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Conjugacy in Permutation Representations of the Symmetric Group

Extended Abstract

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Abstract. Although the conjugacy classes of the general linear group are known, it is not obvious (from the canonic form of matrices) that two permutation matrices are similar if and only if they are conjugate as permutations in the symmetric group, i.e. that conjugacy classes of S_n do not unite under the natural representation. We prove this fact, and give its application to the enumeration of fixed points under a natural action of $S_n \times S_n$. We also consider the permutation representations of S_n which arise from the action of S_n on k-tuples, and classify which of them unite conjugacy classes and which do not.

Résumé. Bien que les classes de conjugaison du groupe linéaire général soient bien connues, il n'est pas évident (à partir de la forme canonique des matrices) que deux oermutations sont similaires si et seulement si elles sont conjuguées comme permutations du groupe symétrique, i.e. que les classes de conjugaison de S_n ne s'unissent pas sous la représentation naturelle.Nous prouvons ici ce fait et nous l'appliquons à l'énumération des points fixes pour une action naturelle de $S_n \times S_n$. We étudions aussi la représentation par permutations de S_n qui découle de l'action de S_n sur les k-uplets, et nous distinguons celles qui unissent les classes de conjugaisons.

1. Introduction

In this extended abstract we study the action of S_n on ordered k-tuples. Denote by ρ_k the corresponding permutation representation over an arbitrary field $\mathbb{F}$. The following problem was presented to us by Lubotzky and Roichman.

Problem 1. *For which $1 \leq k \leq n$ and for which fields $\mathbb{F}$ does the following hold:*
For any two permutations $\pi, \sigma \in S_n$, $\rho_k(\pi)$ is conjugate to $\rho_k(\sigma)$ in $GL(n, \mathbb{F})$ if and only if π and σ are conjugate in S_n.

This problem arises in the enumeration of invertible matrices with respect to a certain natural action of $S_n \times S_n$, see [**BC**] and Section 4 below.

For $k = n$, i.e. the regular representation, a negative solution to Problem 1 was essentially given by Burnside (See [**B**] p. 23-24). In Section 2 it is shown that for $k = 1$ the answer is positive. A full solution is given in Section 3: We find that ρ_1 and ρ_2 do not unite any classes, that ρ_3 unites classes only when n is even, and that ρ_k for $k \geq 4$ always unites some classes. These results do not depend on the choice of the field $\mathbb{F}$. Finally, our results are applied in Section 4 to the enumeration of fixed points of a natural action of $S_n \times S_n$ on invertible matrices. This is an extended abstract: Proofs and full details can be found in [**CS**].

2. The Natural Representation of S_n

There is a natural embedding of S_n in $GL(n, \mathbb{F})$ where $\mathbb{F}$ is any field. Consider a permutation $\pi \in S_n$ as an $n \times n$ matrix obtained from the identity matrix by permutations of the rows. More explicitly: for every permutation $\pi \in S_n$ we identify π with the matrix:

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$$[\pi]_{i,j} = \begin{cases} 1 & i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

This representation can also be realized as the permutation representation which is obtained from the natural action of S_n on $\{1, 2, \ldots, n\}$ defined by $\pi \cdot i = \pi(i)$.

Our first result is that this representation does not unite conjugacy classes of S_n. We shall use the following well known fact:

Fact 2.1. *If σ is a cycle of length n, then σ^k consists of (n, k) cycles, each of length $n/(n, k)$.*[1]

Proposition 2.1. *Let $\mathbb{F}$ be a field of characteristic 0. The conjugacy classes of S_n do not unite in $GL(n, \mathbb{F})$. In other words, if π and σ are permutations with similar matrices in $GL(n, \mathbb{F})$, then they are conjugate in S_n too.*

Proof. Let π and σ be permutations which are similar as matrices. First of all, we note that for any k, π^k and σ^k are also similar.

Each cycle of length k in π contributes the term $x^k - 1$ into the characteristic polynomial of the permutation matrix. Under the above restriction on $\mathrm{char}(\mathbb{F})$ it seems reasonable that the cycle structure of a permutation can be recovered from the characteristic polynomial of the corresponding permutation matrix. However, our proof utilizes the trace of the permutation matrix and the traces of its powers.

Denote by $c_d(\pi)$ the number cycles with length equal to d in π. We shall use induction on d to prove that $c_d(\pi) = c_d(\sigma)$, for all d, and this will show that π and σ are conjugate.

Since π and σ are similar as matrices, we have $\mathrm{trace}(\pi) = \mathrm{trace}(\sigma)$. However, the trace function counts the 1's on the diagonal (here we use the restriction on $\mathrm{char}(\mathbb{F})$), and each such 1 corresponds to a fixed point of the permutation, so $\mathrm{trace}(\pi) = c_1(\pi)$. Therefore, $c_1(\pi) = c_1(\sigma)$, i.e. π and σ have the same number of fixed points. This is the base of our induction.

Now let d be an arbitrary number, and suppose that $c_k(\pi) = c_k(\sigma)$ for all $k < d$. From Lemma 2.1 it follows that a k-cycle in π ends up as a product of k 1-cycles in π^d if and only if k divides d. Therefore, we can conclude that

$$\mathrm{trace}(\pi^d) = \sum_{k|d} k \cdot c_k(\pi) = d \cdot c_d(\pi) + \sum_{k|d, k<d} k \cdot c_k(\pi).$$

Now, by our induction hypothesis, for all proper divisors $k \mid d$ we have $c_k(\pi) = c_k(\sigma)$. On the other hand, $\mathrm{trace}(\pi^d) = \mathrm{trace}(\sigma^d)$. This implies that $c_d(\pi) = c_d(\sigma)$, and completes the induction argument.

We have shown that π and σ have the same cycle structure, so they are conjugate as permutations. $\square$

Note that if $\mathbb{F}$ is such that $\mathrm{char}(\mathbb{F}) < n$ then the trace of a permutation matrix no longer gives the number of fixed points of the permutation, so a more devious route is necessary.

In this case it is impossible to recover the cycle structure of a permutation from the characteristic polynomial of the corresponding permutation matrix: for example, if $\mathrm{char}(\mathbb{F}) = 2$ we have $x^4 + 1 = (x^2 + 1)^2 = (x + 1)^4$, i.e. one cycle of length 4, two cycles of length 2 and four cycles of length 1 all have the same characteristic polynomial.

However, in [**CS**] we extend Proposition 2.1, and prove the following:

Theorem 2.2. *Let $\mathbb{F}$ be an arbitrary field. The conjugacy classes of S_n do not unite in $GL(n, \mathbb{F})$. In other words, if π and σ are permutations with similar matrices in $GL(n, \mathbb{F})$, then they are conjugate in S_n too.*

The proof is obtained by considering certain eigenspaces of powers of π and σ. See [**CS**] for full details.

It should be noted that this property of the natural representation seems to be very "delicate". For example, in the natural representations of the signed permutation groups this property fails to hold. In particular, in B_2, the permutations $\sigma = (1, 2)$ and $\tau = (1, \bar{1})$ are not conjugate, and yet the matrices associated with them, namely

$$P(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P(\tau) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are similar matrices.

[1] We use (n, k) to denote the greatest common divisor of n and k.

3. Other Permutation Representations

3.1. Representations Arising from the Action of S_n on k-tuples. In Section 2 we proved that the natural representation of S_n does not unite conjugacy classes. On the other hand, it is well known (see [**B**] p. 23-24) that the regular representation of S_n (indeed, of any group) unites all elements of equal order. The natural representation can be seen as the permutation representation obtained from the natural action of S_n on the set $\{(1),(2),\ldots,(n)\}$ of 1-tuples. On the other hand, the regular representation can be seen as the permutation representation which arises from the action of S_n on all $n!$ ordered n-tuples of numbers from $\{1,2,\ldots,n\}$. In this section we wish to address the representations in between: the representation arising from the action of S_n on pairs, triplets, etc. and to see where the representations start uniting conjugacy classes.

At first we shall confine ourselves to the complex field, and prove our results there. The results for general fields will follow from these results.

We begin with a general theorem, which holds true for any representation of any finite group.

THEOREM 3.1. *Let G be a group, and $\sigma,\tau \in G$. Let $T : G \to GL(d,\mathbb{C})$ a representation of G, with character χ. Then $T(\sigma) \sim T(\tau)$ as matrices if and only if $\chi(\sigma^k) = \chi(\tau^k)$ for all k.*

Proof is given in [**CS**].

Note that the fact that the regular representation unites all elements of equal order can be derived from this theorem: If χ is the character of the regular representation, then

$$\chi(\sigma^k) = \begin{cases} |G| & |\sigma| \mid k, \\ 0 & \text{otherwise} \end{cases}$$

so obviously $\chi(\sigma^k) = \chi(\tau^k)$ for all k if and only if σ and τ have the same order.

The criterion which we just presented is still rather complicated to use for general groups, but it can be simplified in our case, because of the following simple fact.

FACT 3.2. *Let $\sigma \in S_n$, with $|\sigma| = m$.*

- *If k is relatively prime to m. Then $\sigma^k \sim \sigma$.*
- *For any k, $\sigma^k \sim \sigma^{(m,k)}$.*

CLAIM 3.3. *Let $T : S_n \to GL(d,\mathbb{C})$ be a representation of the symmetric group, with character χ, and $\sigma,\tau \in S_n$ elements of order m. Then $T(\sigma) \sim T(\tau)$ if and only if $\chi(\sigma^k) = \chi(\tau^k)$ for $k \mid m$.*

COROLLARY 3.4. *If σ and τ are of prime order p, then $T(\sigma) \sim T(\tau)$ if and only if $\chi(\sigma) = \chi(\tau)$.*

DEFINITION 3.5. Let $\sigma,\tau \in S_n$ be elements of equal order m, such that if $k \neq 1$ and $k \mid m$ then $\sigma^k \sim \tau^k$. It follows from 3.3 that $T(\sigma) \sim T(\tau)$ if and only if $\chi(\sigma) = \chi(\tau)$. We call such elements *almost similar*. In fact, it is sufficient to require that $\sigma^p \sim \tau^p$ for all prime divisors of m.

We next show that almost similar elements are typical examples of elements that are united by representations, in the following sense:

THEOREM 3.6. *Let $T : S_n \to GL(d,\mathbb{C})$ be a representation. If T unites some two conjugacy classes, then there must exist a pair of almost similar elements which it unites.*

Having proved this, we now have a criterion to check whether a representation unites classes: It is sufficient to show that all pairs of almost similar elements remain non united, i.e. that the character of the representation takes different values on them.

Using this criterion, we show

THEOREM 3.7.

(1) *The natural representation does not unite classes.*
(2) *The representation arising from the action on pairs does not unite classes.*
(3) *The representation arising from the action on triplets unites classes iff n is even.*
(4) *Representations arising from the action on k-tuples, with $k \geq 4$, always unite some classes.*

3.2. Representations Arising from the Action of S_n on Subsets. The natural route to follow now would be to try and generalize these results to other permutation representations, and in particular to those arising from the action of S_n on k-subsets of $\{1, 2, \ldots, n\}$. The general answer eludes us at present, and seems to be pretty unsatisfactory. However, we have managed to show that the representation arising from the action of S_n on *all* subsets of $[n]$ does in fact not unite any classes. In all this section, we shall omit proofs, and refer the interested reader to [**CS**] for full details.

THEOREM 3.8. *The action of S_n on the power set $2^{[n]}$ of $[n]$ does not unite classes.*

Consider now the action of S_n on *even* sized subsets of $[n]$. If n is even, then this action unites some classes. For example, $(1,2)(3,4)\ldots(n-1,n)$ and $(1)(2)(3,4)\ldots(n-1,n)$ get united. (They are almost similar, and both fix $2^{n/2}$ sets.)

However, if n is odd, then this representation does not unite classes.

THEOREM 3.9. *Let n be odd. The action of S_n on the set of even-sized subsets of $[n]$ does not unite classes.*

Finally, we conclude this section by exploring the behavior of the representation arising from the action of S_n on *odd* sized subsets of $[n]$.

THEOREM 3.10. *The action of S_n on the set of odd-sized subsets of $[n]$ does not unite classes. This does not depend on n's parity.*

3.3. General Fields. The proofs in the two previous sections apply only to the complex field $\mathbb{C}$, (in fact, to all fields with characteristic 0.) We shall now show that the same applies to any field. We shall base ourselves on Theorem 2.1 from Section 2, where we proved that the natural representation does not unite classes, regardless the base field.

LEMMA 3.11. *Let $f : G \to H$ and $g : H \to K$ be group homomorphisms.*

1. *If f and g both do not unite classes, then also gf does not unite them.*
2. *If gf does not unite classes, then neither does f.*

THEOREM 3.12. *Let T be any permutation representation of S_n. If T does not unite classes when considered a representation into $GL(m, \mathbb{C})$, then it does not unite classes when considered as a representation into $GL(m, \mathbb{F})$, for any field $\mathbb{F}$.*

PROOF. Any permutation representation can be factored into $S_n \to S_m \to GL(m, \mathbb{C})$, where the first homomorphism is the permutation representation and the second is the natural representation. Now, suppose T does not unite classes. By Lemma 3.11, neither does the permutation representation $S_n \to S_m$. We already know that the natural representation does not unite classes, whatever the field. Tacking these two homomorphisms together gives us the representation in any field, and another appeal to Lemma 3.11 proves that it still doesn't unite any classes. $\square$

4. The action of $S_n \times S_n$ on invertible matrices

In this section we present an application of Theorem 2.1.

DEFINITION 4.1. Let $\mathbb{F}$ be any field. We define an action of $S_n \times S_n$ on the group $GL(n, \mathbb{F})$ by

$$(\pi, \sigma) \bullet A = \pi A \sigma^{-1} \text{ where } (\pi, \sigma) \in S_n \times S_n \text{ and } A \in GL(n, \mathbb{F}) \tag{1}$$

DEFINITION 4.2. Let M be a finite subset of $GL(n, \mathbb{F})$, invariant under the action of $S_n \times S_n$ defined above. We denote by α_M the permutation representation of $S_n \times S_n$ obtained from the action (1) . In the sequel we identify the action (1) with the permutation representation α_M associated with it.

Now we define a generalization of the conjugacy representation of S_n

We present a conjugacy representation of S_n on a subset M of $GL(n, \mathbb{F})$.

DEFINITION 4.3. Denote by β the permutation representation of S_n obtained by the following action on M.

$$\pi \circ A = (\pi, \pi) \bullet A = \pi A \pi^{-1} \tag{2}$$

The connection between α_M and β_M is given by the following easily seen claim:

CLAIM 4.4. *Consider the diagonal embedding of S_n into $S_n \times S_n$. Then*

$$\beta_M = \alpha_M \downarrow_{S_n}^{S_n \times S_n} . \qquad \square$$

COROLLARY 4.5. *For every finite set $M \subseteq GL(n, \mathbb{F})$ invariant under the action (1) of $S_n \times S_n$ defined above:*
If π and σ are conjugate in S_n then

$$\chi_{\alpha_M}((\pi,\sigma)) = \chi_{\alpha_M}((\pi,\pi)) = \chi_{\beta_M}(\pi) = \#\{A \in M \mid \pi A = A\pi\}.$$

If π is not conjugate to σ in S_n then

$$\chi_{\alpha_M}((\pi,\sigma)) = 0.$$

PROOF. If π and σ are conjugate in S_n then (π,σ) is conjugate to (π,π) in $S_n \times S_n$. Since the character is a class function, we have:

$$\chi_{\alpha_M}(\pi,\sigma) = \chi_{\alpha_M}(\pi,\pi) = \#\{A \in M \mid \pi A \pi^{-1} = A\} = \#\{A \in M \mid \pi A = A\pi\}$$

i.e. the value of the character of α_M calculated on the element (π,σ) with π conjugate to σ in S_n is equal to the number of matrices in M which commute with the permutation matrix π.

Now, we know that the character of a permutation representation counts the number of fixed points, so:

$$\chi_{\alpha_M}(\pi,\sigma) = \#\{A \in M \mid \pi A \sigma^{-1} = A\} = \#\{A \in M \mid \pi = A\sigma A^{-1}\}.$$

Note that $\pi = A\sigma A^{-1}$ means that π and σ are similar as invertible matrices. Thus, by Theorem 2.1, if π and σ are not conjugate in S_n they can not be conjugate in $GL(n,\mathbb{F})$ and we have:

$$\{A \in M \mid \pi = A\sigma A^{-1}\} = \varnothing$$

and so

$$\chi_{\alpha_M}(\pi,\sigma) = 0$$

if π and σ are not conjugate in S_n. $\square$

For an application of Corrolary 4.5 to the enumeration of fixed points, see [**BC**].

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