and $\mu(K) \subseteq B_0$.

A coalgebra $C$ is a *graded coalgebra* if there is a direct sum decomposition $C = \bigoplus C_i$ $(i \geq 0)$ such that $\Delta(C_n) \subseteq \bigoplus(C_k \otimes C_{n-k})$ and $\epsilon(C_n) = 0$ if $n \geq 1$.

A bialgebra $H = \bigoplus H_i$ over $K$ is called *graded connected* if it is $\mathbb{Z}$-graded, concentrated in nonnegative degrees, and satisfies $H_0 = K1_H$, where $K$ is a field.

It is a known fact that a connected bialgebra is a connected Hopf algebra [**17**].

The coassociativity and counitary property are dual to associativity and unitary property, respectively. It is natural to expect the dual of a coalgebra to be an algebra and vice versa. In fact, if a module is a graded bialgebra with all homogeneous components finitely generated, then its graded dual is also a graded bialgebra [**6**].

The definition of Grothendieck groups is introduced in [**8**]. Let $B$ be an arbitrary algebra. Denote

$$\begin{aligned} {}_B\mathcal{M} &= \text{the category of all left } B\text{-modules}, \\ {}_B\text{mod} &= \text{the category of all finitely generated left } B\text{-modules}, \\ \mathcal{P}(B) &= \text{the category of all finitely generated projective left } B\text{-modules}. \end{aligned}$$

Definition 2.3. Let $\mathcal{C}$ be one of the above categories. Let $\mathbf{F}$ be the free abelian group generated by symbols $(M)$, one for each isomorphism class of modules $M$ in $\mathcal{C}$. Let $\mathbf{F_0}$ be the subgroup of $\mathbf{F}$ generated by all expressions

$$(M) - (L) - (N)$$

arising from all short exact sequences

$$0 \to L \to M \to N \to 0$$

in $\mathcal{C}$. The *Grothendieck group* $K_0(\mathcal{C})$ of the category $\mathcal{C}$ is defined by

$$K_0(\mathcal{C}) = \mathbf{F}/\mathbf{F_0},$$

an abelian additive group. For $M \in \mathcal{C}$, let $[M]$ denote its image in $K_0(\mathcal{C})$.

Each $x \in K_0(\mathcal{C})$ is expressible as a difference $[M] - [N]$ with $M$, $N \in \mathcal{C}$, though not in a unique manner. Furthermore, it may occur that $x = 0$ even though $M$ is not isomorphic to $N$.

DEFINITION 2.4. The *Grothendieck group* $G_0(B)$ of the algebra $B$ is defined by

$$G_0(B) = K_0({}_B\text{mod}).$$

The *Grothendieck group* $K_0(B)$ of the algebra $B$ is defined by

$$K_0(B) = K_0(\mathcal{P}(B)).$$

Thus, $G_0(B)$ is generated by expressions $[M]$, one for each isomorphism class $(M)$ of finitely generated left $B$-modules $M$, with relations

$$[M] = [M'] + [M'']$$

for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated left $B$-modules.

$K_0(B)$ is generated by expressions $[P]$, one for each isomorphism class $(P)$ of finitely generated left $B$-modules $P$, with relations

$$[P \oplus P'] = [P] + [P']$$

for all $P, P' \in \mathcal{P}(B)$. (Note that each short exact sequence $0 \to P' \to P \to P'' \to 0$ of modules from $\mathcal{P}(B)$ must split, because $P''$ is a projective $B$-module. Hence, the defining relations for $K_0(B)$ can be expressed in the simpler form involving direct sums, rather than exact sequences from $\mathcal{P}(B)$.)

Now let $B$ be a finite-dimensional algebra over a field $K$. Let $\{V_1, \cdots, V_s\}$ be a complete list of nonisomorphic simple $B$-modules. Then their projective covers $\{P_1, \cdots, P_s\}$ are a complete list of nonisomorphic indecomposable projective $B$-modules [**13**]. With these lists, we have

PROPOSITION 2.1.

$$G_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[V_i]$$

*is a free abelian group with basis* $\{[V_1], \cdots, [V_s]\}$. *And*

$$K_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[P_i]$$

*is a free abelian group with basis* $\{[P_1], \cdots, [P_s]\}$.

Let $A$ be an algebra and $B \subseteq A$ a subalgebra. Let $M$ be a (left) $A$-module and $N$ a (left) $B$-module, then the *induction* of $N$ from $B$ to $A$ is $\text{Ind}_B^A N = A \otimes_B N$ an $A$-module and the *restriction* of $M$ from $A$ to $B$ is $\text{Res}_B^A M = \text{Hom}_A(A, M)$ a $B$-module.

## 3. Grothendieck groups of a tower of algebras (Preserving unities)

In this section, first we list all the axioms we need on a graded algebra $(A = \bigoplus_{n \geq 0} A_n, \rho_{m,n})$ with $\rho$ preserving unities. Then we define the inductions and restrictions on their Grothendieck groups $G_0(A)$ and $K_0(A)$ respectively. After this, we use these definitions to construct the multiplications and comultiplications on $G_0(A)$ and $K_0(A)$ and show that $G_0(A)$ and $K_0(A)$ are graded connected Hopf algebras with these operators. Moreover, we define a paring on the Grothendieck groups $G_0(A)$ and $K_0(A)$. It develops that they are dual to each other as Hopf algebras.

Let $A = \bigoplus_{n \geq 0} A_n$, we call it *a tower of algebras* over field $K = \mathbb{C}$ if the following conditions are satisfied:
(1) $A_n$ is a finite-dimensional algebra with unit, for each $n$. $A_0 \cong K$.
(2) There is an external graded multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$ such that
(a) $\rho_{m,n}$ is an injective homomorphism of algebras, for all $m$ and $n$ (sending $1_m \otimes 1_n$ to $1_{m+n}$);
(b) $\rho$ is associative, that is, $\rho_{l+m,n} \cdot (\rho_{l,m} \otimes 1_n) = \rho_{l,m+n} \cdot (1_l \otimes \rho_{m,n}) := \rho_{l,m,n}$, for all $l$, $m$, $n$.
(3) $A_{n+m}$ is a two-sided projective $A_n \otimes A_m$-module by the action defined to be $a \cdot (b \otimes c) = a\rho_{m,n}(b \otimes c)$ and $(b \otimes c) \cdot a = \rho_{m,n}(b \otimes c)a$, for $a \in A_{m+n},\ b \in A_m$ and $c \in A_n$.
(4) For every primitive idempotent $g$ in $A_{m+n}$, $A_{m+n}g \cong \bigoplus (A_m \otimes A_n)(e \otimes f)$ as (left) $A_m \otimes A_n$-modules if and only if $gA_{m+n} \cong \bigoplus (e \otimes f)(A_m \otimes A_n)$ as (right) $A_m \otimes A_n$-modules for the same indexing of idempotents $(e \otimes f)$'s in $A_m \otimes A_n$.

(5) The following equality holds

$$
\begin{aligned}
& [\mathrm{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}}\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}(M\otimes N)] \\
= \ & \sum_{t+s=k}[\widetilde{\mathrm{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(\mathrm{Res}^{A_m}_{A_t\otimes A_{m-t}}M\otimes \mathrm{Res}^{A_n}_{A_s\otimes A_{n-s}}N)]
\end{aligned}
$$

for all $0<k<m+n$, $M$ an $A_m$-module and $N$ an $A_n$-module. We will explain the notations later.

Why we need these conditions? We can give a brief explanation here. Condition (1) guarantees that their Grothendieck groups are grade connected; with conditions (2) and (3) the inductions and restrictions are well defined; with (4) the duality holds; with (5) the multiplication and comultiplication are compatible. We will come up to the details later.

Now we define the inductions on $G_0(A)$ as follows:

$$
\begin{aligned}
i_{m,n} \ : \ G_0(A_m)\bigotimes\nolimits_{\mathbb{Z}}G_0(A_n) &\ \rightarrow\ G_0(A_{m+n}) \\
[M]\otimes[N] &\ \mapsto\ [\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N &= A_{m+n}\bigotimes\nolimits_{A_m\otimes A_n}(M\otimes N) \\
&= \frac{A_{m+n}\otimes M\otimes N}{<a\otimes[(b\otimes c)(w\otimes u)]-[a\rho_{m,n}(b\otimes c)]\otimes w\otimes u>},
\end{aligned}
$$

for $a\in A_{m+n},\ b\in A_m,\ c\in A_n,\ w\in M$ and $u\in N$. Here let $k=t+s$, define the twisted induction

$$
\begin{aligned}
& \widetilde{\mathrm{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(M_1\otimes M_2)\otimes(N_1\otimes N_2) \\
= \ & (A_k\otimes A_{m+n-k})\widetilde{\bigotimes}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}((M_1\otimes M_2)\otimes(N_1\otimes N_2)).
\end{aligned}
$$

This means

$$
\begin{aligned}
& (a\otimes b)\otimes[(c_1\otimes c_2)\cdot(w_1\otimes w_2)\otimes(d_1\otimes d_2)\cdot(u_1\otimes u_2)] \\
\equiv \ & [a\rho_{t,s}(c_1\otimes d_1)\otimes b\rho_{m-t,n-s}(c_2\otimes d_2)]\otimes(w_1\otimes w_2\otimes u_1\otimes u_2),
\end{aligned}
$$

where $a\in A_k,\ b\in A_{m+n-k},\ c_1\in A_t,\ c_2\in A_{m-t},\ d_1\in A_s,\ d_2\in A_{n-s},\ w_i\in M_i,\ u_i\in N_i$. Also define the restrictions

$$
\begin{aligned}
r_{k,l} \ : \ G_0(A_n) &\ \rightarrow\ G_0(A_k)\bigotimes\nolimits_{\mathbb{Z}}G_0(A_l) \text{ with } k+l=n \\
[N] &\ \mapsto\ [\mathrm{Res}^{A_n}_{A_k\otimes A_l}N],
\end{aligned}
$$

where $\mathrm{Res}^{A_n}_{A_k\otimes A_l}N=\mathrm{Hom}_{A_n}(A_n,N)$ is an $A_k\otimes A_l$-module by the action defined to be $((b\otimes c)\cdot f)(a)=f(a\rho_{k,l}(b\otimes c))$, for $a\in A_n,\ b\in A_k,\ c\in A_l$ and $f\in\mathrm{Hom}_{A_n}(A_n,N)$.

Proposition 3.1. *$i$ and $r$ are well defined.*

Proof. Assume $[M]=[M']+[M'']$. Since $A_{m+n}$ is a (right) projective $A_m\otimes A_n$-module, it is not difficult to get that

$$0\rightarrow A_{m+n}\otimes_{A_m\otimes A_n}(M'\otimes N)\rightarrow A_{m+n}\otimes_{A_m\otimes A_n}(M\otimes N)\rightarrow A_{m+n}\otimes_{A_m\otimes A_n}(M''\otimes N)\rightarrow 0$$

is exact as left $A_{m+n}$-modules by the properties of tensor product and short exact sequence. Hence

$$[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N]=[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M'\otimes N]+[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M''\otimes N].$$

Similarly,

$$[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N]=[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N']+[\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N'']$$

for $[N]=[N']+[N'']$. Hence $i$ is well defined on $G_0(A)$.

Assume $[N]=[N']+[N'']$. Since $\mathrm{Hom}_{A_n}(A_n,M)\cong M$ for all $A_n$-modules $M$, it is clear that

$$0\rightarrow\mathrm{Hom}_{A_n}(A_n,N')\rightarrow\mathrm{Hom}_{A_n}(A_n,N)\rightarrow\mathrm{Hom}_{A_n}(A_n,N'')\rightarrow 0$$

is exact, which is also exact as $A_k\otimes A_l$-modules. Hence

$$[\mathrm{Res}^{A_n}_{A_k\otimes A_l}N]=[\mathrm{Res}^{A_n}_{A_k\otimes A_l}N']+[\mathrm{Res}^{A_n}_{A_k\otimes A_l}N''].$$

Therefore, all $r$ are well defined. $\square$

Let $G_0(A)=\bigoplus_{n\geq 0}G_0(A_n)$. We construct the multiplication and comultiplication by $i$ and $r$ and define the unit and counit as follows:

$$\begin{array}{rcl}
\pi & : & G_0(A)\bigotimes_{\mathbb{Z}}G_0(A)\to G_0(A)\\
& & \text{by } \pi|_{G_0(A_k)\bigotimes G_0(A_l)}=i_{k,l}\\
\Delta & : & G_0(A)\to G_0(A)\bigotimes_{\mathbb{Z}}G_0(A)\\
& & \text{by } \Delta|_{G_0(A_n)}=\sum_{k+l=n}r_{k,l}\\
\mu & : & \mathbb{Z}\to G_0(A)\\
& & \text{by } \mu(a)=a[K]\in G_0(A_0),\ \text{for } a\in\mathbb{Z}\\
\epsilon & : & G_0(A)\to\mathbb{Z}\\
& & \text{by } \epsilon([M])=\begin{cases} a & \text{if } [M]=a[K],\ \text{where } a\in\mathbb{Z}\\ 0 & \text{otherwise.}\end{cases}
\end{array}$$

Later we will prove the associativity of $\pi$, the unitary property of $\mu$, the coassociativity of $\Delta$ and the counitary property of $\epsilon$, which imply that $(G_0(A),\pi,\mu)$ is an algebra and $(G_0(A),\Delta,\epsilon)$ is a coalgebra. We will also show the compatibility of the algebra and coalgebra structures to indicate that $(G_0(A),\pi,\mu,\Delta,\epsilon)$ is a graded connected bialgebra.

Now we define the inductions and restrictions on $K_0(A)$ analogously. As before,

$$\begin{array}{rcccl}
i'_{m,n} & : & K_0(A_m)\bigotimes_{\mathbb{Z}}K_0(A_n) & \to & K_0(A_{m+n})\\
& & [P]\otimes[Q] & \mapsto & [\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}}P\otimes Q],
\end{array}$$

where

$$\begin{array}{rcl}
\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}}P\otimes Q & = & A_{m+n}\bigotimes_{A_m\otimes A_n}(P\otimes Q)\\
& = & \dfrac{A_{m+n}\otimes P\otimes Q}{<a\otimes[(b\otimes c)(p\otimes q)]-[a\rho_{m,n}(b\otimes c)]\otimes p\otimes q>},
\end{array}$$

$a\in A_{m+n},\ b\in A_m,\ c\in A_n,\ p\in P$ and $q\in Q$. Let $k=t+s$. Denote

$$\begin{array}{rl}
& \widetilde{\mathrm{Ind}}_{A_t\otimes A_{m-t}\otimes A_s\widetilde{\otimes} A_{n-s}}^{A_k\otimes A_{m+n-k}}(P_1\otimes P_2)\otimes(Q_1\otimes Q_2)\\
= & (A_k\otimes A_{m+n-k})\widetilde{\bigotimes}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}((P_1\otimes P_2)\otimes(Q_1\otimes Q_2))
\end{array}$$

the twisted induction with the same meaning as above. And set

$$\begin{array}{rcccl}
r'_{k,l} & : & K_0(A_n) & \to & K_0(A_k)\bigotimes_{\mathbb{Z}}K_0(A_l) \text{ with } k+l=n\\
& & [R] & \mapsto & [\mathrm{Res}_{A_k\otimes A_l}^{A_n}R],
\end{array}$$

where $\mathrm{Res}_{A_k\otimes A_l}^{A_n}R=\mathrm{Hom}_{A_n}(A_n,R)$ as a left projective $A_k\otimes A_l$-module by the action defined to be $((b\otimes c)\cdot f)(a)=f(a\rho_{k,l}(b\otimes c))$, $a\in A_n,\ b\in A_k,\ c\in A_l$ and $f\in\mathrm{Hom}_{A_n}(A_n,R)$.

PROPOSITION 3.2. *$i'$ and $r'$ are well defined.*

PROOF. To show that $i'$ are well defined, we only need that $\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}}P\otimes Q=A_{m+n}\bigotimes_{A_m\otimes A_n}(P\otimes Q)$ is a projective $A_{m+n}$-module for all projective $A_m$-module $P$ and all projective $A_n$-module $Q$. This is straightforward by the properties of tensor product and short exact sequence and the property of projective modules that there is a module $P'$ such that $P\oplus P'$ is a free module for the projective module $P$.

Assume $R$ is a projective $A_n$-module. Since $\mathrm{Hom}_{A_n}(A_n,M)\cong M$ for all $A_n$-modules $M$, we can get that $\mathrm{Hom}_{A_n}(A_n,R)$ is a summand of some free $A_n$-module by the property of projective modules. Hence, $\mathrm{Hom}_{A_n}(A_n,R)$ is a $A_k\otimes A_l$-module for all $k$ and $l$ with $n=k+l$. Therefore, $r'$ are well defined □

Let $K_0(A) = \bigoplus_{n\geq 0} K_0(A_n)$. Using $i'$ and $r'$ we also define the multiplication, comultiplication, unit and counit on $K_0(A)$.

$$\begin{array}{rcl}
\pi' & : & K_0(A) \bigotimes_{\mathbb{Z}} K_0(A) \to K_0(A) \\
& & \text{by } \pi'|_{K_0(A_k) \bigotimes K_0(A_l)} = i'_{k,l} \\
\Delta' & : & K_0(A) \to K_0(A) \bigotimes_{\mathbb{Z}} K_0(A) \\
& & \text{by } \Delta'|_{K_0(A_n)} = \sum_{k+l=n} r'_{k,l} \\
\mu' & : & \mathbb{Z} \to K_0(A) \\
& & \text{by } \mu'(a) = a[K] \in K_0(A_0), \text{ for } a \in \mathbb{Z} \\
\epsilon' & : & K_0(A) \to \mathbb{Z} \\
& & \text{by } \epsilon'([M]) = \begin{cases} a & \text{if } [M] = a[K], \text{ where } a \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}
\end{array}$$

Similarly, we will realize that $(K_0(A), \pi', \mu')$ is an algebra and $(K_0(A), \Delta', \epsilon')$ is a coalgebra later. It will also be verified that the compatibility of these algebra and coalgebra structures hold, i.e., $(K_0(A), \pi', \mu', \Delta', \epsilon')$ is a graded connected bialgebra.

Theorem 3.1. *(i) $\pi$ is associative and $\big(G_0(A),\ \pi,\ \mu\big)$ is an algebra. So is $\big(K_0(A),\ \pi',\ \mu'\big)$.*

*(ii) $\Delta$ is coassociative and $\big(G_0(A),\ \Delta,\ \epsilon\big)$ is a coalgebra. So is $\big(K_0(A),\ \Delta',\ \epsilon'\big)$.*

*(iii) $\Delta$ and $\epsilon$ are algebra homomorphisms and $G_0(A)$ is a graded connected bialgebra. Hence $G_0(A)$ is a graded Hopf algebra. So is $K_0(A)$.*

Proof. (i) We only need to check the associativity of $\pi$, i.e., $i_{l+m,n} \cdot (i_{l,m} \otimes 1_n) = i_{l,m+n} \cdot (1_l \otimes i_{m,n})$. Form the associativity of $\rho$ and the definition of $i$, we can check it directly. Same for $\pi'$.

(ii) We only need to show the coassociativity of $\Delta$, i.e., $(r_{l,m} \otimes 1) \cdot r_{l+m,n} = (1 \otimes r_{m,n}) \cdot r_{l,m+n}$. Form the definition of $r$ and the Adjointness Theorem [**8**], we can check it directly. Similarly for $\Delta'$.

(iii) Using the definition of compatibility of algebra and coalgebra structures, we show that $G_0(A)$ is a graded bialgebra since condition (5) holds. From condition (1), we know that $G_0(A)$ is a graded connected bialgebra. Hence a graded Hopf algebra. Similarly for $K_0(A)$. $\square$

Next we define a pairing on $K_0(A) \times G_0(A)$. With this pairing we can consider the duality between $K_0(A)$ and $G_0(A)$. The pairing is defined as follows:

$$<,>: K_0(A) \times G_0(A) \to \mathbb{Z}$$

such that

$$<[P],[M]>= \begin{cases} \dim_K\big(\mathrm{Hom}_{A_n}(P,M)\big) & \text{if } [P] \in K_0(A_n) \text{ and } [M] \in G_0(A_n) \\ 0 & \text{otherwise.} \end{cases}$$

and with the same notation $<,>: (K_0(A) \otimes K_0(A)) \times (G_0(A) \otimes G_0(A)) \to \mathbb{Z}$ by

$$<[P]\otimes[Q],[M]\otimes[N]>= \begin{cases} \dim_K\big(\mathrm{Hom}_{A_k\otimes A_l}(P\otimes Q, M\otimes N)\big) & \text{if } [P]\otimes[Q] \in K_0(A_k)\otimes K_0(A_l) \\ & \text{ and } [M]\otimes[N] \in G_0(A_k)\otimes G_0(A_l) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.3. *$<,>$ is a well-defined bilinear pairing on $K_0(A) \times G_0(A)$ satisfying the following identities*

$$\begin{array}{rcl}
<[P]\otimes[Q],[M]\otimes[N]> & = & <[P],[M]><[Q],[N]> \\
<\pi'([P]\otimes[Q]),[M]> & = & <[P]\otimes[Q],\Delta[M]> \\
<\Delta'[P],[M]\otimes[N]> & = & <[P],\pi([M]\otimes[N])> \\
<\mu'(1),[M]> & = & \epsilon([M]) \\
<[P],\mu(1)> & = & \epsilon'([P]).
\end{array}$$

Proof. It is straightforward to check the linearity by the properties of short exact sequences and direst sums of modules .

The identity $<[P]\otimes[Q],[M]\otimes[N]>=<[P],[M]><[Q],[N]>$ is trivial.

To show $<\pi'([P]\otimes[Q]),[M]>=<[P]\otimes[Q],\Delta[M]>$, it is equivalent to prove that $<i'_{k,l}([P]\otimes[Q]),[M]>=<[P]\otimes[Q],r_{k,l}[M]>$, for all $[P] \in K_0(A_k),\ [Q] \in K_0(A_l)$ and $[M] \in G_0(A_{k+l})$, which can be reached by the Adjointness Theorem.

To show $<\Delta'[P],[M]\otimes[N]>=<[P],\pi([M]\otimes[N])>$, we only need to prove that $<r'_{k,l}[P],[M]\otimes[N]>=<[P],i_{k,l}([M]\otimes[N])>$, for all $[P] \in K_0(A_{k+l}),\ [M] \in K_0(A_k)$ and $[N] \in G_0(A_l)$. We can simplify this by proving that $< r'_{k,l}[P], [M]\otimes[N] >=< [P], i_{k,l}([M]\otimes[N]) >$ holds when $P$ is an indecomposable projective $A_{k+l}$-module. We know that each indecomposable projective module corresponding to a primitive idempotent. We establish this identity by the following lemma and condition (4).

LEMMA 3.2. [**15**] *Let $B$ be a finite-dimensional algebra over field $K$, $M$ a left $B$-module and $e$ a primitive idempotent. Then $Hom_B(Be, M)\cong eM$ as vector spaces.*

$< \mu'(1), [M] >= \epsilon([M])$ and $< [P], \mu(1) >= \epsilon([P])$ follow from the definitions of $\mu$ and $\mu'$. □

To get the duality between $G_0(A)$ and $K_0(A)$ these identities are not enough. We should verify that their bases are orthonormal to each other. Let $\{V_1, \cdots, V_s\}$ be a complete list of nonisomorphic simple $A_n$-modules. Then the set of their projective covers $\{P_1, \cdots, P_s\}$ is a complete list of nonisomorphic indecomposable projective $A_n$-modules. Then

PROPOSITION 3.4. $< [P_i], [V_j] >= \delta_{i,j}$ *for* $1\le i, j\le s$.

PROOF. This follows from the property of simple modules and the Schur's Lemma. □

THEOREM 3.3 (Main Result 1). *$(G_0(A), \pi, \mu, \Delta, \epsilon)$ and $(K_0(A), \pi', \mu', \Delta', \epsilon')$ are both graded connected bialgebras. Hence both are graded Hopf algebras. And they are dual to each other with respect to the pairing.*

PROOF. This follows directly from Theorem 3.1 and Propositions 3.3 and 3.4. □

## 4. Grothendieck groups of a tower of algebras (Not preserving unities)

In [**1**], N. Bergeron, C. Holhweg, M. Rosas, and M. Zabrocki consider a semi-tower of algebras with $\rho$ not preserving unities. If we only weaken the condition of $\rho$ and modify the definitions of inductions and restrictions can we get a similar result? In this section, we will do this job. The structure of this section is parallel to Section 3.

Let $A=\bigoplus_{n\ge 0} A_n$, we call it *a tower of algebras* over field $K=\mathbb{C}$ if the following conditions are satisfied:
(1) $A_n$ is a finite-dimensional algebra with unit, for each $n$. $A_0\cong K$.
(2) There is an external graded multiplication $\rho_{m,n}: A_m\otimes A_n\to A_{m+n}$ such that
(a) $\rho_{m,n}$ is an injective homomorphism of algebras, for all $m$ and $n$ (but $\rho_{m,n}(1_m\otimes 1_n)\ne 1_{m+n}$ for some or all $m$ and $n$ );
(b) $\rho$ is associative, that is, $\rho_{l+m,n}\cdot(\rho_{l,m}\otimes 1_n)=\rho_{l,m+n}\cdot(1_l\otimes\rho_{m,n}):=\rho_{l,m,n}$, for all $l$, $m$, $n$.
(3) $A_{n+m}$ is a two-sided projective $A_n\otimes A_m$-module by the action defined to be $a\cdot(b\otimes c)=a\rho_{m,n}(b\otimes c)$ and $(b\otimes c)\cdot a=\rho_{m,n}(b\otimes c)a$, for $a\in A_{m+n},\ b\in A_m$ and $c\in A_n$.
(4) For every primitive idempotent $g$ in $A_{m+n}$, $A_{m+n}g\cong\bigoplus(A_m\otimes A_n)(e\otimes f)$ as (left) $A_m\otimes A_n$-modules if and only if $gA_{m+n}\cong\bigoplus(e\otimes f)(A_m\otimes A_n)$ as (right) $A_m\otimes A_n$-modules for the same indexing of idempotents $(e\otimes f)$'s in $A_m\otimes A_n$.
(5) The following equalities hold

$$\begin{aligned}&[\mathrm{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}}\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}(M\otimes N)]\\ =\ &\textstyle\sum_{t+s=k}[\widetilde{\mathrm{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(\mathrm{Res}^{A_m}_{A_t\otimes A_{m-t}}M\otimes\mathrm{Res}^{A_n}_{A_s\otimes A_{n-s}}N)]\end{aligned}$$

for all $0<k<m+n$, $M$ an $A_m$-module and $N$ an $A_n$-module, and

$$\begin{aligned}&[\mathrm{Res}^{A_{m+n}}_{A_k\otimes A_{m+n-k}}\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}(P\otimes Q)]\\ =\ &\textstyle\sum_{t+s=k}[\widetilde{\mathrm{Ind}}^{A_k\otimes A_{m+n-k}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}(\mathrm{Res}^{A_m}_{A_t\otimes A_{m-t}}P\otimes\mathrm{Res}^{A_n}_{A_s\otimes A_{n-s}}Q)]\end{aligned}$$

for all $0<k<m+n$, $P$ a projective $A_m$-module and $Q$ a projective $A_n$-module. We will explain the notations later.

The definition of inductions on $G_0(A)$ is

$$\begin{aligned}i_{m,n}\ :\ G_0(A_m)\textstyle\bigotimes_{\mathbb{Z}}G_0(A_n)\ &\to\ G_0(A_{m+n})\\ [M]\otimes[N]\ &\mapsto\ [\mathrm{Ind}^{A_{m+n}}_{A_m\otimes A_n}M\otimes N],\end{aligned}$$

which is same as the one in Section 3. Let $k=t+s$, define the twisted induction

$$\begin{array}{rl} & \widetilde{\mathrm{Ind}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}^{A_k\otimes A_{m+n-k}}(M_1\otimes M_2)\otimes(N_1\otimes N_2) \\ = & (A_k\otimes A_{m+n-k})\widetilde{\bigotimes}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}((M_1\otimes M_2)\otimes(N_1\otimes N_2)),\end{array}$$

which is also same as the one in Section 3. Define the restrictions $r$ on $G_0(A)$ by

$$\begin{array}{rcccl} r_{k,l} & : & G_0(A_n) & \to & G_0(A_k)\bigotimes_{\mathbb{Z}} G_0(A_l) \text{ with } k+l=n \\ & & [N] & \mapsto & [\mathrm{Res}_{A_k\otimes A_l}^{A_n} N],\end{array}$$

where $\mathrm{Res}_{A_k\otimes A_l}^{A_n} N=\{u\in N \mid \rho_{k,l}(1_k\otimes 1_l)u=u\}\subseteq N$ is an $A_k\otimes A_l$-module by the action defined to be $(b\otimes c)\cdot u=\rho_{k,l}(b\otimes c)u$, for $u\in \mathrm{Res}_{A_k\otimes A_l}^{A_n} N$, $b\in A_k$ and $c\in A_l$. When $\rho$ preserving unities, we have $\mathrm{Res}_{A_k\otimes A_l}^{A_n} N=N\cong \mathrm{Hom}_{A_n}(A_n,N)$. This coincides with the restrictions $r$ in Section 3.

Proposition 4.1. *$i$ and $r$ are well defined.*

Proof. For $i$, it follows from Proposition 3.1 since they have the same definition.

For $r$, we know $\mathrm{Res}_{A_k\otimes A_l}^{A_n} N=\rho_{k,l}(1_k\otimes 1_l)N$ and $\rho_{k,l}(1_k\otimes 1_l)$ is an idempotent in $A_n$, hence $N=\rho_{k,l}(1_k\otimes 1_l)N\oplus(1-\rho_{k,l}(1_k\otimes 1_l))N$. From the properties of short exact sequence and homomorphisms of modules which can be written as a direct sum, one can get that all $r$ are well defined. □

As in Section 3, we define $\pi,\ \Delta,\ \mu$ and $\epsilon$ by the inductions $i$ and restrictions $r$ on $G_0(A)$. Later we will prove that $G_0(A)$ is a graded bialgebra with these operators.

Now we define inductions and restrictions on $K_0(A)$ as follows:

$$\begin{array}{rcccl} i'_{m,n} & : & K_0(A_m)\bigotimes_{\mathbb{Z}} K_0(A_n) & \to & K_0(A_{m+n}) \\ & & [P]\otimes[Q] & \mapsto & [\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} P\otimes Q],\end{array}$$

where $P=A_m e_m,\ Q=A_n e_n$ for some primitive idempotents $e_m\in A_m$ and $e_n\in A_n$, and

$$\begin{array}{rl} & \mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} P\otimes Q \\ = & \mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} A_m e_m\otimes A_n e_n \\ := & A_{m+n}\rho_{m,n}(e_m\otimes e_n),\end{array}$$

which is a projective $A_m\otimes A_n$-module. Here $i'$ is only defined on the basis of $K_0(A_m)\otimes K_0(A_n)$. To get induction we only need $i'$ to satisfy linearity. i.e.,

$$\begin{array}{rl} & i'((a[P']+b[P''])\otimes(c[Q']+d[Q''])) \\ = & aci'([P']\otimes[Q'])+adi'([P']\otimes[Q''])+bci'([P'']\otimes[Q'])+bdi'([P'']\otimes[Q'']),\end{array}$$

where $a,b,c,d\in\mathbb{Z}, P',P''\in K_0(A_m)$ and $Q',Q''\in K_0(A_n)$ are indecomposable. Hence $i'$ is well defined. And when $\rho$ preserving unities, this $i'$ coincides with the inductions $i'$ in Section 3.

Let $k=t+s$, define the twisted induction

$$\begin{array}{rl} & \widetilde{\mathrm{Ind}}_{A_t\otimes A_{m-t}\otimes A_s\otimes A_{n-s}}^{A_k\otimes A_{m+n-k}}(A_t e_1\otimes A_{m-t}e_2)\otimes(A_s f_1\otimes A_{n-s}f_2) \\ := & A_k\rho_{t,s}(e_1\otimes f_1)\otimes A_{m+n-k}\rho_{m-t,n-s}(e_2\otimes f_2),\end{array}$$

where $e_1,\ e_2,\ f_1$ and $f_2$ are primitive idempotents in $A_t,\ A_{m-t},\ A_s$ and $A_{n-s}$ respectively.

Set

$$\begin{array}{rcccl} r'_{k,l} & : & K_0(A_n) & \to & K_0(A_k)\bigotimes_{\mathbb{Z}} K_0(A_l) \text{ with } k+l=n \\ & & [R] & \mapsto & [\mathrm{Res}_{A_k\otimes A_l}^{A_n} R],\end{array}$$

where $\mathrm{Res}_{A_k\otimes A_l}^{A_n} R=\{x\in R \mid \rho_{k,l}(1_k\otimes 1_l)x=x\}$ as a left projective $A_k\otimes A_l$-module.

Proposition 4.2. *$r'$ is well defined.*

Proof. To show $r'$ well defined, there are three steps. Let $R$ be a projective $A_n$-module.

1. $\rho_{k,l}(1_k\otimes 1_l)$ is an idempotent and $\mathrm{Res}_{A_k\otimes A_l}^{A_n} R=\rho_{k,l}(1_k\otimes 1_l)R$.

2. $\mathrm{Res}_{A_k\otimes A_l}^{A_n} R$ is an $A_k\otimes A_l$-module.

3. $\mathrm{Res}_{A_k\otimes A_l}^{A_n} R$ is a projective $A_k\otimes A_l$-module. Here we verify that $\mathrm{Res}_{A_k\otimes A_l}^{A_n} R$ is a summand of $R$ by lemma 3.2, step 1 and the property of idempotents.

□

As before, using the definitions of inductions $i'$ and restrictions $r'$ we construct $\pi'$, $\Delta'$, $\mu'$ and $\epsilon'$ on $K_0(A)$. Later we will prove that $K_0(A)$ with these operators is a graded bialgebra.

THEOREM 4.1. *(i)* $\pi$ *is associative and* $\big(G_0(A),\ \pi,\ \mu\big)$ *is an algebra. So is* $\big(K_0(A),\ \pi',\ \mu'\big)$.
*(ii)* $\Delta$ *is coassociative and* $\big(G_0(A),\ \Delta,\ \epsilon\big)$ *is a coalgebra. So is* $\big(K_0(A),\ \Delta',\ \epsilon'\big)$.
*(iii)* $\Delta$ *and* $\epsilon$ *are algebra homomorphisms and* $G_0(A)$ *is a graded bialgebra. Hence* $G_0(A)$ *is a graded Hopf algebra. So is* $K_0(A)$.

PROOF. (i) For $G_0(A)$, it holds from Theorem 3.1(i).
For the associativity of $\pi'$ in $K_0(A)$, we need to show $i'_{l+m,n}\cdot(i'_{l,m}\otimes 1_n) = i'_{l,m+n}\cdot(1_l\otimes i'_{m,n})$. One can get it by the associativity of $\rho$ and the definition of $i'$.
(ii) We only need to show the coassociativity of $\Delta$, that is, $(r_{l,m}\otimes 1)\cdot r_{l+m,n} = (1\otimes r_{m,n})\cdot r_{l,m+n}$. This follows from the associativity of $\rho$ and the definition of $r$. Similarly for $\big(K_0(A),\ \Delta',\ \epsilon'\big)$.
(iii) From condition (5), one can prove that $G_0(A)$ is a graded bialgebra by the definition of compatibility of algebra and coalgebra structures. Do the similar work to $K_0(A)$. From condition (1), we know that $G_0(A)$ is a graded connected bialgebra. Hence a graded Hopf algebra. Similarly for $K_0(A)$. $\square$

Define a pairing $<,>: K_0(A)\times G_0(A)\to\mathbb{Z}$ by

$$<[P],[M]>=\begin{cases}\dim_K\big(\mathrm{Hom}_{A_n}(P,M)\big) & \text{if } [P]\in K_0(A_n) \text{ and } [M]\in G_0(A_n)\\ 0 & \text{otherwise.}\end{cases}$$

and with the same notation $<,>: (K_0(A)\otimes K_0(A))\times(G_0(A)\otimes G_0(A))\to\mathbb{Z}$ by

$$<[P]\otimes[Q],[M]\otimes[N]>=\begin{cases}\dim_K\big(\mathrm{Hom}_{A_k\otimes A_l}(P\otimes Q,M\otimes N)\big) & \text{if } [P]\otimes[Q]\in K_0(A_k)\otimes K_0(A_l)\\ & \text{ and } [M]\otimes[N]\in G_0(A_k)\otimes G_0(A_l)\\ 0 & \text{otherwise.}\end{cases}$$

PROPOSITION 4.3. $<,>$ *is a well-defined bilinear pairing on* $K_0(A)\times G_0(A)$ *satisfying the following identities*

$$\begin{aligned}<[P]\otimes[Q],[M]\otimes[N]> &= <[P],[M]><[Q],[N]>\\ <\pi'([P]\otimes[Q]),[M]> &= <[P]\otimes[Q],\Delta[M]>\\ <\Delta'[P],[M]\otimes[N]> &= <[P],\pi([M]\otimes[N])>\\ <\mu'(1),[M]> &= \epsilon([M])\\ <[P],\mu(1)> &= \epsilon'([P]).\end{aligned}$$

PROOF. The bilinearity and the first identity are same as Proposition 3.3.
To show $<\pi'([P]\otimes[Q]),[M]>=<[P]\otimes[Q],\Delta[M]>$, we only need to prove that $<i'_{k,l}([P]\otimes[Q]),[M]>=<[P]\otimes[Q],r_{k,l}[M]>$, for all $[P]\in K_0(A_k)$, $[Q]\in K_0(A_l)$ and $[M]\in G_0(A_{k+l})$. Without loss of generality, let $P=A_ke_k$ and $Q=A_le_l$ for some primitive idempotents $e_k\in A_k$ and $e_l\in A_l$. Using Lemma 3.2 one can get it straightforwardly.
To show $<\Delta'[P],[M]\otimes[N]>=<[P],\pi([M]\otimes[N])>$, we only need to prove that $<r'_{k,l}[P],[M]\otimes[N]>=<[P],i_{k,l}([M]\otimes[N])>$, for all $[P]\in K_0(A_{k+l})$, $[M]\in K_0(A_k)$ and $[N]\in G_0(A_l)$. One can get it from Lemma 3.2 and condition (4).
$<\mu'(1),[M]>=\epsilon([M])$ and $<[P],\mu(1)>=\epsilon([P])$ follow from the definitions of $\mu$ and $\mu'$. $\square$

THEOREM 4.2 (Main Result 2). $(G_0(A),\pi,\mu,\Delta,\epsilon)$ *and* $(K_0(A),\pi',\mu',\Delta',\epsilon')$ *are both graded connected bialgebras. Hence both are graded Hopf algebras. And they are dual to each other with respect to the pairing.*

PROOF. This follows directly from Theorem 4.1 and Propositions 4.3 and 3.4. $\square$

## 5. Some examples

In this section, we will verify that $\bigoplus_{n\geq 0}S_n$, $\bigoplus_{n\geq 0}H_n(0)$ and $\bigoplus_{n\geq 0}HCl_n(0)$ satisfy all the axioms listed in Section 3. They are towers of algebras and we already know that their Grothendieck groups are dual Hopf algebras, respectively. And we will discuss some graded algebras which don't satisfy some axiom are not towers of algebras. Consequently, their Grothendieck groups are not dual Hopf algebras.

Let $A=\bigoplus_{n\geq 0}A_n$ with $A_n=\mathbb{C}S_n$. Here

$$\rho_{m,n}:\mathbb{C}S_m\otimes\mathbb{C}S_n\to\mathbb{C}S_{m+n}$$

is defined to be $\rho_{m,n}(\sigma \otimes \tau) = \sigma(1)\sigma(2)\cdots\sigma(m)(\tau(1)+m)(\tau(2)+m)\cdots(\tau(n)+m)$ if we use the one line notation of permutations, where $\sigma \in S_m$ and $\tau \in S_n$. It is easy to check that $\rho$'s preserve unities and have the associativity. Since $\mathbb{C}S_n$ is a semi-simple algebra, we know that $\mathbb{C}S_{m+n}$ is a two-sided projective $\mathbb{C}S_m \otimes \mathbb{C}S_n$-module and satisfies condition (4). Condition (5) is just the Mackey's Formula [**16**]. Hence $A = \mathbb{C}S_n$ is a tower of algebra and the Grothendieck group $G_0(A) = K_0(A)$ is a self-dual graded Hopf algebra.

For $\bigoplus_{n\geq 0} H_n(0)$ of 0-Hecke algebras, the $\rho$'s are defined by $\rho_{m,n}(T_i \otimes 1) = T_i$ and $\rho_{m,n}(1 \otimes T_j) = T_{j+m}$, where $T_i$'s and $T_j$'s are the generators of $H_m(0)$ and $H_n(0)$, $1 \leq i \leq m-1$ and $1 \leq i \leq n-1$. For $\bigoplus_{n\geq 0} HCl_n(0)$ of 0-Hecke-Clifford algebras, the $\rho$'s are defined by $\rho_{m,n}(T_i \otimes 1) = T_i$, $\rho_{m,n}(C_k \otimes 1) = C_k$, $\rho_{m,n}(1 \otimes T_j) = T_{j+m}$ and $\rho_{m,n}(1 \otimes C_l) = C_{l+m}$, where $T_i$'s with $C_k$'s and $T_j$'s with $C_l$'s are the generators of $HCl_m(0)$ and $HCl_n(0)$ respectively, $1 \leq i \leq m-1$, $1 \leq k \leq m$, $1 \leq i \leq n-1$ and $1 \leq l \leq n$. We will also check that these two satisfy all the axioms listed in section 3. In the introduction we have mentioned that their Grothendieck groups are dual graded Hopf algebras.

Now we describe an example not satisfying condition (5). In [**1**], $(\mathbf{\Pi}, \wedge) = \bigoplus_{n\geq 0}(\mathbb{C}\Pi_n, \wedge)$ of the partition lattice algebras with

$$\rho_{m,n} : (\mathbb{C}\Pi_m, \wedge) \otimes (\mathbb{C}\Pi_n, \wedge) \to (\mathbb{C}\Pi_{m+n}, \wedge)$$

defined by $\rho_{m,n}(A \otimes B) = A|B$, where $A|B = \{A_1, A_2, \ldots, A_{l(A)}, B_1 + m, B_2 + m, \ldots, B_{l(B)} + m\}$. Here $\rho$'s do not preserve unities. Although $\left(\bigoplus_{n\geq 0}(\mathbb{C}\Pi_n, \wedge), \{\rho_{m,n}\}\right)$ satisfies conditions (1)-(4) in section 4, there is no similar Mackey's fomula (5), i.e., the operations of induction and restriction are not compatible as a bialgebra. Hence the Grothendieck groups $G_0(\mathbf{\Pi}, \wedge)$ and $K_0(\mathbf{\Pi}, \wedge)$ do not have the Hopf algebra structure although the operation of restriction on $G_0(\mathbf{\Pi}, \wedge)$ is dual to the operation of induction on $K_0(\mathbf{\Pi}, \wedge)$ and the induction on $G_0(\mathbf{\Pi}, \wedge)$ is dual as graded operations to restriction on $K_0(\mathbf{\Pi}, \wedge)$.

If one consider a direct sum of algebras that does not satisfy conditions (3) then the inductions and restrictions may not be well defined. Hence we can not construct the multiplication and comultiplication. Consequently, its Grothendieck groups are not Hopf algebras. If it does not satisfy condition (4), then its Grothendieck groups are graded Hopf algebras respectively but not necessarily dual to each other.

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