Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

On the characteristic map of finite unitary groups

Nathaniel Thiem and C. Ryan Vinroot

Abstract. In his classic book on symmetric functions, Macdonald describes a remarkable result by Green relating the character theory of the finite general linear group to transition matrices between bases of symmetric functions. This connection allows us to analyze the representation theory of the general linear group via symmetric group combinatorics. Using the work of Ennola, Kawanaka, Lusztig and Srinivasan, this paper describes the analogous setting for the finite unitary group. In particular, we explain the connection between Deligne-Lusztig theory and Ennola's efforts to generalize Green's work, and from this we deduce various representation theoretic results. Applications include finding certain sums of character degrees, and a model of Deligne-Lusztig type for the finite unitary group, which parallels results of Klyachko and Inglis and Saxl for the finite general linear group.

Résumé. Dans son livre classique sur les fonctions symétriques, Macdonald décrit un résultat remarquable dû à Green, qui relie la théorie des caractères du groupe général liéaire fini, aux matrices de transition entre bases de fonctions symétriques. Cette connexion permet d'analyser la théorie de représentation du groupe général linéaire à l'aide de combinatoires de groupes symétriques. En utilisant le travail d'Ennola, Kawanaka, Lusztig et Srinivasan, le présent article décrit le cadre analogue pour le groupe unitaire fini. En particulier, nous expliquons la connexion entre la théorie de Deligne-Lusztig et les efforts d'Ennola concernant la généralisation du travail de Green, et nous en déduisons plusieurs résultats en théorie de représentation. Parmi les applications, nous obtenons certaines sommes de degés de caractères, et un modèle du type Deligne-Lusztig pour le groupe unitaire fini, qui met en parallèle les résultats de Klyachko, Inglis et Saxl pour le groupe général linéaire fini.

1. Introduction

In his seminal work [**7**], Green described a remarkable connection between the class functions of the finite general linear group $\mathrm{GL}(n,\mathbb{F}_q)$ and a generalization of the ring of symmetric functions of the symmetric group S_n. In particular, Green defines a map, called the characteristic map, that takes irreducible characters to Schur-like symmetric functions, and recovers the character table of $\mathrm{GL}(n,\mathbb{F}_q)$ as the transition matrix between these Schur functions and Hall-Littlewood polynomials [**14**, Chapter IV]. Thus, we can use the combinatorics of the symmetric group S_n to understand the representation theory of $\mathrm{GL}(n,\mathbb{F}_q)$. Some of the implications of this approach include an indexing of irreducible characters and conjugacy classes of $\mathrm{GL}(n,\mathbb{F}_q)$ by multi-partitions and a formula for the degrees of the irreducible characters in terms of these partitions.

This paper describes the parallel story for the finite unitary group $\mathrm{U}(n,\mathbb{F}_{q^2})$ by collecting known results for this group and examining some applications of the unitary characteristic map. Inspired by Green, Ennola [**4, 5**] used results of Wall [**16**] to construct the appropriate ring of symmetric functions and characteristic map. Ennola was able to prove that the analogous Schur-like functions correspond to an orthonormal basis for the class functions, and conjectured that they corresponded to the irreducible characters. He theorized that the representation theory of $\mathrm{U}(n,\mathbb{F}_{q^2})$ should be deduced from the representation theory of $\mathrm{GL}(n,\mathbb{F}_q)$ by

2000 *Mathematics Subject Classification.* Primary 20C33; Secondary 05E05.

Key words and phrases. finite unitary group, symmetric functions, models of finite groups.

The first author would like to thank Stanford University for supplying a stimulating research environment, and, in particular, P. Diaconis for acting as a sounding board for many of these ideas.

The second author thanks the Tata Institute of Fundamental Research for an excellent environment for doing much of this work, and Rod Gow for helpful correspondence.

substituting "$-q$" for every occurrence of "q". The general phenomenon of obtaining a polynomial invariant in q for $\mathrm{U}(n,\mathbb{F}_{q^2})$ by this substitution has come to be known as "Ennola duality".

Roughly a decade after Ennola made his conjecture, Deligne and Lusztig [**2**] constructed a family of virtual characters, called Deligne-Lusztig characters, to study the representation theory of arbitrary finite reductive groups. Lusztig and Srinivasan [**13**] then computed an explicit decomposition of the irreducible characters of $\mathrm{U}(n,\mathbb{F}_{q^2})$ in terms of Deligne-Lusztig characters. Kawanaka [**11**] used this composition to demonstrate that Ennola duality applies to Green functions, thereby improving results of Hotta and Springer [**9**] and finally proving Ennola's conjecture.

This paper begins by describing some of the combinatorics and group theory associated with the finite unitary groups. Section 2 defines the finite unitary groups, outlines the combinatorics of multi-partitions, and gives a description of some of the key subgroups. Section 2.4 analyzes the conjugacy classes of $\mathrm{U}(n,\mathbb{F}_{q^2})$ and the Jordan decomposition of these conjugacy classes.

Section 3 outlines the statement and development of the Ennola conjecture from two perspectives. Both points of view define a map from a ring of symmetric functions to the character ring C of $\mathrm{U}(n,\mathbb{F}_{q^2})$. However, the first uses the multiplication for C as defined by Ennola, and the second uses Deligne-Lusztig induction as the multiplicative structure of C. This multiplicative structure on the graded ring of characters of the unitary group was studied by Digne and Michel in [**3**], where the focus is that this multiplication induces a Hopf algebra structure. While some of the results in Section 3 appear in a different form in [**3**], our approach focuses on the explicit map between characters and symmetric functions.

The main results are

I. (Theorem 3.2) The Deligne-Lusztig characters correspond to power-sum symmetric functions via the characteristic map of Ennola.

II. (Corollary 3.2) The multiplicative structure that Ennola defined on C is Deligne-Lusztig induction.

Section 4 computes the degrees of the irreducible characters, and uses this result to evaluate various sums of character degrees (see [**14**, IV.6, Example 5] for the $\mathrm{GL}(n,\mathbb{F}_q)$ analogue of this method). The main results are

III. (Theorem 4.1) An irreducible $\chi^{\boldsymbol{\lambda}}$ character of $\mathrm{U}(m,\mathbb{F}_{q^2})$ corresponds to

$$(-1)^{\lfloor m/2\rfloor+n(\boldsymbol{\lambda})}s_{\boldsymbol{\lambda}} \quad \text{and} \quad \chi^{\boldsymbol{\lambda}}(1)=q^{n(\boldsymbol{\lambda}')}\frac{\displaystyle\prod_{1\leq i\leq m}(q^i-(-1)^i)}{\displaystyle\prod_{\Box\in\boldsymbol{\lambda}}(q^{\mathbf{h}(\Box)}-(-1)^{\mathbf{h}(\Box)})},$$

where $s_{\boldsymbol{\lambda}}$ is a Schur-like function, and both $n(\boldsymbol{\lambda})$ and $\mathbf{h}(\Box)$ are combinatorial statistics on the multi-partition $\boldsymbol{\lambda}$.

IV. (Corollary 4.2) If $\mathcal{P}_n^{\Theta}$ indexes the irreducible characters $\chi^{\boldsymbol{\lambda}}$ of $\mathrm{U}(n,\mathbb{F}_{q^2})$, then

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_n^{\Theta}}\chi^{\boldsymbol{\lambda}}(1)=|\{g\in\mathrm{U}(n,\mathbb{F}_{q^2})\ \mid\ g \text{ symmetric}\}|.$$

V. (Theorem 4.3) We give a subset $\mathcal{X}\subseteq\mathcal{P}_{2n}^{\Theta}$ such that

$$\sum_{\boldsymbol{\lambda}\in\mathcal{X}}\chi^{\boldsymbol{\lambda}}(1)=(q+1)q^2(q^3+1)\cdots q^{2n-2}(q^{2n-1}+1)=\frac{|\mathrm{U}(2n,\mathbb{F}_{q^2})|}{|\mathrm{Sp}(2n,\mathbb{F}_q)|}.$$

Section 5 uses results by Ohmori [**15**] and Henderson [**8**] to adapt a model for the general linear group, found by Klyachko [**12**] and Inglis and Saxl [**10**], to the finite unitary group. The main result is

VI. (Theorem 5.2) Let $U_m=\mathrm{U}(m,\mathbb{F}_{q^2})$, where q is odd, and let Γ_m be the Gelfand-Graev character of U_m, $\mathbf{1}$ be the trivial character of the finite symplectic group $Sp_{2r}=\mathrm{Sp}(2r,\mathbb{F}_q)$, and R_L^G be the Deligne-Lusztig induction functor. Then

$$\sum_{0\leq 2r\leq m}R_{U_{m-2r}\oplus U_{2r}}^{U_m}\left(\Gamma_{m-2r}\otimes\mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1})\right)=\sum_{\boldsymbol{\lambda}\in\mathcal{P}_m^{\Theta}}\chi^{\boldsymbol{\lambda}}.$$

That is, in the theorem of Klyachko, one may replace parabolic induction by Deligne-Lusztig induction to obtain a theorem for the unitary group.

These results give considerable combinatorial control over the representation theory of the finite unitary group, and there are certainly more applications to these results than what we present in this paper. Furthermore, this characteristic map gives some insight as to how a characteristic map might look in general type, using the invariant rings of other Weyl groups.

2. Preliminaries

2.1. The unitary group and its underlying field. Let $\mathbb{K} = \bar{\mathbb{F}}_q$ be the algebraic closure of the finite field with q elements and let $\mathbb{K}_m = \mathbb{F}_{q^m}$ denote the finite subfield with q^m elements. Let $\mathrm{GL}(n, \mathbb{K})$ denote the general linear group over $\mathbb{K}$, and define Frobenius maps

$$\begin{array}{rcl} F : \mathrm{GL}(n,\mathbb{K}) & \longrightarrow & \mathrm{GL}(n,\mathbb{K}) \\ (a_{ij}) & \mapsto & (a_{ji}^q)^{-1}, \end{array} \quad \text{and} \quad \begin{array}{rcl} F' : \mathrm{GL}(n,\mathbb{K}) & \longrightarrow & \mathrm{GL}(n,\mathbb{K}) \\ (a_{ij}) & \mapsto & (a_{n-j,n-i}^q)^{-1}. \end{array} \tag{2.1}$$

Then the unitary group $U_n = \mathrm{U}(n, \mathbb{K}_2)$ is given by

$$U_n = \mathrm{GL}(n,\mathbb{K})^F = \{a \in \mathrm{GL}(n,\mathbb{K}) \mid F(a) = a\} \tag{2.2}$$

$$\cong \mathrm{GL}(n,\mathbb{K})^{F'} = \{a \in \mathrm{GL}(n,\mathbb{K}) \mid F'(a) = a\}. \tag{2.3}$$

We define the multiplicative groups $\mathbb{M}_m$ as

$$\mathbb{M}_m = \mathrm{GL}(1,\mathbb{K})^{F^m} = \{x \in \mathbb{K} \mid x^{q^m - (-1)^m} = 1\}.$$

Note that $\mathbb{M}_m \cong \mathbb{K}_m^\times$ only if m is even. We identify $\mathbb{K}^\times$ with the inverse limit $\varprojlim \mathbb{M}_m$ with respect to the norm maps

$$\begin{array}{rccc} N_{mr}: & \mathbb{M}_m & \longrightarrow & \mathbb{M}_r \\ & x & \mapsto & x x^{-q} \cdots x^{(-q)^{m/r-1}} \end{array}, \quad \text{where } m, r \in \mathbb{Z}_{\geq 1} \text{ with } r \mid m.$$

If $\mathbb{M}_m^*$ is the group of characters of $\mathbb{M}_m$, then the direct limit $\mathbb{K}^* = \varinjlim \mathbb{M}_m^*$ gives the group of characters of $\mathbb{K}^\times$. Let

$$\Theta = \{F\text{-orbits of } \mathbb{K}^*\}.$$

A polynomial $f(t) \in \mathbb{K}_2[t]$ is *F-irreducible* if there exists an F-orbit $\{x, x^{-q}, \ldots, x^{(-q)^d}\}$ of $\mathbb{K}^\times$ such that

$$f(t) = (t - x)(t - x^{-q}) \cdots (t - x^{(-q)^d}).$$

Let

$$\Phi = \{f \in \mathbb{K}_2[t] \mid f \text{ is } F\text{-irreducible}\} \stackrel{1-1}{\longleftrightarrow} \{F\text{-orbits of } \mathbb{K}^\times\}. \tag{2.4}$$

2.2. Combinatorics of Φ-partitions and Θ-partitions. Fix an ordering of Φ and Θ, and let

$$\mathcal{P} = \{\text{partitions}\} \quad \text{and} \quad \mathcal{P}_n = \{\nu \in \mathcal{P} \mid |\nu| = n\}.$$

Let $\mathcal{X}$ be either Φ or Θ. An *$\mathcal{X}$-partition* $\boldsymbol{\nu} = (\boldsymbol{\nu}(x_1), \boldsymbol{\nu}(x_2), \ldots)$ is a sequence of partitions indexed by $\mathcal{X}$. The *size* of an $\mathcal{X}$-partition $\boldsymbol{\nu}$ is

$$||\boldsymbol{\nu}|| = \sum_{x \in \mathcal{X}} |x||\boldsymbol{\nu}(x)|, \quad \text{where} \quad |x| = \begin{cases} |x| & \text{if } \mathcal{X} = \Theta, \\ d(x) & \text{if } \mathcal{X} = \Phi, \end{cases} \tag{2.5}$$

$|x|$ is the size of the orbit $x \in \Theta$, and $d(x)$ is the degree of the polynomial $x \in \Phi$. Let

$$\mathcal{P}_n^{\mathcal{X}} = \{\mathcal{X}\text{-partitions } \boldsymbol{\nu} \mid ||\boldsymbol{\nu}|| = n\}, \quad \text{and} \quad \mathcal{P}^{\mathcal{X}} = \bigcup_{n=1}^{\infty} \mathcal{P}_n^{\mathcal{X}}. \tag{2.6}$$

For $\boldsymbol{\nu} \in \mathcal{P}^{\mathcal{X}}$, let

$$n(\boldsymbol{\nu}) = \sum_{x \in \mathcal{X}} |x| n(\boldsymbol{\nu}(x)), \quad \text{where} \quad n(\nu) = \sum_{i=1}^{\ell(\nu)} (i-1)\nu_i. \tag{2.7}$$

The *conjugate* $\boldsymbol{\nu}'$ of $\boldsymbol{\nu}$ is the $\mathcal{X}$-partition $\boldsymbol{\nu}' = (\boldsymbol{\nu}(x_1)', \boldsymbol{\nu}(x_2)', \ldots)$, where ν' is the usual conjugate partition for $\nu \in \mathcal{P}$.

The *semisimple part* $\boldsymbol{\nu}_s$ of $\boldsymbol{\nu} = (\boldsymbol{\nu}(x_1), \boldsymbol{\nu}(x_2), \ldots) \in \mathcal{P}_n^{\mathcal{X}}$ is

$$\boldsymbol{\nu}_s = ((1^{|\boldsymbol{\nu}(x_1)|}), (1^{|\boldsymbol{\nu}(x_2)|}), \ldots) \in \mathcal{P}_n^{\mathcal{X}}, \tag{2.8}$$

and the *unipotent part* $\boldsymbol{\nu}_u$ of $\boldsymbol{\nu} \in \mathcal{P}_n^{\mathcal{X}}$ is given by

$$\boldsymbol{\nu}_u(\mathbb{1}) \quad \text{has parts} \quad \{|x|\boldsymbol{\nu}(x)_i \mid x \in \mathcal{X}, i = 1, \ldots, \ell(\boldsymbol{\nu}(x))\} \tag{2.9}$$

where

$$\mathbb{1} = \begin{cases} \{\mathbf{1}\} & \text{if } \mathcal{X} = \Theta, \\ t-1 & \text{if } \mathcal{X} = \Phi, \end{cases}$$

$\mathbf{1}$ is the trivial character in $\mathbb{K}^*$, and $\boldsymbol{\nu}_u(x) = \emptyset$ for $x \neq \mathbb{1}$.

2.3. Levi subgroups and maximal tori. Let $\mathcal{X}$ be either Φ or Θ as in Section 2.2. For $\boldsymbol{\nu} \in \mathcal{P}_n^{\mathcal{X}}$, let

$$L_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} L_{\boldsymbol{\nu}}(x), \quad \text{where} \quad \mathcal{X}_{\boldsymbol{\nu}} = \{x \in \mathcal{X} \mid \boldsymbol{\nu}(x) \neq \emptyset\}, \tag{2.10}$$

and for $x \in \mathcal{X}_{\boldsymbol{\nu}}$,

$$L_{\boldsymbol{\nu}}(x) = \begin{cases} \mathrm{U}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{2|x|}) & \text{if } |x| \text{ is odd,} \\ \mathrm{GL}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{|x|}) & \text{if } |x| \text{ is even.} \end{cases} \tag{2.11}$$

Then $L_{\boldsymbol{\nu}}$ is a Levi subgroup of $U_n = \mathrm{U}(n, \mathbb{K}_2)$ (though not uniquely determined by $\boldsymbol{\nu}$). The Weyl group

$$W_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} S_{|\boldsymbol{\nu}(x)|}, \tag{2.12}$$

of $L_{\boldsymbol{\nu}}$ has conjugacy classes indexed by

$$\mathcal{P}_s^{\boldsymbol{\nu}} = \{\boldsymbol{\gamma} \in \mathcal{P}^{\mathcal{X}} \mid \boldsymbol{\gamma}_s = \boldsymbol{\nu}_s\}, \tag{2.13}$$

and the size of the conjugacy class $c_{\boldsymbol{\gamma}}$ is

$$|c_{\boldsymbol{\gamma}}| = \frac{|W_{\boldsymbol{\gamma}}|}{z_{\boldsymbol{\gamma}}}, \quad \text{where} \quad z_{\boldsymbol{\gamma}} = \prod_{x \in \mathcal{X}} z_{\boldsymbol{\gamma}(x)} \quad \text{and} \quad z_\gamma = \prod_{i=1}^{\ell(\gamma)} i^{m_i} m_i!, \tag{2.14}$$

for $\gamma = (1^{m_1} 2^{m_2} \cdots) \in \mathcal{P}$.

For every $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}_n$ there exists a maximal torus (unique up to isomorphism) T_ν of U_n such that

$$T_\nu \cong \mathbb{M}_{\nu_1} \times \mathbb{M}_{\nu_2} \times \cdots \times \mathbb{M}_{\nu_\ell}.$$

For every $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\mu}}$, there exists a maximal torus (unique up to isomorphism) $T_{\boldsymbol{\gamma}} \subseteq L_{\boldsymbol{\nu}}$ such that

$$T_{\boldsymbol{\gamma}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} T_{\boldsymbol{\gamma}}(x), \quad \text{where} \quad T_{\boldsymbol{\gamma}}(x) \cong \mathbb{M}_{|x|\boldsymbol{\gamma}(x)_1} \times \cdots \times \mathbb{M}_{|x|\boldsymbol{\gamma}(x)_\ell}. \tag{2.15}$$

Note that as a maximal torus of U_n, the torus $T_{\boldsymbol{\gamma}} \cong T_{\boldsymbol{\gamma}_u(\mathbb{1})}$.

2.4. Conjugacy classes and Jordan decomposition.

Proposition 2.1. *The conjugacy classes $c_{\boldsymbol{\mu}}$ of U_n are indexed by $\boldsymbol{\mu} \in \mathcal{P}_n^{\Phi}$.*

For $r \in \mathbb{Z}_{\geq 0}$, let $\psi_r(x) = \prod_{i=1}^r (1 - x^i)$.

Proposition 2.2 (Wall). *Let $g \in c_{\boldsymbol{\mu}}$. The order $a_{\boldsymbol{\mu}}$ of the centralizer g in U_n is*

$$a_{\boldsymbol{\mu}} = (-1)^{||\boldsymbol{\mu}||} \prod_{f \in \Phi} a_{\boldsymbol{\mu}(f)}\big((-q)^{d(f)}\big), \text{ where } a_\mu(x) = x^{|\mu| + 2n(\mu)} \prod_j \psi_{m_j}(x^{-1}),$$

for $\mu = (1^{m_1} 2^{m_2} 3^{m_3} \cdots) \in \mathcal{P}$.

For $\boldsymbol{\mu} \in \mathcal{P}^{\Phi}$, let $L_{\boldsymbol{\mu}}$ be as in (2.10). Note that $|L_{\boldsymbol{\mu}}| = a_{\boldsymbol{\mu}_s}$.

Lemma 2.1. *Suppose $g \in c_{\boldsymbol{\mu}}$ with Jordan decomposition $g = su$. Then*

(a) *$s \in c_{\boldsymbol{\mu}_s}$ and $u \in c_{\boldsymbol{\mu}_u}$, where $\boldsymbol{\mu}_s$ and $\boldsymbol{\mu}_u$ are as in (2.8) and (2.9),*
(b) *the centralizer $C_{U_n}(s)$ of s in U_n is isomorphic to $L_{\boldsymbol{\mu}}$.*

3. The Ennola Conjecture

3.1. The characteristic map. Let $X = \{X_1, X_2, \ldots\}$ be an infinite set of variables and let $\Lambda(X)$ be the graded $\mathbb{C}$-algebra of symmetric functions in the variables $\{X_1, X_2, \ldots\}$. Define the *power-sum symmetric function*, $p_\nu(X)$, and the *Schur function*, $s_\lambda(X)$, for $\nu, \lambda \in \mathcal{P}$, as they are in [**14**, Chapter I].

The irreducible characters ω^λ of S_n are indexed by $\lambda \in \mathcal{P}_n$, as in [**14**, Chapter I]. Let $\omega^\lambda(\nu)$ be the value of ω^λ on a permutation with cycle type ν. The relationship between $p_\nu(X)$ and $s_\lambda(X)$ is given by

$$s_\lambda(X) = \sum_{\nu \in \mathcal{P}_{|\lambda|}} \omega^\lambda(\nu) z_\nu^{-1} p_\nu(X), \quad \text{where} \quad z_\nu = \prod_{i \geq 1} i^{m_i} m_i! \tag{3.1}$$

is the order of the centralizer in S_n of the conjugacy class corresponding to $\nu = (1^{m_1} 2^{m_2} \cdots) \in \mathcal{P}$. Let $t \in \mathbb{C}$. For $\mu \in \mathcal{P}$, let the *Hall-Littlewood symmetric function* $P_\mu(X; t)$ be as it is defined in [**14**].

For $\nu, \mu \in \mathcal{P}_n$, the *classical Green function* $Q^\mu_\nu(t)$ is given by

$$p_\nu(X) = \sum_{\mu \in \mathcal{P}_{|\nu|}} Q^\mu_\nu(t^{-1}) t^{n(\mu)} P_\mu(X; t). \tag{3.2}$$

The $p_\nu(X)$, $s_\lambda(X)$, and $P_\mu(X; t)$, are all bases of $\Lambda(X)$ as a $\mathbb{C}$-algebra.

For every $f \in \Phi$, fix a set of independent variables $X^{(f)} = \{X_1^{(f)}, X_2^{(f)}, \ldots\}$, and for any symmetric function h, we let $h(f) = h(X^{(f)})$ denote the symmetric function in the variables $X^{(f)}$. Let

$$\Lambda = \mathbb{C}\text{-span}\{P_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{P}^\Phi\}, \quad \text{where} \quad P_{\boldsymbol{\mu}} = (-q)^{-n(\boldsymbol{\mu})} \prod_{f \in \Phi} P_{\boldsymbol{\mu}(f)}(f; (-q)^{-d(f)}).$$

Then

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n, \quad \text{where} \quad \Lambda_n = \mathbb{C}\text{-span}\{P_{\boldsymbol{\mu}} \mid ||\boldsymbol{\mu}|| = n\},$$

makes Λ a graded $\mathbb{C}$-algebra. Define a Hermitian inner product on Λ by

$$\langle P_{\boldsymbol{\mu}}, P_{\boldsymbol{\nu}} \rangle = a_{\boldsymbol{\mu}}^{-1} \delta_{\boldsymbol{\mu}\boldsymbol{\nu}}.$$

For each $\varphi \in \Theta$ let $Y^{(\varphi)} = \{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \ldots\}$ be an infinite variable set, and for a symmetric function h, let $h(\varphi) = h(Y^{(\varphi)})$. Relate symmetric functions in the X variables to symmetric functions in the Y variables via the transform

$$p_n(\varphi) = (-1)^{n|\varphi|-1} \sum_{x \in \mathbb{M}_{n|\varphi|}} \xi(x) p_{n|\varphi|/d(f_x)}(f_x), \tag{3.3}$$

where $\varphi \in \Theta$, $\xi \in \varphi$, and $f_x \in \Phi$ satisfies $f_x(x) = 0$.

Then

$$\Lambda = \mathbb{C}\text{-span}\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^\Theta\}, \quad \text{where} \quad s_{\boldsymbol{\lambda}} = \prod_{\varphi \in \Theta} s_{\boldsymbol{\lambda}(\varphi)}(\varphi). \tag{3.4}$$

Let C_n denote the set of complex-valued class functions of the group U_n, and for $||\boldsymbol{\mu}|| = n$, let $\pi_{\boldsymbol{\mu}} : U_n \to \mathbb{C}$ be the class function which is 1 on $c_{\boldsymbol{\mu}}$ and 0 elsewhere. Then the $\pi_{\boldsymbol{\mu}}$ form a $\mathbb{C}$-basis for C_n. By Proposition 2.2, the usual inner product on class functions of finite groups, $\langle \cdot, \cdot \rangle : C_n \times C_n \to \mathbb{C}$, satisfies

$$\langle \pi_{\boldsymbol{\mu}}, \pi_{\boldsymbol{\lambda}} \rangle = a_{\boldsymbol{\mu}}^{-1} \delta_{\boldsymbol{\mu}\boldsymbol{\lambda}}.$$

For $\alpha_i \in C_{n_i}$, Ennola [**5**] defined a product $\alpha_1 \star \alpha_2 \in C_{n_1+n_2}$, which takes the following value on the conjugacy class $c_{\boldsymbol{\lambda}}$:

$$\alpha_1 \star \alpha_2(c_{\boldsymbol{\lambda}}) = \sum_{||\boldsymbol{\mu}_i|| = n_i} g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2} \alpha_1(c_{\boldsymbol{\mu}_1}) \alpha_2(c_{\boldsymbol{\mu}_2}),$$

where $g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2}$ is the product of Hall polynomials (see [**14**, Chapter II])

$$g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2} = \prod_{f \in \Phi} g^{\boldsymbol{\lambda}(f)}_{\boldsymbol{\mu}_1(f) \boldsymbol{\mu}_2(f)}((-q)^{d(f)}).$$

Extend the inner product to $C = \bigoplus_{n\geq 0} C_n$, by requiring the components C_n and C_m to be orthogonal for $n \neq m$. This gives C a graded $\mathbb{C}$-algebra structure. The *characteristic map* is

$$\begin{array}{rccll} \mathrm{ch}: & C & \longrightarrow & \Lambda & \\ & \pi_{\boldsymbol{\mu}} & \mapsto & P_{\boldsymbol{\mu}}, & \quad \text{for } \boldsymbol{\mu} \in \mathcal{P}^{\Phi}. \end{array}$$

Proposition 3.1. *Let multiplication in the character ring C of U_n be given by $\star$. Then the characteristic map $\mathrm{ch}: C \to \Lambda$ is an isometric isomorphism of graded $\mathbb{C}$-algebras.*

Following the work of Green [**7**] on the general linear group, Ennola was able to obtain the following result. We may follow the proof in Macdonald [**14**, IV.4] on the general linear group case, making the appropriate changes.

Proposition 3.2 (Ennola). *The set $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^{\Theta}\}$ is an orthonormal basis for Λ.*

Now let $\chi^{\boldsymbol{\lambda}} \in R$ be class functions so that $\chi^{\boldsymbol{\lambda}}(1) > 0$ and $\mathrm{ch}(\chi^{\boldsymbol{\lambda}}) = \pm s_{\boldsymbol{\lambda}}$. Ennola conjectured that $\{\chi^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^{\Theta}\}$ is the set of irreducible characters of U_n. He pointed out that if one could show that the product $\star$ takes virtual characters to virtual characters, then the conjecture would follow. There is no known direct proof of this fact, however. Significant progress on Ennola's conjecture was only made after the work of Deligne and Lusztig [**2**].

3.2. Deligne-Lusztig Induction. Let $T_\nu \cong \mathbb{M}_{\nu_1} \times \cdots \times \mathbb{M}_{\nu_\ell}$ be a maximal torus of U_n. If $t \in T_\nu$, then t is conjugate to

$$J_{(1^{m_1})}(f_1) \oplus \cdots \oplus J_{(1^{m_\ell})}(f_\ell), \quad \text{where } f_i \in \Phi,\ m_i d(f_i) = \nu_i.$$

Define $\boldsymbol{\gamma}_t \in \mathcal{P}^{\Phi}$ by

$$\boldsymbol{\gamma}_t(f) \qquad \text{has parts} \qquad \{m_i \mid f_i = f\}. \tag{3.5}$$

Note that $(\boldsymbol{\gamma}_t)_u(t-1) = \nu$, but in general $t \notin c_{\boldsymbol{\gamma}_t}$.

For $\boldsymbol{\mu} \in \mathcal{P}^{\Phi}$, let $L_{\boldsymbol{\mu}}$, $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\mu}}$ and $T_{\boldsymbol{\gamma}}$ be as in Section 2.3. Let θ be a character of T_ν. The *Deligne-Lusztig character* $R_\nu(\theta) = R_{T_\nu}^{U_n}(\theta)$ is the virtual character of U_n given by

$$\big(R_\nu(\theta)\big)(g) = \sum_{\substack{t \in T_\nu \\ \boldsymbol{\gamma}_t \in \mathcal{P}_s^{\boldsymbol{\mu}}}} \theta(t) Q_{T_{\boldsymbol{\gamma}_t}}^{L_{\boldsymbol{\mu}}}(u),$$

where $g \in c_{\boldsymbol{\mu}}$ has Jordan decomposition $g = su$ (thus, by Lemma 2.1 $C_{U_n}(s) \cong L_{\boldsymbol{\mu}}$), and $Q_{T_{\boldsymbol{\gamma}_t}}^{L_{\boldsymbol{\mu}}}(u)$ is a Green function for the unitary group (see, for example, [**1**]).

Deligne and Lusztig proved that the $R_\nu(\theta)$ span the class functions of U_n,

$$C_n = \mathbb{C}\text{-span}\{R_\nu(\theta) \mid \nu \in \mathcal{P}_n, \theta \in \mathrm{Hom}(T_\nu, \mathbb{C}^\times)\},$$

so we may define *Deligne-Lusztig induction* by

$$\begin{array}{rccc} R_{U_m \oplus U_n}^{U_{m+n}}: & C_m \otimes C_n & \longrightarrow & C_{m+n} \\ & R_\alpha^{U_m}(\theta_\alpha) \otimes R_\beta^{U_n}(\theta_\beta) & \mapsto & R_{T_\alpha \oplus T_\beta}^{U_{m+n}}(\theta_\alpha \otimes \theta_\beta), \end{array} \tag{3.6}$$

for $\alpha \in \mathcal{P}_m$, $\beta \in \mathcal{P}_n$, $\theta_\alpha \in \mathrm{Hom}(T_\alpha, \mathbb{C})$, and $\theta_\beta \in \mathrm{Hom}(T_\beta, \mathbb{C})$.

Let Λ and C be as in Section 3.1, except we now give C a graded $\mathbb{C}$-algebra structure using Deligne-Lusztig induction. That is, we define a multiplication $\circ$ on C by

$$\chi \circ \eta = R_{U_m \oplus U_n}^{U_{m+n}}(\chi \otimes \eta), \quad \text{for } \chi \in C_m \text{ and } \eta \in C_n.$$

We recall the characteristic map defined in Section 3.1,

$$\begin{array}{rccll} \mathrm{ch}: & C & \longrightarrow & \Lambda & \\ & \pi_{\boldsymbol{\mu}} & \mapsto & P_{\boldsymbol{\mu}} & \quad \text{for } \boldsymbol{\mu} \in \mathcal{P}^{\Phi}. \end{array}$$

It is immediate that ch is an isometric isomorphism of vector spaces, but it is not yet clear if ch is also a ring homomorphism when C has multiplication given by Deligne-Lusztig induction.

3.3. The Ennola conjecture. In this section we summarize the remaining steps that are necessary to obtain the proof of the Ennola conjecture. First, we must compute $\mathrm{ch}(R_\nu(\theta))$, and to do so we need to write the Green functions $Q_{T_{\boldsymbol{\gamma}}}^{L_{\boldsymbol{\mu}}}(u)$ as polynomials in q. These Green functions turn out to be those of the general linear group, except with q replaced by $-q$, which is the essence of Ennola's original idea. This fact was proven by Hotta and Springer [**9**] for the case that $p = \mathrm{char}(\mathbb{F}_q)$ is large compared to n, and was finally proven in full generality by Kawanaka [**11**].

Theorem 3.1 (Hotta-Springer, Kawanaka). *The Green functions for the unitary group are given by* $Q_{T_{\boldsymbol{\gamma}}}^{L_{\boldsymbol{\mu}}}(u) = Q_{\boldsymbol{\gamma}}^{\boldsymbol{\mu}}(-q)$, *where*

$$Q_{\boldsymbol{\gamma}}^{\boldsymbol{\mu}}(-q) = \prod_{f\in\Phi_{\boldsymbol{\mu}}} Q_{\boldsymbol{\gamma}(f)}^{\boldsymbol{\mu}(f)}((-q)^{d(f)}),$$

and $Q_\gamma^\mu(q)$ *is the classical Green function as in (3.2).*

For $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$ and $\theta = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_\ell$ a character of T_ν, define

$$p_{\boldsymbol{\mu}_{\nu\theta}} = \prod_{\varphi\in\Theta} p_{\boldsymbol{\mu}_{\nu\theta}(\varphi)}(\varphi), \quad \text{where} \quad \boldsymbol{\mu}_{\nu\theta}(\varphi) = (\nu_i/|\varphi| \ \mid\ \theta_i \in \varphi).$$

From Theorem 3.1 and the machinery of the characteristic map, we obtain the following.

Theorem 3.2. *Let* $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$, $\theta = \theta_1\otimes\theta_2\otimes\cdots\otimes\theta_\ell$ *be a character of* T_ν, *and* $\boldsymbol{\nu} = \boldsymbol{\mu}_{\nu\theta} \in \mathcal{P}^\Theta$. *Then*

$$\mathrm{ch}\big(R_\nu(\theta)\big) = (-1)^{||\boldsymbol{\nu}||-\ell(\boldsymbol{\nu})} p_{\boldsymbol{\nu}}.$$

Corollary 3.1. *Let multiplication in the character ring* C *of* U_n *be given by* $\circ$. *Then the characteristic map* $\mathrm{ch} : C \to \Lambda$ *is an isometric isomorphism of graded* $\mathbb{C}$*-algebras.*

An immediate consequence is that the graded multiplication that Ennola originally defined on C is exactly Deligne-Lusztig induction, or

Corollary 3.2. *Let* $\chi \in C_m$ *and* $\eta \in C_n$. *Then*

$$\chi \circ \eta = \chi \star \eta.$$

We therefore have the advantage of taking either definition when convenience demands.

For $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, let $L_{\boldsymbol{\lambda}}$, $W_{\boldsymbol{\lambda}}$, and $T_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\lambda}}$, be as in Section 2.3.

Note that the combinatorics of $\boldsymbol{\gamma}$ almost specifies character θ_γ of $T_{\boldsymbol{\gamma}}$ in the sense that

$$\theta_{\boldsymbol{\gamma}}(T_{\boldsymbol{\gamma}}(\varphi)) = \theta_\varphi(T_{\boldsymbol{\gamma}}(\varphi)), \qquad \text{for some } \theta_\varphi \in \varphi.$$

In fact, we may define

$$R_{\boldsymbol{\gamma}} = R_{T_{\boldsymbol{\gamma}}}^{U_n}(\theta_{\boldsymbol{\gamma}}) = \mathrm{ch}^{-1}\big((-1)^{||\boldsymbol{\gamma}||-\ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}}\big), \tag{3.7}$$

where $\theta_{\boldsymbol{\gamma}}$ is any choice of the θ_φ's.

For every $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$ there exists a character $\omega^{\boldsymbol{\lambda}}$ of $W_{\boldsymbol{\lambda}}$ defined by

$$\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma}) = \prod_{\varphi\in\Theta} \omega^{\boldsymbol{\lambda}(\varphi)}(\boldsymbol{\gamma}(\varphi)),$$

where $\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})$ is the value of $\omega^{\boldsymbol{\lambda}}$ on the conjugacy class $c_{\boldsymbol{\gamma}}$ corresponding to $\boldsymbol{\gamma} \in \mathcal{P}_s^\Theta$.

In [**13**], Lusztig and Srinivasan decomposed the irreducible characters of U_n as linear combinations of Deligne-Lusztig characters, as follows.

Theorem 3.3 (Lusztig-Srinivasan). *Let* $\boldsymbol{\lambda} \in \mathcal{P}_n^\Theta$. *Then there exists* $\tau'(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$ *such that the class function*

$$R(\boldsymbol{\lambda}) = (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+\sum_{\varphi\in\Theta}|\boldsymbol{\lambda}(\varphi)|+|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} R_{\boldsymbol{\gamma}}$$

is an irreducible character of U_n *(*$z_{\boldsymbol{\gamma}}$ *is as in (2.14)).*

Finally, we obtain the Ennola Conjecture.

Corollary 3.3 (Ennola Conjecture). *For $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$, there exists $\tau(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$ such that*

$$\left\{\mathrm{ch}^{-1}\left((-1)^{\tau(\boldsymbol{\lambda})} s_{\boldsymbol{\lambda}}\right) \;\mid\; \boldsymbol{\lambda} \in \mathcal{P}_n^{\Theta}\right\}$$

is the set of irreducible characters of U_n.

Proof. By Theorem 3.3 and Theorem 3.2,

$$\begin{aligned}
\mathrm{ch}(R(\boldsymbol{\lambda})) &= (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+\sum_{\varphi\in\Theta}|\boldsymbol{\lambda}(\varphi)|+|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}}(-1)^{n-\ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}} \\
&= (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+n+\sum_{\varphi\in\Theta}|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}}(-1)^{\sum_{\varphi\in\Theta}|\boldsymbol{\lambda}(\varphi)|-\ell(\boldsymbol{\gamma})} p_{\boldsymbol{\gamma}}
\end{aligned}$$

Note that the sign character $\omega^{\boldsymbol{\lambda}_s}$ of $W_{\boldsymbol{\lambda}}$ acts by

$$\omega^{\boldsymbol{\lambda}_s}(\boldsymbol{\gamma}) = (-1)^{\sum_{\varphi\in\Theta}|\boldsymbol{\gamma}(\varphi)|-\ell(\boldsymbol{\gamma})},$$

and that $\omega^{\boldsymbol{\lambda}} \otimes \omega^{\boldsymbol{\lambda}_s} = \omega^{\boldsymbol{\lambda}'}$, so since $\boldsymbol{\gamma} \in \mathcal{P}_s^{\boldsymbol{\lambda}}$,

$$\begin{aligned}
\mathrm{ch}(R(\boldsymbol{\lambda})) &= (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+n+\sum_{\varphi\in\Theta}|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{(\omega^{\boldsymbol{\lambda}}\otimes\omega^{\boldsymbol{\lambda}_s})(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} p_{\boldsymbol{\gamma}} \\
&= (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+n+\sum_{\varphi\in\Theta}|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} \sum_{\boldsymbol{\gamma}\in\mathcal{P}_s^{\boldsymbol{\lambda}}} \frac{\omega^{\boldsymbol{\lambda}'}(\boldsymbol{\gamma})}{z_{\boldsymbol{\gamma}}} p_{\boldsymbol{\gamma}},
\end{aligned}$$

and by applying (3.1) to a product over Θ,

$$= (-1)^{\tau'(\boldsymbol{\lambda})+\lfloor n/2\rfloor+n+\sum_{\varphi\in\Theta}|\varphi|\lceil|\boldsymbol{\lambda}(\varphi)|/2\rceil} s_{\boldsymbol{\lambda}'}. \qquad \square$$

Remark. There are at least two natural ways to index the irreducible characters of U_n by Θ-partitions: Theorem 3.3 gives a natural indexing by Θ-partitions, but Corollary 3.3 indicates that the conjugate choice is equally natural. Since we like to think of Schur functions as irreducible characters, we have chosen the latter indexing. However, several references, including Ennola [**5**], Ohmori [**15**], and Henderson [**8**], make use of the former.

4. Characters degrees

In this section, we calculate the degrees of the irreducible characters of the finite unitary group and find several character degree sums.

Let $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$, and suppose $\square \in \boldsymbol{\lambda}$ is in position (i,j) in $\boldsymbol{\lambda}(\varphi)$ for some $\varphi \in \Theta$. The *hook length* $\mathbf{h}(\square)$ of $\square$ is

$$\mathbf{h}(\square) = |\varphi| h(\square), \quad \text{where} \quad h(\square) = \boldsymbol{\lambda}(\varphi)_i - \boldsymbol{\lambda}(\varphi)'_j - i - j + 1,$$

is the usual hook length for partitions.

For $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$, let

$$\eta^{\boldsymbol{\lambda}} = \mathrm{ch}^{-1}(s_{\boldsymbol{\lambda}}).$$

Adapting the computations in [**14**, IV.6], we obtain the following result.

Theorem 4.1. *Let $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$ and let 1 be the identity in $U_{||\boldsymbol{\lambda}||}$. Then*

$$\eta^{\boldsymbol{\lambda}}(1) = (-1)^{\tau(\boldsymbol{\lambda})} q^{n(\boldsymbol{\lambda}')} \frac{\displaystyle\prod_{1\leq i\leq ||\boldsymbol{\lambda}||} (q^i - (-1)^i)}{\displaystyle\prod_{\square\in\boldsymbol{\lambda}} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})},$$

where $\tau(\boldsymbol{\lambda}) = ||\boldsymbol{\lambda}||(||\boldsymbol{\lambda}||+3)/2 + n(\boldsymbol{\lambda}) \equiv \lfloor ||\boldsymbol{\lambda}||/2\rfloor + n(\boldsymbol{\lambda}) \pmod 2$. So for each $\boldsymbol{\lambda}$, we have $\chi^{\boldsymbol{\lambda}} = (-1)^{\tau(\boldsymbol{\lambda})}\eta^{\boldsymbol{\lambda}}$.

The following result follows from Theorem 4.1 and the Littlewood-Richardson rule.

COROLLARY 4.1. *Let* $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^{\Theta}$. *Then* $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ *is a character if and only if every* $\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}$ *such that* $c^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}} > 0$ *satisfies*

$$n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) \equiv n(\boldsymbol{\lambda}) + ||\boldsymbol{\mu}||\,||\boldsymbol{\nu}|| \pmod 2$$

Following a similar approach to [**14**, IV.6, Example 5], we consider the coefficient of t^m in the series

$$S = \sum_{\boldsymbol{\lambda} \in \mathcal{P}^{\Theta}} (-1)^{n(\boldsymbol{\lambda}) + ||\boldsymbol{\lambda}||} \delta(s_{\boldsymbol{\lambda}}) t^{||\boldsymbol{\lambda}||},$$

where $n(\boldsymbol{\lambda})$ is as in (2.7), and we obtain the following result.

THEOREM 4.2. *The sum of the degrees of the complex irreducible characters of* U_m *is given by*

$$\sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots\big(q^m + \frac{(1-(-1)^m)}{2}\big).$$

Write $f_{U_m}(q) = \sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}(1)$. The polynomial $f_{G_m}(q)$ expressing the sum of the degrees of the complex irreducible characters of $G_m = \mathrm{GL}(m, \mathbb{F}_q)$, was computed in [**6**] for odd q and in [**12**] and Example 6 of [**14**, IV.6] for general q. From these results we see that

$$f_{U_m}(q) = (-1)^{m(m+1)/2} f_{G_m}(-q),$$

another example of Ennola duality.

Gow [**6**] and Klyachko [**12**] proved that the sum of the degrees of the complex irreducible characters of G_n is equal to the number of symmetric matrices in G_n. We obtain the same result for U_n by applying Theorem 4.2 and a counting argument.

COROLLARY 4.2. *The sum of the degrees of the complex irreducible characters of* $\mathrm{U}(n, \mathbb{F}_{q^2})$ *is equal to the number of symmetric matrices in* $\mathrm{U}(n, \mathbb{F}_{q^2})$.

A Θ-partition $\boldsymbol{\lambda}$ is *even* if every part of $\boldsymbol{\lambda}(\varphi)$ is even for every $\varphi \in \Theta$.

THEOREM 4.3. *The sum of the degrees of the complex irreducible characters of* U_{2m} *corresponding to* $\boldsymbol{\lambda}$ *such that* $\boldsymbol{\lambda}'$ *is even is given by*

$$\sum_{\substack{||\boldsymbol{\lambda}||=2m \\ \boldsymbol{\lambda}' \text{ even}}} \chi^{\boldsymbol{\lambda}}(1) = (q+1)q^2(q^3+1)\cdots q^{2m-2}(q^{2m-1}+1) = \frac{|\mathrm{U}(2m, \mathbb{F}_{q^2})|}{|\mathrm{Sp}(2m, \mathbb{F}_q)|}.$$

Write $g_{U_m}(q) = \sum_{||\boldsymbol{\lambda}||=2m, \boldsymbol{\lambda}' \text{ even}} \chi^{\boldsymbol{\lambda}}(1)$, and let $g_{G_m}(q)$ denote the corresponding sum for G_m. The polynomial $g_{G_m}(q)$ was calculated in Example 7 of [**14**, IV.6], and similar to the previous example, we see that we have

$$g_{U_m}(q) = (-1)^m g_{G_m}(-q).$$

In the case that q is odd, Proposition 4.3 follows from the following stronger result obtained by Henderson [**8**]. Let $Sp_{2n} = \mathrm{Sp}(2n, \mathbb{F}_q)$ be the symplectic group over the finite field $\mathbb{F}_q$.

THEOREM 4.4 (Henderson). *Let q be odd. The decomposition of* $\mathrm{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1})$ *into irreducibles is given by*

$$\mathrm{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1}) = \sum_{\substack{||\boldsymbol{\lambda}||=2n \\ \boldsymbol{\lambda}' \text{even}}} \chi^{\boldsymbol{\lambda}}.$$

The fact that Proposition 4.3 holds for all q suggests that Theorem 4.4 should as well.

5. A Deligne-Lusztig model

A *model* of a finite group G is a representation ρ, which is a direct sum of representations induced from one-dimensional representations of subgroups of G, such that every irreducible representation of G appears as a component with multiplicity 1 in the decomposition of ρ.

Klyachko [**12**] and Inglis and Saxl [**10**] obtained a model for $\mathrm{GL}(n, \mathbb{F}_q)$, where the induced representations can be written as a Harish-Chandra product of Gelfand-Graev characters and the permutation character of the finite symplectic group.

In this section we show that the same result is true for the finite unitary group, except the Harish-Chandra product is replaced by Deligne-Lusztig induction. The result is therefore not a model for $\mathrm{U}(n, \mathbb{F}_q)$

in the finite group character induction sense, but rather from the Deligne-Lusztig point of view.

Let $U_n' = \mathrm{GL}(n, \mathbb{K})^{F'}$ as in (2.3), and let

$$B_< = \{u \in U_n' \mid u \text{ unipotent and uppertriangular}\} \subseteq U_n'.$$

Fix a nontrivial character $\psi : \mathbb{K}_2^+ \to \mathbb{C}^\times$ of the additive group of the field $\mathbb{K}_2$ such that the restriction to the subgroup $\{x \in \mathbb{K}_2 \mid x^q + x = 0\}$ is also nontrivial. The map $\psi_{(n)} : B_< \to \mathbb{C}$ given by

$$\psi_{(n)}(u) = \psi\big(u_{12} + \cdots + u_{\lfloor n/2\rfloor -1, \lfloor n/2\rfloor} + u_{\lfloor n/2\rfloor, \lceil n/2\rceil+1}\big), \text{ for } u = (u_{ij}) \in B_<,$$

is a linear character of $B_<$. Then

$$\Gamma'_{(n)} = \mathrm{Ind}_{B_<}^{U_n'}(\psi_{(n)})$$

is the *Gelfand-Graev character* of U_n'. Let $\Gamma_{(n)}$ be the corresponding Gelfand-Graev character of $U_n = \mathrm{GL}(n, \mathbb{K})^F$. For $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, define

$$\mathrm{ht}(\boldsymbol{\lambda}) = \max\{\ell(\boldsymbol{\lambda}(\varphi)) \mid \varphi \in \Theta\}.$$

The following appears in Section 5.2 of [**15**], but we can also give a proof using the characteristic map.

Theorem 5.1. *The decomposition of $\Gamma_{(m)}$ into irreducibles is given by*

$$\Gamma_{(m)} = \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_m^\Theta \\ \mathrm{ht}(\boldsymbol{\lambda})=1}} \chi^{\boldsymbol{\lambda}}.$$

For a partition λ, let $o(\lambda)$ denote the number of odd parts of λ, and for $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, let $o(\boldsymbol{\lambda}) = \sum_{\varphi\in\Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi))$.

Theorem 5.2. *Let q be odd. For each r such that $0 \leq 2r \leq m$,*

$$\Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{o(\boldsymbol{\lambda}')=m-2r} \chi^{\boldsymbol{\lambda}}.$$

Furthermore,

$$\sum_{0\leq 2r\leq m} \Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{||\boldsymbol{\lambda}||=m} \chi^{\boldsymbol{\lambda}}$$

Proof. Suppose $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^\Theta$, such that $\mathrm{ht}(\boldsymbol{\mu}) = 1$ and $\boldsymbol{\nu}'$ is even. From the characteristic map, Corollary 3.3, and Pieri's formula [**14**, I.5.16],

$$\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}} = (-1)^{\tau(\boldsymbol{\mu})+\tau(\boldsymbol{\nu})} \sum_{\boldsymbol{\lambda}} \chi^{\boldsymbol{\lambda}}, \tag{5.1}$$

where the sum is taken over all $\boldsymbol{\lambda}$ such that for every $\varphi \in \Theta$, $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip.

We now use Corollary 4.1 to show that $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character. As $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip, the part $\boldsymbol{\lambda}(\varphi)_i'$ is either $\boldsymbol{\nu}(\varphi)_i'$ or $\boldsymbol{\nu}(\varphi)_i' + 1$ for every $i = 1, 2, \ldots, \ell(\boldsymbol{\lambda}(\varphi))$. By assumption, $\boldsymbol{\nu}'$ is even, so $\boldsymbol{\nu}(\varphi)_i'$ is even for every $\varphi \in \Theta$, and so

$$\binom{\boldsymbol{\nu}(\varphi)_i' + 1}{2} = \boldsymbol{\nu}(\varphi)_i' + \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \equiv \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \pmod 2.$$

Thus, $n(\boldsymbol{\lambda}(\varphi)) = \sum_i \binom{\boldsymbol{\lambda}(\varphi)_i'}{2} \equiv n(\boldsymbol{\nu}(\varphi)) \pmod 2$. The assumption $\mathrm{ht}(\boldsymbol{\mu}) = 1$ implies $n(\boldsymbol{\mu}(\varphi)) = 0$, and since $||\boldsymbol{\nu}||$ is even,

$$n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) \equiv n(\boldsymbol{\lambda}) + ||\boldsymbol{\mu}||\,||\boldsymbol{\nu}|| \pmod 2.$$

By Corollary 4.1, $\chi^{\boldsymbol{\mu}} \circ \chi^{\boldsymbol{\nu}}$ is a character.

Use the decompositions of Theorem 4.4 and Theorem 5.1 in the product (5.1) to observe that the irreducible characters $\chi^{\boldsymbol{\lambda}}$ in the decomposition of $\Gamma_{m-2r} \circ \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1})$ are indexed by $\boldsymbol{\lambda} \in \mathcal{P}_m^\Theta$ such that for every φ, $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$-strip, where $||\boldsymbol{\mu}|| = m - 2r$, for some $\boldsymbol{\nu}(\varphi)$ such that $\boldsymbol{\nu}(\varphi)'$ is even. Then the number of odd parts of $\boldsymbol{\lambda}(\varphi)'$ is exactly $|\boldsymbol{\mu}(\varphi)|$, and so the $\boldsymbol{\lambda}$ in the decomposition must satisfy $\sum_{\varphi\in\Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi)') = ||\boldsymbol{\mu}|| = m - 2r$. $\square$

References

1. R. Carter, Finite groups of Lie type: conjugacy classes and complex characters. John Wiley and Sons, 1985.
2. P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* (2) **103** (1976), no. 1, 103–161.
3. F. Digne and J. Michel, Foncteurs de Lusztig et caractères des groupes linéaires et unitaires sur un corps fini, *J. Algebra* **107** (1987), no. 1, 217–255.
4. V. Ennola, On the conjugacy classes of the finite unitary groups, *Ann. Acad. Sci. Fenn. Ser. A I No.* **313** (1962), 13 pages.
5. V. Ennola, On the characters of the finite unitary groups, *Ann. Acad. Sci. Fenn. Ser. A I No.* **323** (1963), 35 pages.
6. R. Gow, Properties of the characters of the finite general linear group related to the transpose-inverse involution, *Proc. London Math. Soc. (3)* **47** (1983), no. 3, 493–506.
7. J.A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), no. 2, 402–447.
8. A. Henderson, Symmetric subgroup invariants in irreducible representations of G^F, when $G = GL_n$, *J. Algebra* **261** (2003), no. 1, 102–144.
9. R. Hotta and T.A. Springer, A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups, *Invent. Math.* **41** (1977), no. 2, 113–127.
10. N.F.J. Inglis and J. Saxl, An explicit model for the complex representations of the finite general linear groups, *Arch. Math. (Basel)* **57** (1991), no. 5, 424–431.
11. N. Kawanaka, Generalized Gel'fand-Graev representations and Ennola duality, In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, 175–206, *Adv. Stud. Pure Math.*, 6, North-Holland, Amsterdam, 1985.
12. A. A. Klyachko, Models for complex representations of the groups GL(n, q), *Mat. Sb. (N.S.)* **120(162)** (1983), no. 3, 371–386.
13. G. Lusztig and B. Srinivasan, The characters of the finite unitary groups, *J. Algebra* **49** (1977), no. 1, 167–171.
14. I.G. Macdonald, Symmetric functions and Hall polynomials. Second edition. With Contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
15. Z. Ohmori, On a Zelevinsky theorem and the Schur indices of the finite unitary groups, *J. Math. Sci. Univ. Tokyo* **2** (1997), no. 2, 417–433.
16. G. E. Wall, On the conjugacy classes in the unitary, orthogonal and symplectic groups, *J. Austral. Math. Soc.* **3** (1962), 1–62.

Department of Mathematics, Stanford University, Stanford, CA 94305-2125
E-mail address: thiem@math.stanford.edu

Department of Mathematics, The University of Arizona, 617 N Santa Rita Ave, Tucson, AZ 85721-0089
E-mail address: vinroot@math.arizona.edu