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Pieri's Formula for Generalized Schur Polynomials

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Abstract. We define a generalization of Schur polynomials as a expansion coefficient of generalized Schur operators. We generalize the Pieri's formula to the generalized Schur polynomials.

Résumé. Nous définissons une généralisation de polynômes de Schur comme un coefficient de l'expansion d'opérateurs de Schur généralisés. Nous généralisons la formule du Pieri aux polynômes de Schur généralisés.

1. Introduction

Young's lattice is a prototypical example of differential posets defined by Stanley [**9**]. Young's lattice has so called the Robinson correspondence, the correspondence between permutations and pairs of standard tableaux whose shapes are the same Young diagram. This correspondence is generalized for differential posets or dual graphs (that is a generalization of differential posets) by Fomin [**3**].

Young's lattice also has the Robinson-Schensted-Knuth correspondence, the correspondence between certain matrices and pairs of semi-standard tableaux. Fomin generalizes the method of the Robinson correspondence to that of the Robinson-Schensted-Knuth correspondence in his paper [**4**]. The operators in Fomin [**4**] are called generalized Schur operators. We can define a generalization of Schur polynomials by generalized Schur operators.

A complete symmetric polynomial is a Schur polynomial associated with a Young diagram consisting of only one row. Schur polynomials satisfy the Pieri's formula, the formula describing products of a complete symmetric polynomial and a Schur polynomial as sums of Schur polynomials like the following;

$$h_i(t_1,\dots,t_n)s_\lambda(t_1,\dots,t_n)=\sum_\mu s_\mu(t_1,\dots,t_n),$$

where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same column, h_i is the i-th complete symmetric polynomial and s_λ is the Schur polynomial associated with λ.

We generalize the Pieri's formula to generalized Schur polynomials (Theorem 3.2 and Proposition 3.3).

2. Definition

We introduce two types of polynomials in this section. One of them is a generalization of Schur polynomials. The other is a generalization of complete symmetric polynomials. We will show Pieri's formula for these polynomials in Section 3.

2.1. Schur Operators. First we recall generalized Schur operators defined by Fomin [**4**]. We define a generalization of Schur function as expansion coefficients of generalized Schur operators.

Let K be a field of characteristic zero that contains all formal power series of variables $t, t', t_1, t_2, \dots$. Let V_i be finite dimensional K-vector spaces for all $i \in \mathbb{Z}$. Fix a basis Y_i of each V_i so that $V_i = KY_i$ and

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$V = KY$ where $Y = \bigcup_i Y_i$. The *rank function* on V which maps $\lambda \in V_i$ to i is denoted by ρ. We say that V *has the minimum* $\varnothing$ if $Y_i = \emptyset$ for $i < 0$ and $Y_0 = \{\varnothing\}$.

For a sequence $\{A_i\}$ and a formal variable x, we write $A(x)$ for the generating function $\sum_{i \geq 0} A_i x^i$.

Hereafter, for $i > 0$, let D_i and U_i be linear operators on V satisfying $\rho(U_i\lambda) = \rho(\lambda) + i$ and $\rho(D_i\lambda) = \rho(\lambda) - i$ for $\lambda \in Y$. In other words, the images $D_j(V_i)$ and $U_j(V_i)$ of V_i by D_j and U_j are contained in V_{i-j} and V_{i+j} for $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ respectively. We call D_i or $D(t)$ and U_i or $U(t)$ down operators and up operators.

DEFINITION 2.1. Let $\{a_i\}$ be a sequence of elements of K. Down and up operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are said to be *generalized Schur operators* if the equation $D(t')U(t) = a(tt')U(t)D(t')$ holds.

We write $*$ for the conjugation with respect to the natural pairing $\langle\ ,\ \rangle$ in KY. For all i, U_i^* and D_i^* act as down and up operators, respectively. By definition, $U^*(t')D^*(t) = a(tt')D^*(t)U^*(t')$ if $D(t')U(t) = U(t)D(t')a(tt')$. Hence down and up operators $U^*(t_n)\cdots U^*(t_1)$ and $D^*(t_1)\cdots D^*(t_n)$ are also generalized Schur operators when $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators.

Let down and up operators $D(t)$ and $U(t)$ be generalized Schur operators with $\{a_i\}$ where $a_0 \neq 0$. Since $a_0 \neq 0$, there exists $\{b_i\}$ such that $a(t)b(t) = 1$. Hence the equation $D(t')U(t) = a(tt')U(t)D(t')$ implies

$$U(t)D(t') = b(tt')D(t')U(t) \tag{2.1}$$

and

$$D^*(t')U^*(t) = b(tt')U^*(t)D^*(t'). \tag{2.2}$$

Let ρ' be $-\rho$. We take ρ' as rank function for the same vertex set V. For this rank function ρ' and the vector space V, D_i^* and U_i^* act as down and up operators, respectively. Since they satisfy the equation (2.2), down and up operators $D^*(t)$ and $U^*(t)$ are generalized Schur operators with $\{b_i\}$. Similarly, it follows from the equation (2.1) that down and up operators $U(t)$ and $D(t)$ are also generalized Schur operators with $\{b_i\}$ for ρ' and V.

DEFINITION 2.2. Let $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ be generalized Schur operators. For $\lambda \in V$ and $\mu \in Y$, we write $s^D_{\lambda,\mu}(t_1,\ldots,t_n)$ and $s_U^{\mu,\lambda}(t_1,\ldots,t_n)$ for the coefficient of μ in $D(t_1)\cdots D(t_n)\lambda$ and $U(t_n)\cdots U(t_1)\lambda$, respectively. We call these polynomials $s^D_{\lambda,\mu}(t_1,\ldots,t_n)$ and $s_U^{\mu,\lambda}(t_1,\ldots,t_n)$ *generalized Schur polynomials*.

Generalized Schur polynomials $s^D_{\lambda,\mu}(t_1,\ldots,t_n)$ are symmetric in the case when $D(t)D(t') = D(t')D(t)$ but not symmetric in general. It follows by definition that

$$\begin{aligned} s^D_{\lambda,\mu}(t_1,\ldots,t_n) &= \langle D(t_1)\cdots D(t_n)\lambda, \mu\rangle \\ &= \langle \lambda, D^*(t_n)\cdots D^*(t_1)\mu\rangle \\ &= s_{D^*}^{\lambda,\mu}(t_1,\ldots,t_n) \end{aligned}$$

for $\lambda, \mu \in Y$.

EXAMPLE 2.3. Our prototypical example is Young's lattice $\mathbb{Y}$ that consists of all Young diagrams. Let a basis Y, K-vector space V and rank function ρ be Young lattice $\mathbb{Y}$, the K-vector space $K\mathbb{Y}$ and the ordinal rank function ρ which maps Young diagram λ to the number of boxes in λ. Young's lattice $\mathbb{Y}$ has the minimum $\varnothing$ the Young diagram with no boxes. Define U_i and D_i by $U_i(\mu) = \sum_\lambda \lambda$, where the sum is over all λ's that are obtained from μ by adding i boxes, with no two in the same column; and by $D_i(\lambda) = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by removing i boxes, with no two in the same column. Then the operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{a_i = 1\}$. In this case, both $s^D_{\lambda,\mu}(t_1,\ldots,t_n)$ and $s_U^{\lambda,\mu}(t_1,\ldots,t_n)$ are equal to the skew Schur polynomial $s_{\lambda/\mu}(t_1,\ldots,t_n)$ for λ and $\mu \in \mathbb{Y}$.

2.2. Weighted Complete Symmetric Polynomials. Next we introduce a generalization of complete symmetric polynomials. We define weighted symmetric polynomials inductively.

DEFINITION 2.4. Let $\{a_m\}$ be a sequence of elements of K. We define the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1,\ldots,t_n)$ by

$$h_m^{\{a_n\}}(t_1,\ldots,t_n)=\begin{cases}\sum_{j=0}^{i} h_j^{\{a_m\}}(t_1,\ldots,t_{n-1})h_{i-j}^{\{a_m\}}(t_n), & (\text{for } n>1)\\ h_i^{\{a_m\}}(t_1)=a_it_1^i & (\text{for } n=1).\end{cases}\tag{2.3}$$

By definition, the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1,\ldots,t_n)$ is a homogeneous symmetric polynomial of degree i.

EXAMPLE 2.5. When a_i equal 1 for all i, $h_j^{\{1,1,\ldots\}}(t_1,\ldots,t_n)$ equals the complete symmetric polynomial $h_j(t_1,\ldots,t_n)$. In this case, the formal power series $\sum_i h_i(t)$ equals the generating function $a(t)=\sum_i t^i=\frac{1}{1-t}$.

EXAMPLE 2.6. When a_i equal $\frac{1}{i!}$ for all i, $h_j^{\{\frac{1}{m!}\}}(t_1,\ldots,t_n)=\frac{1}{i!}(t_1+\cdots+t_n)^i$ and $\sum_j h_j^{\{\frac{1}{m!}\}}(t)=\exp(t)=a(t)$.

In general, the formal power series $\sum_i h_i^{\{a_m\}}(t)$ equals the generating function $a(t)=\sum a_it^i$ by the definition of weighted complete symmetric polynomials. It follows from the equation (2.3) that $a(t_1)a(t_2)=\sum_i h_i^{\{a_m\}}(t_1)\sum_j h_j^{\{a_m\}}(t_2)=\sum_j h_j^{\{a_m\}}(t_1,t_2)$. Since the weighted complete symmetric polynomials satisfy the equation (2.3),

$$\begin{aligned}a(t_1)\cdots a(t_{n-1})a(t_n) &= \sum_i h_i^{\{a_m\}}(t_1,\ldots,t_{n-1})\sum_j h_j^{\{a_m\}}(t_n)\\ &= \sum_i\sum_{k=0}^{i} h_{i-k}^{\{a_m\}}(t_1,\ldots,t_{n-1})h_k^{\{a_m\}}(t_n)\\ &= \sum_i h_i^{\{a_m\}}(t_1,\ldots,t_n)\end{aligned}$$

if $a(t_1)\cdots a(t_{n-1})=\sum h_i^{\{a_m\}}(t_1,\ldots,t_{n-1})$. Hence

$$a(t_1)\cdots a(t_n)=\sum_i h_i^{\{a_m\}}(t_1,\ldots,t_n)$$

as in the case when $a_i=1$ for all i. It follows from this relation that $h_0^{\{a_m\}}(t_1,\ldots,t_n)=a_0^n$.

3. Main Theorem

We show some properties of generalized Schur polynomials and weighted complete symmetric polynomials in this section. We show Pieri's formula (Theorem 3.2 and Proposition 3.3) generalized to our polynomials, the main results in this paper.

First we describe the commuting relation of U_i and $D(t_1)\cdots D(t_n)$. This relation implies Pieri's formula for our polynomials. It also follows from this relation that the weighted complete symmetric polynomials are written as linear combinations of generalized Schur polynomials when V has the minimum.

PROPOSITION 3.1. *Generalized Schur operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ with $\{a_i\}$ satisfy*

$$D(t_1)\cdots D(t_n)U_i=\sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)U_jD(t_1)\cdots D(t_n).$$

In the case when the K-vector space V has the minimum $\varnothing$, weighted complete symmetric polynomials are written as linear combinations of generalized Schur polynomials.

PROPOSITION 3.2. *For generalized Schur operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ with $\{a_i\}$ on V with the minimum $\varnothing$, the following equations hold for all $i\geq 0$;*

$$s^D_{U_i\varnothing,\varnothing}(t_1,\ldots,t_n)=h_i^{\{a_m\}}(t_1,\ldots,t_n)d_0^nu_0,$$

where u_0 and $d_0\in K$ satisfy $D_0\varnothing=d_0\varnothing$ and $U_0\varnothing=u_0\varnothing$.

EXAMPLE 3.1. In the prototypical example $\mathbb{Y}$, Proposition 3.2 means that the Schur polynomial $s_{(i)}$ corresponding to Young diagram with only one row equals the complete symmetric polynomial h_i.

Next we consider the case when Y may not have a minimum. It follows from Proposition 3.1 that

$$\langle D(t_1)\cdots D(t_n)U_i\lambda,\mu\rangle = \langle\sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\dots,t_n)U_jD(t_1)\cdots D(t_n)\lambda,\mu\rangle$$

for $\lambda\in V$ and $\mu\in Y$. This equation implies Theorem 3.2, the main result in this paper.

THEOREM 3.2 (Pieri's formula). *For any $\mu\in Y_k$ and any $\lambda\in V$, generalized Schur operators satisfy*

$$s_{U_i\lambda,\mu}^{D}(t_1,\dots,t_n) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\dots,t_n)\sum_{\nu(\in Y_{k-j})}\langle U_j\nu,\mu\rangle s_{\lambda,\nu}^{D}(t_1,\dots,t_n).$$

If Y has the minimum $\varnothing$, this theorem implies the following proposition.

PROPOSITION 3.3. *For all $\lambda\in V$, the following equations hold;*

$$\begin{aligned} s_{U_i\lambda,\varnothing}^{D}(t_1,\dots,t_n) &= h_i^{\{a_m\}}(t_1,\dots,t_n)u_0s_{\lambda,\varnothing}^{D}(t_1,\dots,t_n)\\ &= s_{U_i\varnothing,\varnothing}^{D}(t_1,\dots,t_n)u_0s_{\lambda,\varnothing}^{D}(t_1,\dots,t_n),\end{aligned}$$

where $U_0\varnothing = u_0\varnothing$.

EXAMPLE 3.3. In the prototypical example $\mathbb{Y}$, for any $\lambda\in\mathbb{Y}$, $U_i\lambda$ means the sum of all Young diagrams obtained from λ by adding i boxes, with no two in the same column. Thus Proposition 3.3 is nothing but the classical Pieri's formula. Theorem 3.2 means Pieri's formula for skew Schur polynomials; for a skew Young diagram λ/μ and $i\in\mathbb{N}$,

$$\sum_{\kappa} s_{\kappa/\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i}\sum_{\nu} h_{i-j}(t_1,\dots,t_n)s_{\lambda/\nu}(t_1,\dots,t_n),$$

where the first sum is over all κ's that are obtained from λ by adding i boxes, with no two in the same column; the last sum is over all ν's that are obtained from μ by removing j boxes, with no two in the same column.

4. More Examples

In this section, we see some examples of generalized Schur operators.

4.1. Shifted Shapes. This example is the same as Fomin [**4**, Example 2.1].
Let Y be the set of all shifted shapes. (i.e., $Y = \{\{(i,j)|i\le j<\lambda_i+i\}|\lambda=(\lambda_1>\lambda_2>\cdots),\lambda_i\in\mathbb{N}\}$.)
Down operators D_i are defined for $\lambda\in Y$ by

$$D_i\lambda = \sum_{\nu} 2^{cc_0(\lambda\setminus\nu)}\nu,$$

where $cc_0(\lambda\setminus\nu)$ is the number of connected components of $\lambda\setminus\nu$ which do not intersect the main diagonal; and the sum is over all ν's that are satisfying $\nu\subset\lambda$, $\rho(\nu)=\rho(\lambda)-i$ and $\lambda\setminus\nu$ contains at most one box on each diagonal.

Up operators U_i are defined for $\lambda\in Y$ by

$$U_i\lambda = \sum_{\mu} 2^{cc(\mu\setminus\lambda)}\mu,$$

where $cc(\lambda\setminus\mu)$ is the number of connected components of $\lambda\setminus\nu$; and the sum is over all μ's that are satisfying $\lambda\subset\mu$, $\rho(\mu)=\rho(\lambda)+i$ and $\lambda\setminus\mu$ contains at most one box on each diagonal.

In this case, since down and up operators $D(t)$ and $U(t)$ satisfy

$$D(t')U(t) = \frac{1+tt'}{1-tt'}U(t)D(t'),$$

down and up operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $a_0 = 1$, $a_i = 2$ for $i \geq 1$. In this case, for λ, $\mu \in Y$, generalized Schur polynomials $s^D_{\lambda,\mu}$ and $s^{\lambda,\mu}_U$ are respectively $Q_{\lambda/\mu}(t_1,\ldots,t_n)$ and $P_{\lambda/\mu}(t_1,\ldots,t_n)$, where $P\cdots$ and $Q\cdots$ are the shifted skew Schur polynomials.

In this case, Proposition 3.2 means

$$h_i^{\{1,2,2,2,\ldots\}}(t_1,\ldots,t_n) = \begin{cases} 2Q_{(i)}(t_1,\ldots,t_n) & i > 0 \\ Q_{\varnothing}(t_1,\ldots,t_n) & i = 0 \end{cases}.$$

It also follows that

$$h_i^{\{1,2,2,2,\ldots\}}(t_1,\ldots,t_n) = P_{(i)}(t_1,\ldots,t_n).$$

Proposition 3.3 means

$$h_i^{\{1,2,2,2,\ldots\}}Q_\lambda(t_1,\ldots,t_n) = \sum_\kappa 2^{cc(\lambda\setminus\mu)}Q_\kappa(t_1,\ldots,t_n),$$

where $cc(\lambda\setminus\mu)$ is the number of connected components of $\lambda\setminus\nu$; and the sum is over all μ's that are satisfying $\lambda \subset \mu$, $\rho(\mu) = \rho(\lambda) + i$ and $\lambda \setminus \mu$ contains at most one box on each diagonal.

4.2. Young's Lattice: Dual Identities. This example is the same as Fomin [**4**, Example 2.4]. We take Young's lattice $\mathbb{Y}$ for Y. Up operators U_i are the same as in the prototypical example, (i.e., $U_i\lambda = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same column.) Down operators D'_i are defined by $D'_i = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same row. (In other words, down operators D'_i remove a vertical strip, while up operators U_i add a horizontal strip.)

In this case, since down and up operators $D'(t)$ and $U(t)$ satisfy

$$D'(t')U(t) = (1 + tt')U(t)D'(t'),$$

down and up operators $D'(t_1)\cdots D'(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $a_0 = a_1 = 1$ $a_i = 0$ for $i \geq 2$. In this case, for λ, $\mu \in Y$, generalized Schur polynomials $s^{D'}_{\lambda,\mu}$ equal $s_{\lambda'/\mu'}(t_1,\ldots,t_n)$, where λ' and μ' are the transposes of λ and μ, $s_{\lambda'/\mu'}(t_1,\ldots,t_n)$ are the shifted Schur polynomials.

In this case, Proposition 3.2 means

$$h_i^{\{1,1,0,0,0,\ldots\}}(t_1,\ldots,t_n) = s_{(1^i)}(t_1,\ldots,t_n) = e_i(t_1,\ldots,t_n),$$

where $e_i(t_1,\ldots,t_n)$ stands for the i-th elementally symmetric polynomials.

Proposition 3.3 means

$$e_i(t_1,\ldots,t_n)s_\lambda(t_1,\ldots,t_n) = \sum_\mu s_\mu(t_1,\ldots,t_n),$$

where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same row.

For a skew Young diagram λ/μ and $i \in \mathbb{N}$, Theorem 3.2 means

$$\sum_\kappa s_{\kappa/\mu}(t_1,\ldots,t_n) = \sum_{j=0}^{i}\sum_\nu h_{i-j}(t_1,\ldots,t_n)s_{\lambda/\nu}(t_1,\ldots,t_n),$$

where the first sum is over all κ's that are obtained from λ by adding i boxes, with no two in the same row; the last sum is over all ν's that are obtained from μ by removing j boxes, with no two in the same row.

4.3. Planar Binary Trees. Let F be the monoid of words generated by the alphabet $\{1,2\}$ and 0 denotes the word of length 0. We identify F with a poset by $v \leq vw$ for $v, w \in F$. We call an ideal of poset F a *planar binary tree* or shortly a *tree*. An element of a tree is called a *node* of the tree. We write $\mathbb{T}$ for the set of trees and $\mathbb{T}_i$ for the set of trees of i nodes. For $T \in \mathbb{T}$ and $v \in F$, we define T_v by $T_v := \{w \in T | v \leq w\}$.

Definition 4.1. Let T be a tree and m a positive integer. We call a map $\varphi : T \to \{1,\ldots,m\}$ a *left-strictly-increasing labeling* if

- $\varphi(w) < \varphi(v)$ for $w \in T$ and $v \in T_{w1}$ and
- $\varphi(w) \leq \varphi(v)$ for $w \in T$ and $v \in T_{w2}$.

We call a map $\varphi : T \to \{1,\ldots,m\}$ a *right-strictly-increasing labeling* if

- $\varphi(w) \leq \varphi(v)$ for $w \in T$ and $v \in T_{w1}$ and
- $\varphi(w) < \varphi(v)$ for $w \in T$ and $v \in T_{w2}$.

We call a map $\varphi : T \to \{1, \dots, m\}$ a *binary-searching labeling* if

- $\varphi(w) \geq \varphi(v)$ for $w \in T$ and $v \in T_{w1}$ and
- $\varphi(w) < \varphi(v)$ for $w \in T$ and $v \in T_{w2}$.

First we consider a presentation of increasing labelings as sequences of trees. For a tree $T \in \mathbb{T}$, we call a node $w \in T$ an *l-node* in T if $T_w \subset \{w1^n | n \in \mathbb{N}\}$. A node $w \in T$ is called an *r-node* in T if $T_w \subset \{w2^n | n \in \mathbb{N}\}$. By the definition of increasing labelings φ, the inverse image $\varphi^{-1}(\{1, \dots, n\})$ is a tree for each n. For a right-strictly-increasing labeling φ, $\varphi^{-1}(\{1, \dots, n+1\}) \setminus \varphi^{-1}(\{1, \dots, n\})$ consists of some l-nodes in $\varphi^{-1}(\{1, \dots, n+1\})$. Conversely, for a left-strictly-increasing labeling φ, $\varphi^{-1}(\{1, \dots, n+1\}) \setminus \varphi^{-1}(\{1, \dots, n\})$ consists of some r-nodes in $\varphi^{-1}(\{1, \dots, n+1\})$. Hence we respectively identify right-strictly-increasing and left-strictly-increasing labelings φ with sequences $(\emptyset = T^0, T^1, \dots, T^m)$ of $m+1$ trees such that $T^{i+1} \setminus T^i$ consists of some l-nodes and r-nodes in T^{i+1} for all i.

We define linear operators D and D' on $K\mathbb{T}$ by

$$DT := \sum_{T' \subset T;\ T \setminus T' \text{ consists of some l-nodes}} T',$$
$$D'T := \sum_{T' \subset T;\ T \setminus T' \text{ consists of some r-nodes}} T'.$$

Next we consider binary-searching trees. For $T \in \mathbb{T}$, let s_T be $\{w \in T |$ If $w = v1w'$ then $v2 \notin T$. $w2 \notin T.\}$. The set s_T is a chain. We define S_T by the set of ideals of s_T. For $s \in S_T$, we define $T \ominus s$ by

$$T \ominus s := \begin{cases} T & (s = \emptyset) \\ (T - \max(s)) \ominus (s \setminus \{\max(s)\}) & (s \neq \emptyset), \end{cases}$$

where

$$T - w = (T \setminus T_w) \cup \{wv | w1v \in T_w\}$$

for $w \in T$ such that $w2 \notin T$. There exists the natural inclusion ν from $T - w$ to T defined by

$$\nu(v') = \begin{cases} w1v & v' = wv \in T_w \\ v' & v' \notin T_w. \end{cases}$$

This inclusion induces the inclusion $\nu : T \ominus s \to T$. For a binary-searching labeling φ from $T \in \mathbb{T}$ to $\{1, \dots, m\}$, by the definition of binary-searching labeling, the inverse image $\varphi^{-1}(\{m\})$ is in S_T. The map $\varphi \circ \nu$ induced from φ by the natural inclusion $\nu : T \ominus \varphi^{-1}(\{m\}) \to T$ is a binary-searching labeling from $T \ominus \varphi^{-1}(\{m\})$ to $\{1, \dots, m-1\}$. Hence we identify binary-searching labelings φ with sequences $(\emptyset = T^0, T^1, \dots, T^m)$ of $m+1$ trees such that there exists $s \in S_{T^{i+1}}$ satisfying $T^i = T^{i+1} \ominus s$ for each i.

We define linear operators U on $K\mathbb{T}$ by

$$UT := \sum_{s \in S_T} T \ominus s.$$

These operators $D(t')$, $D'(t')$ and $U(t)$ satisfy the following equations;

$$D(t')U(t) = \frac{1}{1 - tt'} U(t)D(t'),$$
$$D'(t')U(t) = (1 + tt')U(t)D'(t').$$

Hence the generalized Schur polynomials for these operators satisfy the same Pieri's formula as in the case of the classical Young's lattice and its dual construction.

In this case, generalized Schur polynomials are not symmetric in general. For example, since

$$\begin{aligned}
&U^*(t_1)U^*(t_2)\{0, 1, 12\} \\
&= U^*(t_1)(\{0, 1, 12\} + t_2\{0, 2\} + t_2^2\{0\}) \\
&= (\{0, 1, 12\} + t_1\{0, 2\} + t_1^2\{0\}) + t_2(\{0, 2\} + t_1\{0\}) + t_2^2(\{0\} + t_1\emptyset),
\end{aligned}$$

$s^{U^*}_{\{0,1,12\},\emptyset}(t_1,t_2) = s^{\{0,1,12\},\emptyset}_{U}(t_1,t_2) = t_1t_2^2$ is not symmetric.

For a labeling φ from T to $\{1,\dots,m\}$, we define $t^{\varphi} = \prod_{w\in T} t_{\varphi(w)}$. For a tree T, it follows that

$$s^{T,\emptyset}_{U}(t_1,\dots,t_n) = \sum_{\varphi;\text{ a binary-searching labeling}} t^{\varphi},$$

$$s^{D}_{T,\emptyset}(t_1,\dots,t_n) = \sum_{\varphi;\text{ a right-strictly-increasing labeling}} t^{\varphi},$$

$$s^{D'}_{T,\emptyset}(t_1,\dots,t_n) = \sum_{\varphi;\text{ a left-strictly-increasing labeling}} t^{\varphi}.$$

These generalized Schur polynomials $s^{T,\emptyset}_{U}(t_1,\dots,t_n)$ in this case are the commutativizations of the basis elements $\mathbf{P}_T$ of **PBT** in Hivert-Novelli-Thibon [**7**].

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