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A Spectral Approach to Pattern-Avoiding Permutations

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Abstract. We study the number of permutations in the symmetric group on n elements that avoid consecutive patterns S. We show that the spectrum of an associated integral operator on the space $L^2[0,1]^m$ determines the asymptotic behavior of such permutations. Moreover, using an operator version of the classical Frobenius-Perron theorem due to Kreĭn and Rutman, we prove asymptotic results for large classes of patterns S. This extends previously known results of Elizalde.

Résumé. Nous étudions le nombre de permutations dans le groupe symétrique sur n éléments qui évitent des motifs S consécutifs. Nous montrons que le spectre d'un opérateur intégral associé sur $L^2[0,1]^m$ détermine le comportement asymptotique de telles permutations. Utilisant de plus une version d'opérateur du théorme classique de Frobenius-Perron en raison de Kreĭn et Rutman, nous donnons des résultats asymptotiques pour les grandes classes de motifs S. Ceci étend résultats précédemment des connus de Elizalde.

1. Introduction

In this paper, we study integral operators of the form

$$\begin{aligned} T : L^2\left([0,1]^m\right) &\longrightarrow L^2\left([0,1]^m\right) \\ f &\longmapsto \int_0^1 \chi(t, x_1, \ldots, x_m) f(t, x_1, \ldots, x_{m-1})\, dt \end{aligned} \tag{1.1}$$

and their applications to the theory of pattern avoidance in permutations. Here χ is a real-valued function on $[0,1]^{m+1}$ which takes the values 0 or 1 on each of the simplices in the standard triangulation of $[0,1]^{m+1}$, i.e., the partition

$$[0,1]^k = \bigcup_{\pi \in \mathfrak{S}_k} \Delta_\pi$$

where the simplex Δ_π is given by

$$\Delta_\pi = \left\{(x_1, \ldots, x_k) : x_{\pi^{-1}(1)} \leq x_{\pi^{-1}(2)} \leq \cdots \leq x_{\pi^{-1}(k)}\right\}$$

We will show how integral operators of this type arise naturally in counting pattern-avoiding permutations where the pattern has length $m+1$.

Recall that a *pattern* of length $m+1$ is an element $\sigma \in \mathfrak{S}_{m+1}$. A permutation $\pi \in \mathfrak{S}_n$, $n \geq m+1$, avoids the consecutive pattern σ if there is no integer j, $0 \leq j \leq n-m-1$, with the property that $\pi_{j+\sigma^{-1}(1)} < \pi_{j+\sigma^{-1}(2)} < \cdots < \pi_{j+\sigma^{-1}(m+1)}$. More generally, if S is a subset of $\mathfrak{S}_{m+1}$, we say that π avoids S if π avoids each $\sigma \in S$.

Fix a subset S of $\mathfrak{S}_{m+1}$ and, for $n \geq m+1$, let a_n denote the number of permutations $\pi \in \mathfrak{S}_n$ that avoid S. Let χ_S: $[0,1]^{m+1} \to \{0,1\}$ be given by

$$\chi_S(x_1, \ldots, x_{m+1}) = \begin{cases} 0 \text{ if } x_{\sigma^{-1}(1)} \leq x_{\sigma^{-1}(2)} \leq \cdots \leq x_{\sigma^{-1}(m+1)} \text{ for some } \sigma \in S; \\ 1 \text{ otherwise.} \end{cases} \tag{1.2}$$

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Let T_S be the integral operator on $L^2([0,1]^m)$ given by

$$(T_S f)(x_1,\ldots,x_m) = \int_0^1 \chi_S(t,x_1,\ldots,x_m) f(t,x_1,\ldots,x_{m-1})\, dt. \tag{1.3}$$

THEOREM 1.1. *The formula*

$$\frac{a_n}{n!} = (1, T_S^{n-m} 1) \tag{1.4}$$

holds for any $n \geq m+1$, where 1 *denotes the constant function with value* 1 *and* $(\,\cdot\,,\,\cdot\,)$ *denotes the usual inner product on $L^2([0,1]^m)$. Moreover, we have the inequality*

$$\frac{a_n}{n!} \leq C_S \left(\frac{a_{2m}}{(2m)!}\right)^{n/m}. \tag{1.5}$$

The inequality (1.5) is not optimal.

It is natural to attempt a large-n asymptotic expansion of the right-hand side of (1.4) using the spectral theory of the operator T_S. Recall that, if A is a bounded operator, the *resolvent set* of the operator A is the set $\rho(A)$ of complex numbers with the property that $(A-zI)^{-1}$ exists as a bounded operator. The *spectrum* of A is the set $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$. The *spectral radius* of a bounded operator A is the quantity

$$r(A) = \limsup_{n\to\infty} \|A^n\|^{1/n}.$$

Note that $\sigma(A)$ is contained in the closed disc of radius $r(A)$ about 0 in $\mathbb{C}$ (see, for example, §VI.3 of [**12**]).

THEOREM 1.2. *Let T_S be an integral operator of the form (1.3) for some nonempty pattern S. Then $\sigma(T_S)$ is a discrete set with* 0 *as its only possible accumulation point. Moreover, $r(T_S) < 1$ strictly.*

The proof uses the fact that, although T_S is not compact, the operator T_S^k is compact–in fact Hilbert-Schmidt–for any $k \geq m$. We will show that the Hilbert-Schmidt norm of T_S^m is strictly less than 1, from which the statement about the spectral radius follows. We will also give examples of sets of patterns S for which $r(T_S) = 0$, and the ratio $a_n/n!$ converges to zero as $n \to \infty$.

Our main interest is in patterns for which $r(T_S) > 0$. With an additional condition on S, we can use spectral theory to obtain an asymptotic formula for a_n. Below (Theorem 1.5), we will give a sufficient condition on a pattern S so that the hypotheses of Theorem 1.3 hold. To state this condition, recall that an operator A on the space $L^2(X,\mu)$ of complex-valued measurable functions on the measure space (X,μ) is called *strongly positive* if for every $f \geq 0$ there is an integer n so that $(T^n f)(x) > 0$ for almost every x. As we show through examples below, there are patterns S for which T_S is not strongly positive.

THEOREM 1.3. *Suppose that T_S is an operator of the form (1.3) for some set of patterns S, and that T_S is strongly positive. Then T_S has a unique simple eigenvalue $\rho > 0$ with positive eigenfunction ϕ, and all other eigenvalues $\lambda \in \sigma(T_S)$ satisfy $|\lambda| < \rho$ strictly. Moreover, the adjoint operator T_S^* has ρ as its unique positive eigenvalue and a positive eigenfunction ψ of T_S^* with eigenvalue ρ.*

It is important to note that the strong positivity of T_S implies that T_S has nonzero spectral radius, *and* that the positive eigenvalue is the only eigenvalue on the circle $|z| = \rho$. The existence of such a "spectral gap" and the associated positive eigenfunctions follows from an operator version of the celebrated Perron-Frobenius Theorem (see, e.g., Gantmacher [**8**], vol. 2, §XIII.2) due to Kreĭn and Rutman (see Theorem 6.3 of [**10**]). Under the assumption of Theorem 1.3, let

$$r_2(T_S) = \sup_{\lambda\in\sigma(T_S),\lambda\neq\rho} |\lambda| \tag{1.6}$$

Using spectral theory, we obtain:

THEOREM 1.4. *Suppose that T_S is a strongly positive operator of the form (1.3). Let ρ be the largest eigenvalue of T_S with associated eigenfunction ϕ. Let ψ be the eigenfunction of the adjoint operator T_S^* with eigenvalue ρ. Finally, let r_2 be given by equation (1.6). Then we have*

$$\frac{a_n}{n!} = \rho^{n-m}\frac{(\psi,1)(1,\phi)}{(\psi,\phi)} + \mathcal{O}(r_2^{n-m}).$$

Here $(\,\cdot\,,\,\cdot\,)$ denotes the usual inner product on $L^2\left([0,1]^m\right)$. Note that the leading term in this expansion is strictly positive since ϕ and ψ are positive functions of T_S and T_S^*. Higher-order terms in the expansion can be computed if further eigenvalues and eigenfunctions of the operator T_S are known (see, for example, Section 3 in what follows); see Section 2.3 for a statement of the full expansion.

We can give a sufficient condition in combinatorial terms for a pattern S to have a spectral gap in the sense of Theorem 1.3. To do so we associate to a pattern S a directed graph, G_S, defined as follows. If $x \in \mathbb{Z}^m$ is a vector of positive integers define $\Pi(x)$ to be the permutation $\pi \in \mathfrak{S}_m$ with the property that $x_i < x_j$ if and only if $\pi(i) < \pi(j)$ for all $1 \leq i < j \leq m$. The vertices of G_S are the elements of $\mathfrak{S}_m$, and the edge $\sigma = (\sigma_1, \ldots, \sigma_{m+1}) \in \mathfrak{S}_{m+1} - S$ goes from the permutation $\Pi(\sigma_1, \ldots, \sigma_m)$ to the permutation $\Pi(\sigma_2, \ldots, \sigma_{m+1})$. The graph G_S is *strongly connected* if any point of G_S is connected to any other point of G_S by a directed path. A strongly connected graph is *aperiodic* if there exists a positive integer k and two vertices u and v such that there exists a directed path from u to v of any length greater than or equal to k. The condition that two such vertices exist is equivalent to the statement that between any two vertices in the graph, one can find directed paths of any length greater than or equal to k.

THEOREM 1.5. *Let $S \subset \mathfrak{S}_{m+1}$ and suppose that G_S is strongly connected and the two monotone permutations $12\cdots m+1$ and $m+1\cdots 21$ do not belong to the set S. Then T_S is strongly positive. Hence we conclude that there exist three positive constants ρ, r_2 and c such that $r_2 < \rho$ and*

$$\frac{a_n}{n!} = c\rho^{n-m} + \mathcal{O}(r_2^{n-m}).$$

EXAMPLE 1.6. Let S be the set $\{132, 231\}$. Hence, S-avoiding permutations are permutations without a peak, and there are 2^{n-1} such permutations in $\mathfrak{S}_n$. In this case, the operator T_S has no eigenvalues and our spectral methods do not apply. Also, observe that the graph G_S is not strongly connected, so Theorem 1.5 does not apply.

EXAMPLE 1.7. Let S be the set $\{123, 213, 231, 321\}$. The directed graph G_S is strongly connected, but not aperiodic. Again Theorem 1.5 does not apply. In fact, in this case, $a_n = 2$ for all $n \geq 2$.

In many cases of interest the leading term is explicitly computable. Using Theorem 1.4, we will prove the following asymptotic formulas.

THEOREM 1.8. *The number a_n of 123-avoiding permutations in $\mathfrak{S}_n$ obeys the asymptotic formula*

$$\frac{a_n}{n!} = \lambda_0^{n+1} \exp\left(\frac{1}{2\lambda_0}\right) + \mathcal{O}\left(\lambda_{-1}^n\right)$$

where

$$\lambda_0 = \frac{3\sqrt{3}}{2\pi}, \quad \lambda_{-1} = \frac{3\sqrt{3}}{4\pi}\ .$$

In this case, all of the eigenvalues of T_S are real and T_S has empty kernel. We can easily obtain higher-order terms in the expansion from the spectral methods used there since, in fact, all of the eigenvalues and eigenfunctions of the operator T_S and its adjoint can be computed explicitly: see Theorem 3.3.

We also have the following result for 213-avoiding permutations.

THEOREM 1.9. *The number b_n of 213-avoiding permutations in the symmetric group $\mathfrak{S}_n$ obeys the asymptotic formula*

$$b_n/n! = \exp\left(\frac{1}{2\lambda_0^2}\right) \cdot \lambda_0^{n+1} + \mathcal{O}\left(\left(\frac{1}{\sqrt{2}}\right)^{n-2}\right),$$

where $\lambda_0 = 0.7839769312\ldots$ is the unique real root to the equation

$$\operatorname{erf}\left(\frac{1}{\lambda\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}}\ .$$

In this case, the other eigenvalues of T_S are not real and the kernel of T_S has infinite dimension.

We close our introduction by a brief overview on the subject of pattern avoidance in permutations (for more details we refer to [**2**]). The "classical" definition of a pattern is slightly different than one provided above. We say that a permutation π avoids a pattern σ if π does not contain a *subsequence* which is order-isomorphic to σ. The study of such patterns originated in theoretical computer science by Donald Knuth [**9**].

However, the first systematic study was done by Simon and Schmidt [**13**], who completely classified the avoidance of patterns of length three. Since then several hundred papers related to the field have been published.

One of the most important results in the subject is the proof by Marcus and Tardos [**11**] of the so-called Stanley-Wilf conjecture related to the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any pattern S there exists a constant c (depending on σ) such that the number of the permutations of length n that avoid S is less than c^n.

In this paper we also study asymptotic behavior of permutations avoiding patterns, but we consider *consecutive* patterns, occurrences of which correspond to (contiguous) factors, rather than subsequences, anywhere in permutations. Suppose $\alpha_n(S)$ is the number of permutations avoiding a consecutive pattern S. It is known [**5**] that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(S)/n!}$ is a nonnegative constant. Moreover, in [**6**] asymptotics for the following consecutive patterns is given: 123, 132, 1342, 1234, and 1243. These results are obtained by representation of permutations as increasing binary trees, then using symbolic methods followed by solving certain linear differential equations with polynomial coefficients to get corresponding exponential generating functions, and, finally, using the following result:

THEOREM 1.10. *[See* [**7**, Chapter 4] *for a discussion] Let $A(z)$ be a meromorphic function on a domain of the complex plane including the origin, and let ρ be the unique pole of $A(z)$ such that $|\rho|$ is minimum. Then the following asymptotic estimate holds:*

$$[z^n]A(z) \sim \gamma \cdot \rho^{-n}$$

where γ is the residue of A in ρ.

In our paper we develop a general method (not involving generating functions) that gives detailed asymptotic expansions and allows for explicit computation of leading terms in many cases. As special cases of our results, we get a more detailed asymptotics for some of the results of Elizalde and Noy [**6**].

The outline of this paper is as follows. In § 2 we prove Theorems 1.1, 1.2, 1.3, and 1.4. We also note some symmetries of the operator T_S for certain patterns S, and consider the case of descent pattern avoidance. We use Theorem 1.4 to give the proof of Theorem 1.8 in Section 3 and the proof of Theorem 1.9 in Section 4.

2. The Operator T

LEMMA 2.1. *Let T be an operator of the form (1.1) with $0 \le \chi(x) \le 1$ for all $x \in [0,1]^m$. Then $\|T\| \le 1$ and T^m is compact.*

The adjoint operator of T is given by the expression

$$T^*(f) = \int_0^1 \chi(x_1, \ldots, x_m, u) f(x_2, \ldots, x_m, u)\, du.$$

2.1. Symmetries.

Let J and R be the following two involutions on the space $L^2([0,1]^m)$:

$$(Jf)(x_1, x_2, \ldots, x_m) = f(1-x_m, \ldots, 1-x_2, 1-x_1), \tag{2.1}$$

$$(Rf)(x_1, x_2, \ldots, x_m) = f(x_m, \ldots, x_2, x_1). \tag{2.2}$$

Observe that both J and R are self adjoint operators.

LEMMA 2.2. *Assume that χ has the symmetry*

$$\chi(x_1, x_2, \ldots, x_m, x_{m+1}) = \chi(1-x_{m+1}, 1-x_m, \ldots, 1-x_2, 1-x_1).$$

Then the adjoint of the associated operator T is given by $T^* = JTJ$. Moreover, if ϕ is an eigenfunction of the operator T with eigenvalue λ then $J\phi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ.

Similarly to Lemma 2.2 we have the next lemma.

LEMMA 2.3. *Assume that χ has the symmetry*

$$\chi(x_1, x_2, \ldots, x_m, x_{m+1}) = \chi(x_{m+1}, x_m, \ldots, x_2, x_1).$$

Then we have that the adjoint of the associated operator T is given by $T^* = RTR$. Moreover, if ϕ is an eigenfunction of the operator T with eigenvalue λ then $R\phi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ.

Finally, we have the following relation between T_S and T_S^*. For a permutation $\pi \in \mathfrak{S}_n$, let π^* be the reverse permutation, that is, if $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ then $\pi^* = (\pi_n, \pi_{n-1}, \dots, \pi_1)$. Similarly, if $S \subset \mathfrak{S}_n$, then $S^* = \{\pi \in \mathfrak{S}_n : \pi^* \in S\}$.

Lemma 2.4. *The equality*

$$T_S^* = R T_{S^*} R$$

holds, where R is given by (2.2).

2.2. Connection with Pattern Avoidance. Here we show how operators of the form (1.1) arise naturally in the study of pattern-avoiding permutations, proving Theorem 1.1. We recall that the standard triangulation of the unit cube $[0,1]^n$ into n-simplices is in one-to-one correspondence with permutations $\sigma \in \mathfrak{S}_n$: a given σ corresponds to the simplex

$$\left\{(x_1, \dots, x_n) : x_{\sigma^{-1}(1)} \le x_{\sigma^{-1}(2)} \le \cdots \le x_{\sigma^{-1}(n)}\right\}$$

which has Euclidean volume $(n!)^{-1}$.

Choose and fix a nonempty subset S of $\mathfrak{S}_{m+1}$ (the set of patterns to be avoided), and define χ_S and T_S respectively as in (1.2) and (1.3). For $n \ge m+1$, let

$$\chi_n(x_1, \dots, x_n) = \prod_{j=1}^{n-m} \chi_S(x_j, \dots, x_{m+j}) \tag{2.3}$$

Then $\chi_n(x)$ is 0 if x belongs to an n-simplex of $[0,1]^n$ corresponding to a permutation containing a forbidden pattern (starting at any j between 1 and $n-m$), and 1 otherwise. From this observation, the following lemma is immediate.

Lemma 2.5. *The formula*

$$a_n = n! \int_{[0,1]^n} \chi_n(x)\ dx$$

holds for any $n \ge m$.

Now define a sequence of functions $\{f_n\}_{n=m}^{\infty}$ on $[0,1]^m$ by the formulas

$$\begin{aligned} f_m(y_1, \dots, y_m) &= 1 \\ f_n(y_1, \dots, y_m) &= \int_{[0,1]^{n-m}} \chi_n(x_1, \dots, x_{n-m}, y_1, \dots, y_m)\ dx. \end{aligned}$$

Lemma 2.6. *For any $n \ge m$, the formula*

$$f_{n+1}(y_1, \dots, y_m) = (T_S f_n)(y_1, \dots, y_m)$$

holds.

We can also estimate the norm of $\|T_S^m\|$. The following estimate shows that $\|T_S^m\| < 1$ strictly, when S is non-empty.

Lemma 2.7. *The estimate*

$$\|T_S^m\| \le \left(\frac{a_{2m}}{(2m)!}\right)^{1/2}$$

holds.

Proof of Theorem 1.1. From Lemma 2.5 and the definition of f_n, it is easy to see that for any $n \ge m+1$,

$$\frac{a_n}{n!} = (1, f_n)$$

where the right-hand side is the inner product of the constant function 1 and the function f_n in $L^2\left([0,1]^m\right)$. From Lemma 2.6 it follows that $f_{m+n} = T_S^n f_m = T_S^n 1$ from which we conclude that for any $n \ge m+1$,

$$\frac{a_n}{n!} = \left(1, T_S^{n-m} 1\right).$$

It easily follows that

$$\frac{a_n}{n!} \le \left\|T_S^{n-m}\right\|$$

and if $n = km + r$ with $0 \le r \le m-1$ we have by Lemma 2.7 that

$$\|T_S^{n-m}\| \le \left(\frac{a_{2m}}{(2m)!}\right)^{(k-1)/2} \|T\|^r$$

from which it follows that

$$\frac{a_n}{n!} \le C_S \left(\frac{a_{2m}}{(2m)!}\right)^{n/m}.$$

□

2.3. Spectral Theory: The Spectral Gap. In this subsection, we prove Theorem 1.2.

Suppose that T is a bounded operator on a Hilbert space $\mathcal{H}$ with the property that T^m is compact for some positive integer m. For a bounded operator A, let $\sigma(A)$ denote the spectrum of A, i.e., the set of all $\lambda \in \mathbb{C}$ for which $(A - \lambda I)^{-1}$ does not exist as a bounded operator on $\mathcal{H}$. Recall that the spectral mapping theorem (see Dunford and Schwarz [**3**], chapter VII, Theorem 11, p. 569) implies that if f is an analytic function and T is a bounded operator, then the spectrum of $f(T)$ is the image under f of $\sigma(T)$. Here $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi i}\int_\gamma f(z)(T - zI)^{-1} dz$$

where γ is any contour surrounding $\sigma(T)$; is is easy to see that if $f(z) = z^m$, then this coincides with the usual definition of T^m. Since $\sigma(T^m)$ is at most a countable set with 0 as the only possible accumulation point, we immediately obtain:

Lemma 2.8. *Suppose that T is a bounded operator on a Hilbert space $\mathcal{H}$ and that T^m is compact for some positive integer m. Then the spectrum of T is at most countable and has zero as the only possible accumulation point.*

Proof of Theorem 1.2. All of the statements except the assertion that $r(T_S) < 1$ follow from Lemma 2.8. From Lemma 2.7 we have $\|T_S^m\| < 1$ The discreteness of the spectrum of T_S implies that $r(T_S) = \sup\{|\lambda| : \lambda \in \sigma(T_S)\}$. Since $\sigma(T_S^m) = \{\lambda^m : \lambda \in \sigma(T_S)\}$ it follows from this estimate that $\sigma(T_S)$ is contained in a closed disc of radius $(a_{2m}/(2m)!)^{1/(2m)} < 1$. □

To give the proof of Theorem 1.3, we note the following result which is a special case of Theorem 6.3 in Kreĭn and Rutman [**10**].

Theorem 2.9. *(see [**10**], Theorem 6.3) Let (X, μ) be a measure space and A be a compact operator on $L^2(X, \mu)$. Suppose that A is strongly positive. Then:*
(a) There is a unique strictly positive function $\phi \in L^2(X, \mu)$ and $\rho > 0$ with $A\phi = \rho\phi$ and $\|\phi\| = 1$,
(b) There is a unique nonnegative function $\psi \in L^2(X, \mu)$ with $A^*\psi = \rho\psi$ and $\|\psi\| = 1$, and
(c) If λ is any other eigenvalue of A, then $|\lambda| < \rho$ strictly.

Proof of Theorem 1.3. It follows from the hypothesis and Theorem 2.9(a) and (c) that the operator T_S^k has a positive eigenvalue α of maximum modulus with associated positive eigenfunction ϕ. Let ρ be the unique positive kth root of α. By the spectral mapping theorem, $\omega\rho$ is an eigenvalue of T_S for some kth root of unity $\omega = \exp(2\pi ij/k)$, $0 \le j \le k-1$. From the spectral mapping theorem again, it follows that $\omega^n\rho^n$ is an eigenvalue of T_S^n for any positive integer n. Moreover, since $\omega^k\rho^k$ is an eigenvalue of maximum modulus for T_S^k, it follows from the spectral mapping theorem that $\omega^n\rho^n$ will be an eigenvalue of maximum modulus for T_S^n if $n \ge k$. But T_S^n is positivity improving for any such n, so ω^n is real for all $n \ge k$. Hence $\omega = 1$ and ρ is an eigenvalue of T_S. We may now identify ϕ as the unique positive eigenfunction of T_S whose real eigenvalue $\rho > 0$ has maximum modulus, and applying the spectral mapping theorem again we see that all other eigenvalues of T_S have modulus strictly less than ρ. The statements about T_S^* follow from Theorem 2.9(b) and (c) and a similar argument. □

To prove Theorem 1.5, we will need the following lemma. In what follows, Δ_π denotes the simplex in $[0,1]^n$ corresponding to $\pi \in \mathfrak{S}_n$.

LEMMA 2.10. *Let $S \subset \mathfrak{S}_{m+1}$ and suppose that G_S is strongly connected and the two monotone permutations $12\cdots m+1$ and $m+1\cdots 21$ do not belong to the set S. Then there exist a positive integer k such that for any two permutations σ and π in $\mathfrak{S}_n$ and any function $f \in L^2\left([0,1]^m\right)$ such that $f|_{\Delta_\pi}$ is nonnegative and nonzero, the function $T^k f|_{\Delta_\sigma}$ is strictly positive.*

Now consider the adjoint operator T_S^*. Since $(T_S^*)^m$ is compact it follows that $\sigma(T_S^*)$ is a discrete set whose only accumulation point is 0. It is not difficult to see that $\sigma(T_S^*)\setminus\{0\}$ consists of those λ with $\overline{\lambda} \in \sigma(T_S)$. Indeed, if $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then λ is an isolated singularity of T. Hence $\ker(T-\lambda I)$ is nonempty since $\ker(T-\lambda I)$ is the range of the projection given by the residue of $(T-zI)^{-1}$ at $z=\lambda$. Since any eigenvector of T with eigenvalue λ is also an eigenvector of T^m with eigenvalue λ^m and T^m is compact, it follows that $V_\lambda = \ker(T-\lambda I)$ has finite dimension N_λ for any $\lambda \neq 0$. A similar argument applies to T^*, and the identity

$$\left[(T-zI)^{-1}\right]^* = (T^* - \overline{z}I)^{-1}$$

shows that the finite-dimensional space $W_\lambda = \ker(T^* - \overline{\lambda}I)$ has the same dimension as V_λ. Recall that

$$P = \operatorname*{Res}_{z=\lambda} (T-zI)^{-1}$$

projects onto V_λ, so clearly P^* projects onto W_λ.

Now let $\{\varphi^i\}_{i=1}^{N_\lambda}$ be an orthogonal basis for V_λ. By the Riesz representation theorem, the functional $\psi \mapsto (\varphi^i, P\psi)$ is represented by a vector ψ_i so that

$$P = \sum_{i=1}^{N_\lambda} (\psi^i, \ \cdot\ )\varphi^i. \tag{2.4}$$

Since P^* is the projection onto $\ker\left(T^* - \overline{\lambda}I\right)$, the vectors ψ_i are eigenvectors of T^* with eigenvalue $\overline{\lambda}$. The condition that $P^2 = P$ implies that

$$\left(\psi^i, \varphi^j\right) = \delta_{ij} \tag{2.5}$$

These conditions suffice to determine the ψ^j given a choice of $\left\{\varphi^j\right\}$.

2.4. Spectral Theory: The Expansion Theorem. We now consider the spectral expansion of T^n, assuming now that $\sigma(T)$ is contained in the interior of the unit disc. From the analytic functional calculus we have

$$T^n = \frac{1}{2\pi i}\int_{|z|=1} (T-zI)^{-1}\ z^n\ dz$$

If we write $\sigma(T) = \{\lambda_k\}_{k=1}^\infty$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq 0$ and let $r_k = |\lambda_k|$ we then have

$$T^n = \sum_{j=1}^{k} \lambda_j^n P_j + \mathcal{O}\left(r_{k+1}^n\right) \tag{2.6}$$

by shrinking the contour. Here P_j is the projection for $\lambda = \lambda_j$ and the remainder estimate depends on

$$\sup_{|z|=r} \left\|(T-zI)^{-1}\right\|$$

where $r > 0$ is chosen so that (i) all the eigenvalues $\{\lambda_j\}_{j=1}^k$ lie in the exterior of the disc of radius r and (ii) the circle $|z| = r$ contains no eigenvalues of T. This choice is possible since $\sigma(T)$ is discrete.

Note that, in case $\sigma(T) = \{0\}$, we do not obtain a meaningful formula–there must be at least one nonzero eigenvalue for the expansion to make sense.

PROOF OF THEOREM 1.4. We take $T = T_S$ and note that, by hypothesis, the eigenvalue of T_S having greatest modulus is positive and simple. From (2.6), (2.4) and the simplicity of ρ we get

$$(1, T_S^k 1) = \rho^k(\psi,1)(\varphi,1) + \mathcal{O}\left(r_2^k\right)$$

provided $(\psi,\varphi) = 1$; here φ and ψ are respectively the eigenfunctions of T_S and T_S^* associated with eigenvalue ρ. The conclusion is immediate. $\square$

From (2.6) and (2.4), one can refine the expansion as follows if other eigenvalues and eigenvectors are known. Ordering the eigenvalues as above we have for any integer N that

$$(1, T_S^k 1) = \sum_{j=1}^{N} c_j \lambda_j^k + \mathcal{O}(r_{N+1}^k) \tag{2.7}$$

where

$$\begin{aligned} c_j &= (1, P_j 1) \\ &= \sum_{m=1}^{N_j} (\psi_j^m, 1)(\varphi_j^m, 1) \end{aligned}$$

where $\left\{\varphi_j^m\right\}$ and $\left\{\psi_j^m\right\}$ are bases for the $\lambda = \lambda_j$ eigenspaces of T_S and T_S^*, respectively, so chosen that the normalization (2.5) holds.

2.5. Descent pattern avoidance. The descent set of a permutation π in the symmetric group on n elements is the subset of $\{1, \ldots, n-1\}$, given by $\{i : \pi_i > \pi_{i+1}\}$. An equivalent notion is the descent word, defined as follows. The descent word of the permutation π is the word $u = u_1 \cdots u_{n-1}$ where $u_i = a$ if $\pi_i < \pi_{i+1}$ and $u_i = b$ otherwise.

Let U be a collection of ab-words of length m. The permutation π avoids the set U if there is no consecutive subword of the descent word of π contained in the collection U.

Descent pattern avoidance is a special case of consecutive pattern avoidance. For instance, permutations avoiding the word aab is the permutations avoiding the set $S = \{1243, 1342, 2341\}$, since these three permutations are the permutations with descent word aab.

For an ab-word u of length $m-1$ define the descent polytope P_u to be the subset of the unit cube $[0,1]^m$ corresponding to all vectors with descent word u. That is,

$$P_u = \{(x_1, \ldots, x_m) \in [0,1]^m \ : \ x_i \leq x_{i+1} \text{ if } u_i = a \text{ and } x_i \geq x_{i+1} \text{ if } u_i = b\}.$$

Observe that the m-dimensional unit cube is the union of the 2^{m-1} descent polytopes P_u. Now the operator T corresponding to the descent pattern avoidance of the set U has the following form. For an ab-word u of length $m-2$ and $y \in \{a, b\}$ we have

$$\begin{aligned} T(f)|_{P_{uy}} &= \int_0^{x_1} \chi(auy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{au}} \, dt \\ &\quad + \int_{x_1}^{1} \chi(buy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{bu}} \, dt, \end{aligned} \tag{2.8}$$

where by abuse of notation we let $\chi(w) = 1$ if w does not belong to the set U and $\chi(w) = 0$ otherwise.

PROPOSITION 2.11. *Let T be the operator associated with a descent pattern avoidance and let k be an integer such that $1 \leq k \leq m-1$. Let u be an ab-word of length $m-1$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

COROLLARY 2.12. *Let T be the operator associated with a descent pattern avoidance and let ϕ be an eigenfunction associated with a non-zero eigenvalue λ. Let u be an ab-word of length $m-1$. Then the eigenfunction restricted to the descent polytope P_u only depends on the variable x_1.*

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f that only depend on the variable x_1 when restricted to each of the descent polytopes P_u. Observe that the subspace V is invariant under the operator T. That is, the operator T restricts to the subspace V. Moreover the constant function 1 belongs to V. Hence to understand the behavior of $T^n(1)$ it is enough to study this restricted operator.

In order to describe the subspace V more explicitly define for an ab-word u of length $m-1$ the polynomial $f(u; x_1)$ as follows:

$$f(u; x_1) = \int_{(x_1, x_2, \ldots, x_m) \in P_u} 1 dx_2 \cdots dx_m.$$

This polynomial was first introduced and studied in [**4**], with a different indexing.

Let p be a vector $(p_u(x_1))_{u\in\{a,b\}^{m-1}}$. That is, the vector p consists of one-variable functions in the variable x_1 and is indexed by ab-words of length $m-1$. Consider the function f on $[0,1]^m$ defined by

$$f(x_1,\ldots,x_m)|_{P_u} = p_u(x_1)$$

for all ab-words u of length $m-1$. Observe that the function f belongs to $L^2([0,1]^m)$, and hence to the invariant subspace V, if and only if

$$\int_0^1 f(u;x_1)\cdot |p_u(x_1)|^2 \; dx_1 < \infty$$

for all ab-words u of length $m-1$. For two functions f and g in the subspace V, corresponding to the two vectors $(p_u(x_1))_{u\in\{a,b\}^{m-1}}$ and $(q_u(x_1))_{u\in\{a,b\}^{m-1}}$, the inner product is given by

$$(f,g) = \sum_{u\in\{a,b\}^{m-1}} \int_0^1 f(u;x_1)\cdot p_u(x_1)\cdot q_u(x_1) dx_1 .$$

This discussion leads to the following structural result about the subspace V.

Proposition 2.13. *The invariant subspace V is isometrically isomorphic to the space $L^2\left([0,1]\right)^{2^{m-1}}$.*

3. 123-Avoiding Permutations

A 123-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ with no index j so that $\pi_j < \pi_{j+1} < \pi_{j+2}$, where $1 \le j \le n-2$. We denote by a_n the number of 123-avoiding permutations in $\mathfrak{S}_n$. Thus, in the notation of the introduction S consists of the single permutation 123 and

$$\chi_S(x_1,x_2,x_3) = \begin{cases} 0 \text{ if } x_1 \le x_2 \le x_3; \\ 1 \text{ otherwise.} \end{cases} \tag{3.1}$$

We will obtain an asymptotic formula for a_n by computing the eigenvalues and eigenfunctions of the corresponding operator T_S and using the spectral expansions of Section 2.3. As we will see, in this case the operator T_S has real eigenvalues and a trivial kernel. This is related to the fact that the eigenvalue problem for T_S can be recast as an eigenvalue problem for a first-order system of differential equations.

3.1. Eigenfunctions and Eigenvectors. Since 123-avoiding permutations can be viewed as permutations with no double descents Corollary 2.12 allows us to recast then problem of finding eigenfunctions in two variables into finding two one-variable functions.

Proposition 3.1. *The eigenvalues λ_k of the operator T on $L^2([0,1]^2)$ are given by*

$$\lambda_k = \frac{\sqrt{3}}{2\pi\cdot\left(k+\frac{1}{3}\right)}, \tag{3.2}$$

where $k \in \mathbb{Z}$ and the associated eigenfunctions $\phi_k = \begin{cases} p_k(x) & \text{if } 0 \le x \le y \le 1 \\ q_k(x) & \text{if } 0 \le y \le x \le 1 \end{cases}$ are given by

$$\phi_k = \exp\left(-\frac{x}{2\lambda}\right)\cdot \begin{cases} \cos\left(\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cdot\frac{x}{\lambda}\right) & \text{if } 0 \le x \le y \le 1, \\ \sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\cdot\frac{x}{\lambda}\right) & \text{if } 0 \le y \le x \le 1. \end{cases} \tag{3.3}$$

Note that the eigenvalues are ordered by

$$\lambda_0 > -\lambda_{-1} > \lambda_1 > -\lambda_{-2} > \lambda_2 > -\lambda_{-3} > \lambda_3 > \cdots > 0.$$

By applying the involution J we obtain the adjoint eigenfunction

$$\psi_k = \exp\left(\frac{y-1}{2\lambda}\right)\cdot \begin{cases} \cos\left(\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cdot\frac{1-y}{\lambda}\right) & \text{if } 0 \le x \le y \le 1, \\ \sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\cdot\frac{1-y}{\lambda}\right) & \text{if } 0 \le y \le x \le 1. \end{cases} \tag{3.4}$$

Proposition 3.2. *For the eigenfunctions $\phi_k = \phi$ of T and $\psi_k = \psi$ of T^* with eigenvalue $\lambda_k = \lambda = \sqrt{3}/(2\pi(k+1/3))$,*

$$(1,\phi) = (1,\psi) = \frac{\sqrt{3}}{2}\lambda^2 \tag{3.5}$$

$$(\psi,\phi) = \frac{3}{4}(-1)^k \lambda \exp\left(-\frac{1}{2\lambda}\right) \tag{3.6}$$

In particular

$$\frac{(1,\phi)\,(1,\psi)}{(\phi,\psi)} = (-1)^k \lambda^3 \exp\left(\frac{1}{2\lambda}\right) \tag{3.7}$$

3.2. Asymptotics. The above computations show that all eigenvalues of T_S are simple and give explicit formulas. We thus obtain the following expansion for $a_n/n!$ as an immediate consequence of (2.7), Propositions 3.1, and 3.2.

Theorem 3.3. *For any positive integer $n \geq 2$ and any positive integer K, the formula*

$$\frac{a_n}{n!} = \sum_{|k|\leq K} (-1)^k \lambda_k^{n+1} \exp\left(\frac{1}{2\lambda_k}\right) + \mathcal{O}\left(r_{K+1}^n\right)$$

holds, where λ_k is given by (3.2) and

$$r_k = \frac{\sqrt{3}}{2\pi \cdot \left(k - \frac{1}{3}\right)}$$

4. 213-Avoiding Permutations

A 213-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ which contains no sequence of the form

$$\pi_{j+1} < \pi_j < \pi_{j+2}$$

for any j with $1 \leq j \leq n-2$. We denote the number of 213-avoiding permutations of $\mathfrak{S}_n$ by b_n. Thus, S consists of the single permutation (213) and

$$\chi_S(x_1,x_2,x_3) = \begin{cases} 0 & \text{if } x_2 \leq x_1 \leq x_3, \\ 1 & \text{otherwise.} \end{cases}$$

By symmetry, the study of 213-avoiding permutations is equivalent to 132-avoiding permutations, 231-avoiding permutations and 312-avoiding permutations. However the case of 213-avoiding permutations gives the most straightforward equations.

We will compute the eigenvalues and eigenfunctions of the operator T_S and obtain an asymptotic expansion for b_n using spectral methods. In this case, it turns out that T_S has a nontrivial kernel and its eigenvalues need not be real. However, its eigenvalue of largest modulus is real and isolated, as we will show, so that we can still obtain an asymptotic formula for b_n.

4.1. Eigenfunctions and Eigenvectors. In what follows, we will make use of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt \tag{4.1}$$

which extends to an entire function on $\mathbb{C}$, and the function

$$q(x) = \exp\left(-\frac{x^2}{2\lambda^2}\right). \tag{4.2}$$

Let

$$f(x,y) = \begin{cases} p(x,y) & \text{if } 0 \leq x \leq y \leq 1, \\ q(x,y) & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Then

$$(Tf)(x,y) = \begin{cases} \int_0^x p(t,x)dt + \int_y^1 q(t,x)dt & \text{if } 0 \leq x \leq y \leq 1, \\ \int_0^x p(t,x)dt + \int_x^1 q(t,x)dt & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Now we characterize the nonzero eigenvalues and eigenfunctions.

Proposition 4.1. *The non-zero eigenvalues λ of the operator T satisfies the equation*

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}\cdot\lambda}\right)=\frac{\sqrt{2}}{\sqrt{\pi}} \tag{4.3}$$

and the corresponding eigenfunctions are

$$\varphi(x,y)=\begin{cases} q(x)-\dfrac{1}{\lambda}\int_x^y q(t)\,dt & \text{if } x\le y,\\ q(x) & \text{if } x>y,\end{cases}$$

where $q(x)$ is given by (4.2).

The adjoint operator T^* is given by

$$T^*(f(x,y))=\begin{cases} \int_0^y q(y,u)du+\int_y^1 p(y,u)du & \text{if } 0\le x\le y\le 1,\\ \int_0^y q(y,u)du+\int_y^x p(y,u)du & \text{if } 0\le y\le x\le 1.\end{cases}$$

Proposition 4.2. *For a non-zero eigenvalue λ of the operator T the corresponding eigenfunction of the adjoint operator T^* is*

$$\psi(x,y)=\begin{cases} p^*(y) & \text{if } 0\le x\le y\le 1,\\ p^*(y)-\frac{1}{\lambda}\cdot\int_x^1 p^*(u)du & \text{if } 0\le y\le x\le 1.\end{cases}$$

where

$$p^*(y)=-2\cdot y\cdot\exp\left(\frac{y^2}{2\lambda^2}\right)+2\cdot\lambda+\sqrt{2\pi}\cdot y\cdot\exp\left(\frac{y^2}{2\lambda^2}\right)\cdot\operatorname{erf}\left(\frac{y}{\sqrt{2}\lambda}\right). \tag{4.4}$$

Proposition 4.3. *For a non-zero eigenvalue λ with eigenvector ϕ and adjoint eigenvector ψ, we have*

$$\begin{aligned}(1,\phi)&=\lambda^2,\\ (1,\psi)&=2\cdot\lambda^3,\\ (\psi,\phi)&=2\cdot\lambda^2\cdot\exp(-1/(2\lambda^2)).\end{aligned}$$

In particular,

$$\frac{(1,\phi)\cdot(1,\psi)}{(\psi,\phi)}=\lambda^3\cdot\exp(1/(2\lambda^2)).$$

4.2. Asymptotics. To obtain leading asymptotics for b_n, we need to compute the eigenvalue of greatest modulus of the operator T_S and show that all other eigenvalues of T have strictly smaller moduli. From the eigenvalue condition (4.3), it suffices to study the roots of the equation $\operatorname{erf}(z)=\sqrt{2/\pi}$.

Since the error function is an increasing function on the real axis, the equation $\operatorname{erf}(z)=\sqrt{2}/\sqrt{\pi}$ has a unique real root $z_0=0.9019484541\ldots$. Hence the eigenvalue equation (4.3) has the unique real root $\lambda_0=0.7839769312\ldots$. Since the error function is an odd function we know by the strong version of the little Picard theorem that the equation $\operatorname{erf}(z)=\sqrt{2}/\sqrt{\pi}$ has infinitely many roots. The location of these roots is the subject of the next result.

Proposition 4.4. *The equation $\operatorname{erf}(z)=\sqrt{2}/\sqrt{\pi}$ has exactly one root in the interior of the unit disc, namely the unique real root $z_0=0.9019484541\ldots$, and all other (infinitely many) roots lie in the complement of the closed unit disc.*

As a corollary we have:

Corollary 4.5. *The eigenvalue equation (4.3) has the unique real root*

$$\lambda_0=0.7839769312\ldots$$

outside the disc of radius $1/\sqrt{2}$ centered at the origin, and all other (infinitely many) roots lie inside this disc.

Combining Propositions 4.1 through 4.3 and Corollary 4.5 using Theorem 1.4 we obtain Theorem 1.9.

5. Concluding remarks

In the case of descent pattern avoidance, can one prove that T restricted to the invariant subspace V is compact? We have done so in the case of 123-avoiding permutations.

It is straightforward to design a Viennot "pyramid" to compute the number a_n of S-avoiding permutations. For the original Viennot triangle, see [**14, 15**]. Let the entry $a_n^{i_1,\ldots,i_m}$ of the pyramid be the number of permutations in the symmetric group on n elements, avoiding the set S and ending with the m entries $i_1, \ldots, i_m$. Then the entry $a_n^{i_1,\ldots,i_m}$ is a sum of entries of the form $a_{n-1}^{j,i_1,\ldots,i_{m-1}}$. This sum being a discrete analogue of the operator T. How far does this analogue between the discrete model and the continuous one go? Does the function $f_n = T^{n-m}(1)$ approximate the n-th level of the pyramid? More exactly, how well does the integer $a_n^{i_1,\ldots,i_m}$ compare with $n! \cdot f_n(i_1/n, \ldots, i_m/n)$?

The next four largest roots to the eigenvalue equation in the 213-avoiding permutation case are:

$$\lambda_1 = 0.2141426360\ldots \pm 0.2085807022\ldots \cdot i$$
$$\lambda_2 = -0.1677323922\ldots \pm 0.2418627350\ldots \cdot i$$

Knowing these roots enables us to give an explicit error estimate in Theorem 1.9.

In this paper our object is to understand consecutive pattern avoidance. Generalized pattern avoidance was introduced by Babson and Steingrímsson [**1**]. Is there an analytic approach to obtain asymptotics for these classes of permutations? Lastly, it would be daring to ask for an analytic proof of the former Stanley-Wilf conjecture, recently proved in [**11**].

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