Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Coincidences among skew Schur functions

Victor Reiner, Kristin M. Shaw, and Stephanie van Willigenburg

Abstract. We define an equivalence relation on skew diagrams such that two skew diagrams are equivalent if and only if they give rise to equal skew Schur functions. Then we derive some necessary and sufficient conditions for equivalence.

Résumé. Nous étudions quand deux fonctions de skew Schur sont égales. Avec plus précision, nous derivons quelques règles nécessaires et quelques règles suffisantes pour l'égalité.

1. Introduction

Schur functions are ubiquitous in algebraic combinatorics. They have recently been connected to branching rules for classical Lie groups [**8, 11**], and eigenvalues and singular values of sums of Hermitian and of complex matrices [**1, 5, 8**] via the study of *inequalities* among products of skew Schur functions.

With this in mind, a natural problem is to describe all *equalities* among products of skew Schur functions, or equivalently, to describe all *binomial syzygies* among skew Schur functions. As we shall see in Section 2 this is equivalent to describing all equalities among individual skew Schur functions indexed by *connected* skew diagrams. This is a more tractable instance of a problem that currently seems intractable: describe *all* syzygies among skew Schur functions. Famous non-binomial syzygies include various formulations of the Littlewood-Richardson rule, which give some indication of the complexity that any such description would involve.

The study of equalities among skew Schur functions can also be regarded as part of the calculus of shapes. For an arbitrary subset D of $\mathbb{Z}^2$, there are polynomial representations $\mathcal{S}^D$ and $\mathcal{W}^D$ of $GL_N(\mathbb{C})$ known as a *Schur* or *Weyl modules* respectively. These $GL_N(\mathbb{C})$-representations are obtained by row-symmetrizing and column-antisymmetrizing tensors whose tensor positions are indexed by the cells of D. In general, these representations have $GL_N(\mathbb{C})$-character equal to a symmetric function $s_D(x_1, \ldots, x_N)$; when D is a skew diagram, this is a skew Schur function. Therefore, the question of when two skew Schur or Weyl modules are equivalent in characteristic zero is precisely the question of equalities among skew Schur functions.

Thus we wish to study the following equivalence relation.

Definition 1.1. Given two skew diagrams D_1 and D_2 we say they are *skew-equivalent* denoted $D_1 \sim D_2$ if and only if $s_{D_1} = s_{D_2}$.

For the sake of brevity, in this abstract we assume that the reader is familiar with the basic tenets of algebraic combinatorics such as skew diagrams and Schur functions. If this is not the case, then we refer them to the excellent texts [**9, 12, 13**]*, whose lead we follow by using english notation throughout. One further indispensable tool for us will be the more recent Hamel-Goulden determinant, expressing a skew Schur function s_D, for a skew diagram D, in terms of a determinant based on an outside decomposition of D, and the cutting strip associated to the decomposition; see* [**4, 7**] *for further details.*

2000 *Mathematics Subject Classification.* Primary 05E05, 20C30.

Key words and phrases. Symmetric function, skew Schur function, ribbon Schur function, Weyl module.

The first author was supported by NSF grant DMS-0245379. The second and third authors were supported in part by the National Sciences and Engineering Research Council of Canada.

2. Reduction to connected diagrams

We begin by explaining two easy reductions:

A. Understanding all binomial syzygies among the skew Schur functions is equivalent to understanding the equivalence relation $\sim$ on all skew diagrams, and

B. the latter is equivalent to understanding $\sim$ among *connected* skew diagrams.

These reductions follow from some observations about the matrix

$$JT(\lambda/\mu) := (h_{\lambda_i-\mu_j-i+j})_{i,j=1}^{\ell(\lambda)},$$

which appears in the Jacobi-Trudi formula for a skew diagram λ/μ. We collect these observations in the following proposition, whose straightforward proof is omitted in this abstract.

Proposition 2.1. *Let λ/μ be a skew diagram with $\ell := \ell(\lambda)$.*

(i) The largest subscript k occurring on any nonzero entry h_k in the Jacobi-Trudi matrix $JT(\lambda/\mu)$ is

$$L := \lambda_1 + \ell - 1$$

and this subscript occurs exactly once, on the $(1, \ell)$-entry h_L.

(ii) The subscripts on the diagonal entries in $JT(\lambda/\mu)$ are exactly the row lengths

$$(r_1, \ldots, r_\ell) := (\lambda_1 - \mu_1, \ldots, \lambda_\ell - \mu_\ell)$$

and the monomial $h_{r_1} \cdots h_{r_\ell}$ occurs in the determinant s_D

(a) with coefficient $+1$, and

(b) as the monomial whose subscripts rearranged into weakly decreasing order give the smallest partition of $|\lambda/\mu|$ in dominance order among all nonzero monomials.

(iii) The subscripts on the nonzero subdiagonal entries in $JT(\lambda/\mu)$ are exactly one less than the adjacent row overlap lengths:

$$(\lambda_2 - \mu_1, \lambda_3 - \mu_2, \ldots, \lambda_\ell - \mu_{\ell-1}).$$

Corollary 2.1. *For a disconnected skew diagram $D = D_1 \oplus D_2$, one has the factorization $s_D = s_{D_1} s_{D_2}$. For a connected skew diagram D, the polynomial s_D is irreducible in $\mathbb{Z}[h_1, h_2, \ldots]$.*

Proof. (sketch) The first assertion of the proposition is well-known, and follows, for example, immediately from the definition of skew Schur functions using tableaux.

For the second assertion, let $D = \lambda/\mu$ with $\ell := \ell(\lambda)$ and $L := \lambda_1 + \ell - 1$. Then the Jacobi-Trudi formula and Proposition 2.1(i) imply that the expansion of s_D as a polynomial in the h_r is of the form

$$s \cdot h_L + r \tag{2.1}$$

where s, r are polynomials containing no occurrences of h_L. Proposition 2.1(ii) implies that r is not the zero polynomial, and hence if one can show that s is also nonzero, Equation (2.1) would exhibit s_D as a linear polynomial in h_L with nonzero constant term, and hence clearly irreducible in $\mathbb{Z}[h_1, h_2, \ldots]$. The latter is argued using the fact that D is connected, so that its adjacent row overlaps are all positive, along with Proposition 2.1(iii). $\square$

We can now infer reductions A and B from the beginning of the section. Given a binomial syzygy

$$c\ s_{D_1} s_{D_2} \cdots s_{D_m} - c'\ s_{D'_1} s_{D'_2} \cdots s_{D'_m} = 0$$

among the skew Schur functions, one can rewrite this as $c\, s_D = c' s_{D'}$, where

$$D := D_1 \oplus D_2 \oplus \cdots \oplus D_m$$
$$D' := D'_1 \oplus D'_2 \oplus \cdots \oplus D'_m.$$

Then Proposition 2.1 (ii) implies the unitriangular expansion $s_D = h_\rho + \sum_{\mu : \mu >_{dom} \rho} c_\mu h_\mu$ in which ρ is the weakly decreasing rearrangement of the row lengths in D. This forces $c = c'$ and hence $s_D = s_{D'}$, achieving reduction A.

For reduction B, use the fact that $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ is a unique factorization domain, along with Corollary 2.1.

3. Sufficient conditions

The most basic skew-equivalence is the following well-known fact.

Proposition 3.1. [**13**, Exercise 7.56(a)] *If D is a skew diagram then $D \sim D^*$, where D^* is the antipodal rotation of D.*

It transpires that there are several other constructions and operations on skew diagrams that give rise to more skew-equivalences.

3.1. Composition with ribbons. Recall the subset of skew diagrams that contain no 2×2 subdiagram, often known as *ribbons*. In this subsection we generalize in two different ways the composition operation $\alpha \circ \beta$ on ribbons α, β that was introduced in [**2**].

Given two skew diagrams D_1, D_2, aside from their disjoint sum $D_1 \oplus D_2$, there are two closely related important operations called their *concatentation* $D_1 \cdot D_2$ and their *near-concatenation* $D_1 \odot D_2$. The concatentation $D_1 \cdot D_2$ (resp. near concatentation $D_1 \odot D_2$) is obtained from the disjoint sum $D_1 \oplus D_2$ by moving all cells of D_2 one column west (resp. one row south), so that the same column (resp. row) is occupied by the rightmost column (resp. topmost row) of D_1 and the leftmost column (resp. bottommost row) of D_2. For example, if $D_1 = (2,2), D_2 = (3,2)/(1)$ then

$$D_1 \oplus D_2 = \begin{array}{ccccc} & & & 2 & 2 \\ & & 2 & 2 & \\ 1 & 1 & & & \\ 1 & 1 & & & \end{array} \qquad D_1 \cdot D_2 = \begin{array}{cccc} & & 2 & 2 \\ & 2 & 2 & \\ 1 & 1 & & \\ 1 & 1 & & \end{array} \qquad D_1 \odot D_2 = \begin{array}{ccccc} & & & 2 & 2 \\ 1 & 1 & 2 & 2 & \\ 1 & 1 & & & \end{array}.$$

Observe we have used the numbers 1 and 2 to distinguish between those cells in D_1 and those cells in D_2. The reason for the names "concatentation" and "near-concatentation" becomes clearer when we restrict to ribbons. Observe that in this case there exists a natural correspondence that identifies a composition $\alpha = \alpha_1 \dots \alpha_k$ with the ribbon whose row lengths are $\alpha_1, \dots, \alpha_k$ read from the bottom. Hence, if we identify ribbons with compositions via this natural correspondence, to get $\alpha = (\alpha_1, \dots, \alpha_\ell)$ and $\beta = (\beta_1, \dots, \beta_m)$, we have

$$\begin{aligned} \alpha \cdot \beta &= (\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m) \\ \alpha \odot \beta &= (\alpha_1, \dots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \dots, \beta_m), \end{aligned}$$

which are the definitions for concatenation and near concatenation given in [**6**].

Note that the operations $\cdot$ and $\odot$ are each associative, and associate with each other:

$$\begin{aligned} (D_1 \cdot D_2) \cdot D_3 &= D_1 \cdot (D_2 \cdot D_3) \\ (D_1 \odot D_2) \odot D_3 &= D_1 \odot (D_2 \odot D_3) \\ (D_1 \odot D_2) \cdot D_3 &= D_1 \odot (D_2 \cdot D_3) \\ (D_1 \cdot D_2) \odot D_3 &= D_1 \cdot (D_2 \odot D_3) \end{aligned}. \tag{3.1}$$

Consequently a string of operations $D_1 \star_1 D_2 \star_2 \cdots \star_{k-1} D_k$ in which each $\star_i$ is either $\cdot$ or $\odot$ is well-defined without any parenethesization. Also note that ribbons are exactly the skew diagrams that can be written uniquely as a string of the form

$$\alpha = \square \star_1 \square \star_2 \cdots \star_{k-1} \square \tag{3.2}$$

where $\square$ is the Ferrers diagram with exactly one cell.

Given a ribbon α and a skew diagram D, define $\alpha \circ D$ to be the result of replacing each cell $\square$ in the expression (3.2) for α with D:

$$\alpha \circ D := D \star_1 D \star_2 \cdots \star_{k-1} D.$$

For example, if

$$\alpha = \begin{array}{cccc} & & & \times \\ & \times & \times & \times \\ \times & \times & & \end{array} \quad \text{and} \quad D = \begin{array}{cc} \times & \times \\ \times & \times \end{array}$$

then

$$\alpha = \square \odot \square \cdot \square \odot \square \odot \square \cdot \square$$
$$\alpha \circ D = D \odot D \cdot D \odot D \odot D \cdot D$$

$$= \begin{array}{cccccccccc}
 & & & & & & & & 6 & 6 \\
 & & & & & & & & 6 & 6 \\
 & & & & & & & 5 & 5 & \\
 & & & & & 4 & 4 & 5 & 5 & \\
 & & & 3 & 3 & 4 & 4 & & & \\
 & & & 3 & 3 & & & & & \\
 & & 2 & 2 & & & & & & \\
1 & 1 & 2 & 2 & & & & & & \\
1 & 1 & & & & & & & &
\end{array}$$

where we have used numbers to distinguish between copies of D.

It is easily seen that when $D = \beta$ is a ribbon, then $\alpha \circ \beta$ is also a ribbon, and agrees with the definition in [**2**].

Similarly, given a skew diagram D and a ribbon β, we can also define $D \circ \beta$ as follows. Create a copy $\beta^{(i)}$ of the ribbon β for each of the cells of D, numbered $i = 1, 2, \ldots, n$ arbitrarily. Then assemble the ribbons $\beta^{(i)}$ into a disjoint decomposition of $D \circ \beta$ by translating them in the plane in such a way that $\beta^{(i)} \sqcup \beta^{(j)}$ forms a copy of

$$\begin{cases} \beta^{(i)} \odot \beta^{(j)} & \text{if } i \text{ is just left of } j \text{ in some row of } D, \\ \beta^{(i)} \cdot \beta^{(j)} & \text{if } i \text{ is just below } j \text{ in some column of } D. \end{cases}$$

For example, if

$$D = \begin{array}{ccc} & 1 & 2 \\ 3 & 4 & 5 \end{array}, \qquad \beta = \begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \end{array}$$

then $D \circ \beta$ is the skew diagram

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & 2 & 2 & 2 \\
 & & & & & & & & 1 & 1 & 1 & 2 & 2 & & \\
 & & & & & & & 1 & 1 & 5 & 5 & 5 & & & \\
 & & & & & 4 & 4 & 4 & 5 & 5 & & & & & \\
 & 3 & 3 & 3 & 4 & 4 & & & & & & & & & \\
3 & 3 & & & & & & & & & & & & &
\end{array}$$

where we have used numbers to distinguish between copies of β.

Again it is clear that when $D = \alpha$ is a ribbon, then $\alpha \circ \beta$ is another ribbon agreeing with that in [**2**]. The following distributivity properties should also be clear.

Proposition 3.2. *For a skew diagram D and ribbons α and β the operation $\circ$ distributes over $\cdot$ and $\odot$, that is*

$$(\alpha \cdot \beta) \circ D = (\alpha \circ D) \cdot (\beta \circ D)$$
$$(\alpha \odot \beta) \circ D = (\alpha \circ D) \odot (\beta \circ D)$$

and

$$(D_1 \cdot D_2) \circ \beta = (D_1 \circ \beta) \cdot (D_2 \circ \beta)$$
$$(D_1 \odot D_2) \circ \beta = (D_1 \circ \beta) \odot (D_2 \circ \beta).$$

Remark 3.1. Observe that $D_1 \circ D_2$ has not been defined for both D_1 and D_2 being non-ribbons, and we invite the reader to investigate this situation in order to appreciate the complexities that can arise.

In the meantime, we show that the notation for the operations $\alpha \circ D$ and $D \circ \beta$ is consistent with the notation for skew Schur functions. These operations also lead to nontrivial skew-equivalences, generalizing the constructions of [**2**].

We begin by reviewing the presentation of the ring Λ of symmetric functions by the generating set of ribbon Schur functions s_α, that is, those skew Schur functions indexed by ribbons. Let $\mathcal{Q}[z_\alpha]$ denote a polynomial algebra in infinitely many variables z_α indexed by all compositions α.

PROPOSITION 3.3. [**2**, Proposition 2.2]. *The algebra homomorphism*

$$\begin{array}{ccc} \mathcal{Q}[z_\alpha] & \to & \Lambda \\ z_\alpha & \mapsto & s_\alpha \end{array}$$

is a surjection, whose kernel is the ideal generated by the relations

$$z_\alpha z_\beta - (z_{\alpha\cdot\beta} + z_{\alpha\odot\beta}). \tag{3.3}$$

In fact, this same syzygy is well-known to be satisfied [**9**, Chapter 1.5, Example 21 part (a)] by *all* skew diagrams D_1, D_2:

$$s_{D_1} s_{D_2} = s_{D_1\cdot D_2} + s_{D_1\odot D_2}. \tag{3.4}$$

As a consequence, one deduces the following.

COROLLARY 3.2. *For a fixed skew diagram D the map*

$$\begin{array}{ccc} \mathcal{Q}[z_\alpha] & \stackrel{(-)\circ s_D}{\longrightarrow} & \Lambda \\ z_\alpha & \longmapsto & s_{\alpha\circ D} \end{array}$$

descends to a well-defined map $\Lambda \longrightarrow \Lambda$. In other words, for any symmetric function f, one can arbitrarily write f as a polynomial in ribbon Schur functions $f = p(s_\alpha)$ and then set $f \circ s_D := p(s_{\alpha\circ D})$.

We are abusing notation here by using $\circ$ both for the map $(-) \circ s_D$ on symmetric functions, as well as the two diagrammatic operations $\alpha \circ D$ and $D \circ \beta$. The previous corollary says that it is well-defined to set

$$s_\alpha \circ s_D = s_{\alpha\circ D} \tag{3.5}$$

so that we are at least consistent with one of the diagrammatic operations. The next result says that we are also consistent with the other.

PROPOSITION 3.4. *For any skew diagram D and ribbon β*

$$s_{D\circ\beta} = s_D \circ s_\beta.$$

PROOF. (sketch) One uses the Hamel-Goulden determinant for s_D, which starts with an outside decomposition of D into ribbons $(\theta_1, \ldots, \theta_m)$. The induced outside decomposition $(\theta_1 \circ \beta, \ldots, \theta_m \circ \beta)$ for $D \circ \beta$ leads to a Hamel-Goulden determinant for $s_{D\circ\beta}$. The proposition then follows because various operations commute with each other. □

THEOREM 3.3. *Assume one has ribbons α, α' and skew diagrams D, D' satisfying $\alpha \sim \alpha'$ and $D \sim D'$. Then*

(i) $\alpha \circ D \sim \alpha' \circ D$,
(ii) $D \circ \alpha \sim D' \circ \alpha$,
(iii) $D \circ \alpha \sim D \circ \alpha'$, *and*
(iv) $\alpha \circ D \sim \alpha \circ D^*$.

PROOF. Assertions (i) and (ii) both follow from the fact that if E is any skew diagram, then $D \sim D'$ means $s_D = s_{D'}$, and hence

$$s_D \circ s_E = s_{D'} \circ s_E. \tag{3.6}$$

The third follows by Proposition 3.4 if one can show it when $D = \alpha$ is a ribbon. This special case $\alpha\circ\beta_1 \sim \alpha\circ\beta_2$ was shown in [**2**]. Assertion (iv) follows from assertion (i) and Proposition 3.1:

$$\alpha \circ D \sim (\alpha \circ D)^* = \alpha^* \circ D^* \sim \alpha \circ D^*.$$

□

REMARK 3.4. The last skew-equivalence begs the question of whether $D_1 \sim D_2$ for skew diagrams D_1, D_2 implies $\alpha \circ D_1 \sim \alpha \circ D_2$ for any ribbon α. This turns out to be false. For example, one can check that

$$D_1 = \begin{array}{cccc} & \times & \times & \times \\ & \times & \times & \\ \times & \times & & \\ \times & & & \end{array} \quad \sim \quad \begin{array}{cccc} & & \times & \times \\ \times & \times & \times & \\ \times & \times & & \\ \times & & & \end{array} = D_2,$$

e.g. by Corollary 3.20. However, if one takes $\alpha = (2)$, that is, the ribbon having one row with two cells, then we find

$$\alpha \circ D_1 = \begin{array}{cccccccc} & & & & & \times & \times & \times \\ & & & & & \times & \times & \\ & & & & \times & \times & & \\ & \times & \times & \times & \times & & & \\ & \times & \times & & & & & \\ \times & \times & & & & & & \\ \times & & & & & & & \end{array}$$

$$\alpha \circ D_2 = \begin{array}{cccccccc} & & & & & & \times & \times \\ & & & & \times & \times & \times & \\ & & & & \times & \times & & \\ & & \times & \times & \times & & & \\ \times & \times & \times & & & & & \\ \times & \times & & & & & & \\ \times & & & & & & & \end{array}$$

and $\alpha \circ D_1 \not\sim \alpha \circ D_2$, e.g. by Theorem 4.4.

3.2. Amalgamation and amalgamated composition of ribbons. Now in a third way we generalize the operation $\alpha \circ \beta$ to an operation $\alpha \circ_\omega \beta$, which we will call the *amalgamated composition* of α and β with respect to ω.

Definition 3.5.
Given a skew diagram D and a nonempty ribbon ω, say that ω *lies in the top (resp. bottom) of* D if the restriction of D to its $|\omega|$ northeasternmost (resp. southwesternmost) diagonals is (a translated copy of) the ribbon ω.

Given two skew diagrams D_1, D_2 and a nonempty ribbon ω lying in the top of D_1 and the bottom of D_2, the *amalgamation of* D_1 *and* D_2 *along* ω, denoted $D_1 \amalg_\omega D_2$, is the new ribbon obtained from the disjoint union $D_1 \oplus D_2$ by identifying the copy of ω in the northeast of D_1 with the copy of ω in the southwest of D_2.

Say that ω *protrudes from the top (resp. bottom) of* D if there is another ribbon ω^+ having $|\omega^+| = |\omega|+1$ such that both ω, ω^+ lie at the top (resp. bottom) of D. Equivlalently, ω protrudes from the top (resp. bottom) of D if it lies at the top (resp. bottom) of D and the restriction of D to its $|\omega|+1$ northeasternmost (resp. southwesternmost) diagonals is also a ribbon, namely ω^+.

Example 3.6.
Consider the skew diagram

$$D = \begin{array}{cccc} & \times & \times & \times \\ \times & \times & \times & \end{array}.$$

Then D has $\omega_1 = \times$ protruding from the top and bottom. It has $\omega_2 = \times\times$ lying in its top and bottom, but protruding from neither top nor bottom. Furthermore,

$$D \amalg_{\omega_1} D = \begin{array}{ccccccc} & & & & \times & \times & \times \\ & \times & \times & o & \times & \times & \\ \times & \times & \times & & & & \end{array}, \qquad D \amalg_{\omega_2} D = \begin{array}{cccccc} & & & \times & \times & \times \\ & \times & o & o & \times & \\ \times & \times & \times & & & \end{array}$$

in which the copies of ω_1 and ω_2 that have been amalgamated are indicated with the letter o.

Definition 3.7.
When ω lies in the top of D_1 and bottom of D_2, one can form the *outer (resp. inner) projection of* D_1 *onto* D_2 *with respect to* ω. This is a new diagram in the plane, not necessarily skew, obtained from the disjoint union $D_1 \oplus D_2$ by translating D_2 until it is underneath and to the right (resp. above and to the left) of D_1, in such a way that the two copies of ω in D_1, D_2 are adjacent and occupy the same set of diagonals.

One can see that if ω not only lies in the top of D_1 and bottom of D_2, but actually *protrudes* from the top of D_1 and from the bottom of D_2, then at most one of these two projections can be a skew diagram (and possibly neither one is). When one of them is a skew diagram, call it $D_1 \cdot_\omega D_2$, and say that $D_1 \cdot_\omega D_2$ *is defined* in this case.

Example 3.8.
Let D, ω_1, ω_2 be as in the previous example. Then the outer and inner projections of D onto D with respect to ω_2 are

$$\begin{array}{ccccccc} & \times & o & o & \times & \times & \times \\ \times & \times & \times & o & o & \times & \end{array} \qquad \begin{array}{ccccc} & & \times & \times & \times \\ & o & o & \times & \\ & \times & o & o & \\ \times & \times & \times & & \end{array},$$

which are both skew diagrams. On the other hand, the outer and inner projections of D onto D with respect to ω_1 are

$$\begin{array}{cccccccc} & \times & \times & o & & \times & \times & \times \\ \times & \times & \times & & o & \times & \times & \end{array} \qquad \begin{array}{cccccc} & & & \times & \times & \times \\ & & o & \times & \times & \\ & \times & \times & o & & \\ \times & \times & \times & & & \end{array} = D \cdot_{\omega_1} D$$

in which only the latter is a skew diagram.

Definition 3.9.
Given a skew diagram D, and ω a ribbon lying in both the top and bottom of D, one can define

$$D^{\amalg_\omega n} = \underbrace{D \amalg_\omega D \amalg_\omega \cdots \amalg_\omega D}_{n \text{ factors}} := ((D \amalg_\omega D) \amalg_\omega D) \amalg_\omega \cdots \amalg_\omega D.$$

If one assumes that $D \cdot_\omega D$ is also defined so that, in particular, ω protrudes from the top and bottom of D, then one can check that this will imply that for any positive integers m, n, we have $(D^{\amalg_\omega m}) \cdot_\omega (D^{\amalg_\omega n})$ is also defined. Under this assumption, for any ribbon $\alpha = (\alpha_1, \ldots, \alpha_k)$, define the *amalgamated composition* of α and D with respect to ω to be the diagram

$$\alpha \circ_\omega D := (D^{\amalg_\omega \alpha_1}) \cdot_\omega \ldots \cdot_\omega (D^{\amalg_\omega \alpha_k}). \tag{3.7}$$

The following theorems are obtained by consideration of an appropriate Hamel-Goulden determinant.

Theorem 3.10. *Let D be a connected skew diagram, and ω a ribbon which protrudes from the top and bottom of D, with $D \cdot_\omega D$ defined. Assume further that the two copies of ω in the top and bottom of D are separated by at least one diagonal,that is, there is a nonempty diagonal in D intersecting neither copy of ω.*

Then for any ribbon α one has

$$s_{\alpha \circ_\omega D} = s_\alpha \circ_\omega s_D.$$

Theorem 3.11. *Let α, α' be ribbons with $\alpha \sim \alpha'$, and assume that D, ω satisfy the hypotheses of Theorem 3.10. Then one has the following skew-equivalences:*

$$\alpha' \circ_\omega D \;\sim\; \alpha \circ_\omega D \;\sim\; \alpha \circ_{\omega^*} D^*.$$

Theorem 3.12. *Let $\{\beta_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k$ be ribbons, and for each i either $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$. If the skew diagrams D, ω satisfy the hypotheses of Theorem 3.10, then*

$$\begin{aligned} &\gamma_1 \circ_\omega \gamma_2 \circ_\omega \ldots \circ_\omega \gamma_k \circ_\omega D \\ &\quad \sim \beta_1 \circ_\omega \beta_2 \circ_\omega \ldots \circ_\omega \beta_k \circ_\omega D \\ &\qquad \sim \beta_1 \circ_{\omega^*} \beta_2 \circ_{\omega^*} \ldots \circ_{\omega^*} \beta_k \circ_{\omega^*} D^* \end{aligned}$$

where all the operations $\circ_\omega$ or $\circ_{\omega^*}$ are performed from right to left.

Remark 3.13. Theorem 3.11 is analogous to [**2**, Theorem 4.4 parts 1 and 2], whereas Theorem 3.12 is analogous to the reverse direction of [**2**, Theorem 4.1].

3.3. Conjugation and ribbon staircases. The goal here is to construct skew diagrams D that are skew-equivalent to their conjugates D^t. We first define two decompositions of a connected skew diagram D into ribbons; when one of these decompositions takes on a very special form, we will show that implies $D \sim D^t$.

Definition 3.14.
Given a connected skew diagram D define the *southeast decomposition* to be the following unique decomposition into ribbons. The first ribbon θ is the unique ribbon that starts at the cell on the lower left, traverses the *southeast* border of D, and ends at the cell on the upper right. Now consider D with θ removed, which

may decompose into several connected component skew shapes, and iterate the above procedure on each of these shapes. The *northwest decomposition* is similarly defined, starting with a ribbon θ that traverses the *northwest* border of D.

Note that both of these are outside decompositions of D, and hence give rise to Hamel-Goulden determinants for s_D. In both cases, the associated cutting strip for the decomposition coincides with its first and largest ribbon θ. We will be interested in the case where all of the ribbons in the southeast or northwest decomposition of D arise in a very special way from the amalgamation construction of Section 3.2.

Definition 3.15.
Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_\ell)$ be ribbons. For an integer $m \geq 1$, say that the *m-intersection $\alpha \cap_m \beta$ exists* if there is a ribbon $\omega = (\omega_1, \dots, \omega_m)$ with m rows lying in the top of α and the bottom of β for which $\omega_1 = \beta_1$ and $\omega_m = \alpha_k$; when $m = 1$, we set $\omega_1 := \min\{\alpha_k, \beta_1\}$. In this case, define the *m-intersection* $\alpha \cap_m \beta$ and the *m-union* $\alpha \cup_m \beta$ to be

$$\begin{aligned} \alpha \cap_m \beta &:= \omega \\ \alpha \cup_m \beta &:= \alpha \amalg_\omega \beta^{\cdot} \end{aligned}$$

If $\alpha \cup_m \beta = \alpha$ or β (resp. or $\alpha \cap_m \beta = \alpha$ or β) then we say the m-union (resp. m-intersection) is *trivial.* If α is a ribbon such that $\alpha \cap_m \alpha$ exists and is non-trivial then

$$\varepsilon_m^k(\alpha) := \underbrace{\alpha \cup_m \alpha \cup_m \ldots \cup_m \alpha}_{k \text{ factors}}$$

is the *ribbon staircase* of height k and depth m generated by α.

Example 3.16.
Let α be the ribbon $(2,3)$. Then

$$\varepsilon_1^3(\alpha) = \varepsilon_1^3 \left(\begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \end{array} \right) = \begin{array}{cccccccc} & & & & & \times & \times & \times \\ & & & \times & \times & \times & & \\ & \times & \times & \times & & & & \\ \times & \times & & & & & & \end{array} .$$

Definition 3.17.
Say that a skew diagram D has a *southeast ribbon staircase decomposition* if there exists an $m < \ell(\alpha)$ and a ribbon α such that all ribbons in the southeast decomposition of D are of the forms $\alpha \cap_m \alpha$ or $\varepsilon_m^p(\alpha)$ for various integers $p \geq 1$.

In this situation, let k be the maximum value of p occurring among the $\varepsilon_m^p(\alpha)$ above, so that the largest ribbon θ equals $\varepsilon_m^k(\alpha)$. We will think of θ as containing k copies of α, numbered $1, 2, \dots, k$ from southwest to northeast. We now wish to define the *nesting* $\mathcal{N}$ associated to this decomposition. The nesting $\mathcal{N}$ is a word of length $k-1$ using as letters the four symbols, *dot* ".", *left parenthesis* "(", *right parenthesis* ")" and *vertical slash* "|". Considering the ribbons in the southeast decomposition of D,

- a ribbon of the form $\varepsilon_m^p(\alpha)$ creates a pair of left and right parentheses in positions i and j if the ribbon occupies the same diagonals as the $i+1, i+2, \dots, j-1, j$ copies of α in θ, while
- a ribbon of the form $\alpha \cap_m \alpha$ creates a vertice slash in position i if it occupies the same diagonals as the intersection of the $i, i+1$ copies of α in θ, and
- all other letters in $\mathcal{N}$ are dots.

With this notation, say that $D = (\varepsilon_m^k(\alpha), \mathcal{N})_{se}$. Analogously define the notation $D = (\varepsilon_m^k(\alpha), \mathcal{N})_{nw}$ using the northwest decomposition.

Lastly, given a nesting $\mathcal{N}$, denote the *reverse nesting*, which is the reverse of the word $\mathcal{N}$, by $\mathcal{N}^*$.

Example 3.18. Consider the following skew diagram D, with its southeast decomposition into ribbons $\theta_1, \theta_2, \theta_3, \theta_4$ distinguished by the numbers $1, 2, 3, 4$ respectively:

$$D = \begin{array}{ccccccc} & & & 4 & 3 & 3 & 1 & 1 \\ & & & 3 & 3 & 1 & 1 & \\ & & & 3 & 1 & 1 & & \\ & & 2 & 1 & 1 & & & \\ & & 1 & 1 & & & & \\ & 1 & 1 & & & & & \\ & 1 & & & & & & \end{array}.$$

This happens to be a southeast ribbon staircase decomposition, in which

$$\alpha = \begin{array}{cc} \times & \times \\ \times & \end{array}, \quad m = 1, \quad k = 6, \quad \mathcal{N} = .\,|\,(\,|\,),$$

that is, $D = (\varepsilon_1^6(\alpha), \mathcal{N})_{se}$. Here $\mathcal{N}^* = (\,|\,)\,|\,.$, and additionally note that the skew diagram $D' = (\varepsilon_1^6(\alpha), \mathcal{N}^*)$ is the same as the conjugate skew diagram D^t.

The following is a consequence of the Hamel-Goulden determinant associated to the southeast (or northwest) decomposition of D, when its decomposition is a ribbon staircase decomposition.

Theorem 3.19. *Let α be a ribbon, and let*

$$D_1 = (\varepsilon_m^k(\alpha), \mathcal{N})_x$$
$$D_2 = (\varepsilon_m^k(\alpha), \mathcal{N}^*)_x$$

where $m < \ell(\alpha)$ and $x = se$ or nw. Then $D_1 \sim D_2$.

This leads to the following interesting corollary.

Corollary 3.20. *Let $D = (\varepsilon_m^k(\alpha), \mathcal{N})_x$ where α is a self-conjugate ribbon, $m < l(\alpha)$ and $x = se$ or nw. Then $D \sim D^t$. Furthermore, for any Ferrers diagram μ contained in the staircase partition $\delta_n := (n-1, n-2, \ldots, 1) \vdash \binom{n}{2}$, one has*

$$\delta_n/\mu \sim (\delta_n/\mu)^t.$$

We conjecture the following converse, which has been verified for all skew diagrams D with $|D| \leq 18$.

Conjecture 3.21. *If a skew diagram D satisfies $D \sim D^t$, then $D = (\varepsilon_m^k(\alpha), \mathcal{N})_x$ for some self-conjugate ribbon α, some $m < \ell(\alpha)$ and $x = se$ or nw.*

3.4. Adding a full column/row, and complementation within a rectangle. Let D be thought of as any finite subset of the plane $\mathbb{Z}^2$. We wish to consider two operations on D, which turn out to be closely related.

- *Adding a full column (resp. row)*: Add to the shape a new column (resp. row) which has a cell in every previously nonempty row (resp. column), and possibly in some new rows (resp. columns).
- *Complementation within a rectangle*: If R is a rectangular Ferrers diagram containing D, consider the complementary shape $R \backslash D$.

When D is a Ferrers diagram λ, it is not hard to see that the result of the latter is at least a skew diagram. However, when D is only assumed to be a *skew* diagram, after performing either of these operations, it is generally *not* true that the result is another skew diagram. Nevertheless, in some cases, after performing these operations, one may be able to reorder the columns (resp. rows) so as to obtain a skew diagram again. This combined with the following definition allows us to derive another sufficiency.

Definition 3.22.
A skew diagram D has *spinal columns* if it contains either a single column or a union of two adjacent columns whose union intersects every nonempty row of D. One can similarly define when D has *spinal rows*.

The following can be proved using results from [**10**].

Theorem 3.23. *Let D_i for $i = 1, 2$ be skew diagrams, both having spinal columns, and ℓ nonempty rows. Let R be a rectangle with ℓ rows that contains D_1, D_2. Let D_i^+ be obtained from D_i by adding a full column of length ℓ to form a skew diagram. Then*

$$D_1 \sim D_2 \text{ if and only if } D_1^+ \sim D_2^+ \text{ if and only if } R \backslash D_1 \sim R \backslash D_2.$$

4. Necessary conditions

We now present some combinatorial invariants for the skew-equivalence relation $D_1 \sim D_2$ on connected skew diagrams.

4.1. Frobenius rank. Recall that the *Durfee* or *Frobenius rank* of a skew diagram D is defined to be the minimum number of ribbons in any decomposition of D into ribbons. It was recently conjectured by Stanley [**14**], and proven by Chen and Yang [**3**], that the rank coincides with the highest power of t dividing the polynomial $s_D(1, 1, \ldots, 1, 0, 0, \ldots)$, where t of the variables have been set to 1, and the rest to zero. This implies the following.

Corollary 4.1. *Frobenius rank is an invariant of skew-equivalence, that is, two skew-equivalent diagrams must have the same Frobenius rank.*

In particular, skew-equivalence restricts to the subset of ribbons as they are the skew diagrams of Frobenius rank 1.

4.2. Overlaps. Data about the amount of overlap between sets of rows or columns in the skew diagram D can be recovered from its skew Schur function s_D.

Definition 4.2.
Let D be a skew diagram occupying r rows. For each k in $\{1, 2, \ldots, r\}$, define the *k-row overlap composition* $r^{(k)} = (r_1^{(k)}, \ldots, r_{r-k+1}^{(k)})$ to be the sequence where $r_i^{(k)}$ is the number of columns occupied in common by the rows $i, i+1, \cdots, i+k-1$. Let $\rho^{(k)}$ be the *k-row overlap partition* that is the weakly decreasing rearrangement of $r^{(k)}$. Similarly define column overlap compositions $c^{(k)}$ and column overlap partitions $\gamma^{(k)}$.

Example 4.3. If $D = \begin{array}{ccc} & \times & \times \\ \times & \times & \times \\ \times & & \end{array}$, then the 1-row, 2-row and 3-row overlap compositions are

$$\begin{aligned} r^{(1)} &= (2, 3, 1) \\ r^{(2)} &= (2, 1) \\ r^{(3)} &= (0). \end{aligned}$$

With this is mind we are able to prove

Theorem 4.4. *If $D_1 \sim D_2$ then D_1, D_2 have the same k-row overlap partitions and the same k-column overlap partitions for all k.*

It transpires that the row overlap partitions $(\rho^{(k)})_{k \geq 1}$ and the column overlap partitions $(\gamma^{(k)})_{k \geq 1}$ determine each other uniquely. To see this, we define a third form of data on a skew diagram D, which mediates between the two, and which is more symmetric under conjugation.

Proposition 4.1. *Given a skew diagram D, consider the doubly-indexed array $(a_{k,\ell})_{k,\ell \geq 1}$ where $a_{k,\ell}$ is defined to be the number of $k \times \ell$ rectangular subdiagrams contained inside D. Then we have*

$$\begin{aligned} a_{k,\ell} &= \sum_{\ell' \geq \ell} \left(\rho^{(k)}\right)^t_{\ell'} \\ &= \sum_{k' \geq k} \left(\gamma^{(\ell)}\right)^t_{k'} . \end{aligned}$$

Consequently, any one of the three forms of data

$$(\rho^{(k)})_{k \geq 1}, \quad (\gamma^{(k)})_{k \geq 1}, \quad (a_{k,\ell})_{k,\ell \geq 1}$$

on D determines the other two uniquely.

Remark 4.5. Unfortunately, having the same row and column overlap partitions $\rho^{(k)}, \gamma^{(k)}$ is not sufficient for the skew-equivalence of two skew diagrams as

$$\begin{array}{cccc} & & \times & \times \\ \times & \times & \times & \\ \times & & & \end{array} \quad \not\sim \quad \begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{array}$$

even though they have the same row and column overlap partitions $\rho^{(k)}, \gamma^{(k)}$ for every k.

5. Complete classification

The sufficient conditions discussed in this abstract explain all but six of the skew-equivalences among skew diagrams with up to 18 cells. For example, the following skew-equivalence cannot yet be explained:

$$\begin{array}{cccccc} & & & & \times & \times \\ & & \times & \times & \times & \\ & \times & \times & \times & \times & \\ & \times & \times & & & \\ \times & \times & & & & \\ \times & \times & & & & \end{array} \quad \sim \quad \begin{array}{cccccc} & & & & \times & \times \\ & & & \times & \times & \\ & & & \times & \times & \\ & \times & \times & \times & & \\ \times & \times & \times & \times & & \\ \times & \times & & & & \end{array} .$$

We end with the following conjectures.

Conjecture 5.1. *The skew-equivalence relation $\sim$, when restricted to skew diagrams of Frobenius rank at most* 3*, is explained by all of the constructions in this paper. In other words, it is the equivalence relation generated by the equivalences listed in*

- *Proposition 3.1,*
- *Theorem 3.3,*
- *Theorem 3.11,*
- *Theorem 3.19, and*
- *Theorem 3.23.*

Conjecture 5.2. *Every skew-equivalence class of skew diagrams has cardinality a power of 2.*

References

[1] F. Bergeron, R. Biagioli, and M.H. Rosas, Inequalities between Littlewood-Richardson coefficients, `math.CO/0403541`.
[2] L.J. Billera, H. Thomas, and S. van Willigenburg, Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions, `math.CO/0405434`.
[3] W.Y.C. Chen and A.L.B. Yang, Stanley's Zrank problem on skew partitions, `math.CO/0509181`.
[4] W. Chen, G.-G. Yan, and A. Yang, Transformations of border strips and Schur function determinants, *J. Algebraic Combin.* **21** (2005), 379–394.
[5] S. Fomin, W. Fulton, C.-K. Li, and Y.-T. Poon, Eigenvalues, singular values, and Littlewood-Richardson coefficients, `math.AG/0301307`.
[6] I. M. Gel'fand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, Noncommutative symmetric functions, *Adv. Math.* **112** (1995), 218–348.
[7] A. Hamel and I. Goulden, Planar decompositions of tableaux and Schur function determinants, *European J. Combin.* **16** (1995), 461–477.
[8] T. Lam, A. Postnikov, and P. Pylyavskyy, Schur positivity and Schur log-concavity, `math.CO/0502446`.
[9] I. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd Edition*, Oxford University Press, New York, USA, 1995.
[10] P. Magyar, Borel-Weil theorem for configuration varieties and Schur modules, *Adv. Math.* **134** (1998), 328–366.
[11] A. Okounkov, Log-concavity of multiplicities with application to characters of $U(\infty)$, *Adv. Math.* **127** (1997), 258-282.
[12] B.E. Sagan, *The symmetric group: Representations, combinatorial algorithms, and symmetric functions (2nd edition), Graduate Texts in Mathematics,* **203**, Springer-Verlag, New York, USA, 2001.
[13] R.P. Stanley, *Enumerative Combinatorics, vol. 2*, Cambridge University Press, Cambridge, UK, 1999.
[14] R.P. Stanley, The rank and minimal border strip decompositions of a skew partition, *J. Combinatorial Theory (A)* **100** (2002), 349–375.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: reiner@math.umn.edu

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: krishaw@math.ubc.ca

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: steph@math.ubc.ca