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On the Number of Factorizations of a Full Cycle

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Abstract. We give a new expression for the number of factorizations of a full cycle into an ordered product of permutations of specified cycle types. This is done through purely algebraic means, extending recent work of Biane [*Nombre de factorisations d'un grand cycle*, Sém. Lothar. de Combinatoire **51** (2004)]. We deduce from our result a remarkable formula of Poulalhon and Schaeffer [*Factorizations of large cycles in the symmetric group*, Discrete Math. **254** (2002), 433–458] that was previously derived through an intricate combinatorial argument.

Résumé. Nous proposons une nouvelle formule pour le nombre de factorisations d'un grand cycle en un produit ordonné de permutations de types cycliques donnés. Nous utilisons des arguments purement algébriques, étendant un travail récent de Biane [*Nombre de factorisations d'un grand cycle.*, Sém. Lothar. de Combinatoire **51** (2004)]. Nous déduisons de notre résultat une formule remarquable de Poulalhon et Schaeffer [*Factorizations of large cycles in the symmetric group*, Discrete Math. **254** (2002), 433–458] obtenue précédemment à l'aide d'arguments combinatoires complexes.

1. Notation

Our notation is generally consistent with Macdonald [**5**]. We write $\lambda \vdash n$ (or $|\lambda| = n$) and $\ell(\lambda) = k$ to indicate that λ is a partition of n into k parts; that is, $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 + \ldots + \lambda_k = n$. If λ has exactly m_i parts equal to i then we write $\lambda = [1^{m_1} 2^{m_2} \cdots]$, suppressing terms with $m_i = 0$. We also define $z_\lambda = \prod_i i^{m_i} m_i!$ and $\mathrm{Aut}(\lambda) = \prod_i m_i!$. A *hook* is a partition of the form $[1^b,\, a+1]$ with $a, b \geq 0$. We use Frobenius notation for hooks, writing $(a|b)$ in place of $[1^b,\, a+1]$.

The conjugacy class of the symmetric group $\mathfrak{S}_n$ consisting of all $n!/z_\lambda$ permutations of cycle type $\lambda \vdash n$ will be denoted by $\mathcal{C}_\lambda$. The irreducible characters χ^λ of $\mathfrak{S}_n$ are naturally indexed by partitions λ of n, and we use the usual notation χ^λ_μ for the common value of χ^λ at any element of $\mathcal{C}_\mu$. We write f^λ for the degree $\chi^\lambda_{[1^n]}$ of χ^λ.

For vectors $\mathbf{j} = (j_1, \ldots, j_m)$ and $\mathbf{x} = (x_1, \ldots, x_m)$ we use the abbreviations $\mathbf{j}! = j_1! \cdots j_m!$ and $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \cdots x_m^{j_m}$. Finally, if $\alpha \in \mathbb{Q}$ and $f \in \mathbb{Q}[[\mathbf{x}]]$ is a formal power series, then we write $[\alpha \mathbf{x}^{\mathbf{j}}]\, f(\mathbf{x})$ for the coefficient of the monomial $\alpha \mathbf{x}^{\mathbf{j}}$ in $f(\mathbf{x})$.

2. Factorizations of Full Cycles

Given $\lambda, \alpha_1, \ldots, \alpha_m \vdash n$, let $c^\lambda_{\alpha_1,\ldots,\alpha_m}$ be the number of factorizations in $\mathfrak{S}_n$ of a given permutation $\pi \in \mathcal{C}_\lambda$ as an ordered product $\pi = \sigma_1 \ldots \sigma_m$, with $\sigma_i \in \mathcal{C}_{\alpha_i}$ for all i. The problem of evaluating $c^\lambda_{\alpha_1,\ldots,\alpha_m}$ for various λ and α_i has attracted a good deal of attention and is linked to various questions in algebra, geometry, and physics. For details on the history of this problem and its connections to other areas of

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mathematics, we direct the reader to [**4**] and the references therein. Here we focus on the particularly well-studied case $\lambda = (n)$, which corresponds to counting factorizations of the full cycle $(1\,2\,\cdots\,n) \in \mathfrak{S}_n$ into factors of specified cycle types.

While it is straightforward to express $c^{(n)}_{\alpha_1,\dots,\alpha_m}$ as a character sum (see (3.1) below), the appearance of alternating signs in this sum — and resulting cancellations — preclude asymptotic analysis. Goupil and Schaeffer [**4**, FPSAC'98] overcame this difficulty in the case $m = 2$ by interpreting certain characters combinatorially (*viz.* the Murnaghan-Nakayama rule) and employing a sequence of bijections in which a sign-reversing involution accounts for cancellations. This leads to an expression for $c^{(n)}_{\alpha,\beta}$ as a sum of *positive* terms, which in turn permits nontrivial asymptotics. Poulalhon and Schaeffer [**6**] later extended this argument to arrive at a similar formula for $c^{(n)}_{\alpha_1,\dots,\alpha_m}$.

Biane [**1**] has recently given a remarkably succinct algebraic derivation of Goupil and Schaeffer's formula for $c^{(n)}_{\alpha,\beta}$. Our purpose here is to extend his method to give a new expression for $c^{(n)}_{\alpha_1,\dots,\alpha_m}$ as a sum of positive contributions. In particular, if for $\gamma = (\gamma_1, \gamma_2, \dots) \vdash n$ we define the polynomial $R_\gamma(x, y)$ and the *nonnegative* constants $r^\gamma_{j,k}$ by

$$R_\gamma(x,y) := \frac{1}{2y}\prod_{i\geq 1}((x+y)^{\gamma_i} - (x-y)^{\gamma_i}) = \sum_{j+k=n-1} r^\gamma_{j,k} x^j y^k, \tag{2.1}$$

then our main result is the following:

Theorem 2.1. *Let $\alpha_1, \dots, \alpha_m \vdash n$ and, for $\lambda = [1^{m_1}2^{m_2}3^{m_3}\cdots]$, let $2\lambda - 1 = [1^{m_1}3^{m_2}5^{m_3}\cdots]$. Set $\mathbf{x} = (x_1, \dots, x_m)$ and let $e_\lambda(\mathbf{x})$ denote the elementary symmetric function in $x_1, \dots, x_m$ indexed by λ. Then*

$$c^{(n)}_{\alpha_1,\dots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)}\prod_i z_{\alpha_i}} \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}-\mathbf{1}} [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)} \cdot \prod_{i=1}^m j_i!\, k_i!\, r^{\alpha_i}_{j_i,k_i},$$

where the outer sum extends over all vectors $\mathbf{j} = (j_1, \dots, j_m)$ and $\mathbf{k} = (k_1, \dots, k_m)$ of nonnegative integers such that $j_i + k_i = n - 1$ for all i, and the inner sum over all partitions λ with $n - 1$ parts.

A proof of Theorem 2.1 is given in the next section. In §4, we use this result to deduce Poulalhon and Schaeffer's formula for $c^{(n)}_{\alpha_1,\dots,\alpha_m}$ (listed here as Theorem 4.1), thereby giving a purely algebraic derivation that avoids the detailed combinatorial constructions in [**6**].

3. Proof of the Main Result

It is well known that the class sums $\mathsf{K}_\lambda = \sum_{\sigma\in\mathcal{C}_\lambda}\sigma$ (for $\lambda \vdash n$) form a basis of the centre of the group algebra $\mathbb{C}\mathfrak{S}_n$. Indeed, the linearization relations $\mathsf{K}_{\alpha_1}\cdots\mathsf{K}_{\alpha_m} = \sum_{\lambda\vdash n} c^\lambda_{\alpha_1,\dots,\alpha_m}\mathsf{K}_\lambda$ identify the constants $c^\lambda_{\alpha_1,\dots,\alpha_m}$ as the connection coefficients of $\mathbb{C}\mathfrak{S}_n$. By using character theory to express K_λ in terms of central idempotents of $\mathbb{C}\mathfrak{S}_n$ (see [**7**], Problem 7.67b) one finds that

$$c^\lambda_{\alpha_1,\dots,\alpha_m} = \frac{n!^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{\beta\vdash n} \frac{\chi^\beta_{\alpha_1}\cdots\chi^\beta_{\alpha_m}}{(f^\beta)^{m-1}}\chi^\beta_\lambda.$$

This sum is generally intractable but simplifies considerably in the case $\lambda = (n)$, since there χ^β_λ vanishes when β is not a hook; in particular, the Murnaghan-Nakayama rule [**7**] implies $\chi^\beta_{(n)} = (-1)^b$ if $\beta = (a|b)$, while $\chi^\beta_{(n)} = 0$ otherwise. Moreover, the hook-length formula gives $f^{(a|b)} = \binom{a+b}{b}$, so

$$c^{(n)}_{\alpha_1,\dots,\alpha_m} = \frac{n^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{a+b=n-1} (a!\,b!)^{m-1}\chi^{(a|b)}_{\alpha_1}\cdots\chi^{(a|b)}_{\alpha_m}(-1)^b. \tag{3.1}$$

Let μ be the measure on $\mathbb{C}$ defined by the density $d\mu(z) = \frac{1}{\pi}e^{-|z|^2}dz$, where dz is the standard Lebesgue density (*i.e.* $dz = ds\,dt$ for $z = s + t\sqrt{-1}$). Following Biane [**1**], we shall make use of the formula

$$\int_{\mathbb{C}} z^j\bar{z}^k d\mu(z) = j!\,\delta_{jk}, \tag{3.2}$$

which is easily verified by changing to polar form.

Proof of Theorem 2.1: For $\gamma \vdash n$, let $F_\gamma(u,v) = \sum \chi_\gamma^{(a|b)} u^a v^b$ be the generating series for hook characters, where the sum extends over all pairs (a,b) of nonnegative integers with $a+b=n-1$. Then

$$\begin{aligned}
&\frac{1}{(n-1)!}(u_1\cdots u_m - v_1\cdots v_m)^{n-1}\prod_{i=1}^m F_{\alpha_i}(\bar{u}_i,\bar{v}_i)\\
&\qquad = \sum_{a+b=n-1}\frac{u_1^a\cdots u_m^a\cdot v_1^b\cdots v_m^b}{a!\,b!}(-1)^b\prod_{i=1}^m\sum_{a_i+b_i=n-1}\chi_{\alpha_i}^{(a_i|b_i)}\bar{u}_i^{a_i}\bar{v}_i^{b_i}.
\end{aligned}$$

Consider the effect of integrating the RHS with respect to $d\mu(\mathbf{u},\mathbf{v}) := \prod_{i=1}^m d\mu(u_i)d\mu(v_i)$. Using (3.2), note that all monomials $\frac{(-1)^b}{a!b!}\prod_i \chi_{a_i}^{(a_i|b_i)} u_i^a\bar{u}_i^{a_i}v_i^b\bar{v}_i^{b_i}$ vanish except those with $a_i=a$ and $b_i=b$ for all i, and each monomial of this special form is replaced by $(-1)^b(a!\,b!)^{m-1}\prod_i\chi_{\alpha_i}^{(a|b)}$. Thus we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^{2m}}(u_1\cdots u_m - v_1\cdots v_m)^{n-1}\prod_{i=1}^m F_{\alpha_i}(\bar{u}_i,\bar{v}_i)\,d\mu(\mathbf{u},\mathbf{v})\\
&\qquad\qquad = (n-1)!\sum_{a+b=n-1}(a!\,b!)^{m-1}\chi_{\alpha_1}^{(a|b)}\cdots\chi_{\alpha_m}^{(a|b)}(-1)^b.
\end{aligned}$$

Let I be the integral on the LHS, and change variables by letting $u_i = (y_i+x_i)/\sqrt{2}$, $v_i=(y_i-x_i)/\sqrt{2}$. As an immediate consequence of the Murnaghan-Nakayama rule we have

$$F_\gamma(u,v) = \frac{1}{u+v}\prod_{i\geq 1}(u^{\gamma_i} - (-v)^{\gamma_i})$$

for a partition $\gamma = (\gamma_1,\gamma_2,\ldots)$. Thus (2.1) gives $F_\gamma(y+x,y-x) = R_\gamma(x,y)$, and since F_{α_i} is homogeneous of degree $n-1$ the change of variables yields $F_{\alpha_i}(\bar{u}_i,\bar{v}_i) = 2^{-(n-1)/2}R_{\alpha_i}(\bar{x}_i,\bar{y}_i)$ for all i. Furthermore, it is easy to check that $d\mu(\mathbf{u},\mathbf{v}) = d\mu(\mathbf{x},\mathbf{y})$ and

$$u_1\cdots u_m - v_1\cdots v_m = \frac{1}{\sqrt{2^m}}\left(\prod_{i=1}^m(y_i+x_i) - \prod_{i=1}^m(y_i-x_i)\right) = \frac{2y_1\cdots y_m}{\sqrt{2^m}}\sum_{s\geq 1}e_{2s-1}(\mathbf{x}/\mathbf{y}),$$

where $\mathbf{x}/\mathbf{y} = (\frac{x_1}{y_1},\ldots,\frac{x_m}{y_m})$. Thus, with the aid of (3.2), we get

$$\begin{aligned}
I &= \frac{1}{2^{(n-1)(m-1)}}\int_{\mathbb{C}^{2m}}\left(y_1\cdots y_m\sum_{s\geq 1}e_{2s-1}(\mathbf{x}/\mathbf{y})\right)^{n-1}\prod_{i=1}^m R_{\alpha_i}(\bar{x}_i,\bar{y}_i)\,d\mu(\mathbf{x},\mathbf{y})\\
&= \frac{1}{2^{(n-1)(m-1)}}\sum_{\mathbf{j},\mathbf{k}}\mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}]\left(y_1\cdots y_m\sum_{s\geq 1}e_{2s-1}(\mathbf{x}/\mathbf{y})\right)^{n-1}\cdot[\bar{\mathbf{x}}^{\mathbf{j}}\bar{\mathbf{y}}^{\mathbf{k}}]\prod_{i=1}^m R_{\alpha_i}(\bar{x}_i,\bar{y}_i)\\
&= \frac{1}{2^{(n-1)(m-1)}}\sum_{\mathbf{j}+\mathbf{k}=\mathbf{n-1}}\mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}]\left(\sum_{s\geq 1}e_{2s-1}(\mathbf{x})\right)^{n-1}\prod_{i=1}^m r_{j_i,k_i}^{\alpha_i}\\
&= \frac{(n-1)!}{2^{(n-1)(m-1)}}\sum_{\mathbf{j}+\mathbf{k}=\mathbf{n-1}}\mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}]\sum_{\ell(\lambda)=n-1}\frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}\prod_{i=1}^m r_{j_i,k_i}^{\alpha_i}.
\end{aligned}$$

The result now follows from (3.1). □

4. Recovery of Poulalhon & Schaeffer's Formula

We require some extra notation to state the Poulalhon-Schaeffer formula for $c_{\alpha_1,\ldots,\alpha_m}^{(n)}$. First, we define symmetric polynomials $S_p(x_1,\ldots,x_l)$ by setting $S_0(x_1,\ldots,x_l)=1$ and

$$S_p(x_1,\ldots,x_l) = \sum_{p_1+\cdots+p_l=p}\prod_{i=1}^l\frac{1}{x_i}\binom{x_i}{2p_i+1}$$

for $p > 0$. Note that these have the simple generating series

$$\sum_{p\geq 0} S_p(x_1,\ldots,x_l)t^{2p} = \prod_{i=1}^{l} \frac{(1+t)^{x_i}-(1-t)^{x_i}}{2x_i t}, \tag{4.1}$$

which is obviously closely related to our series $R_\gamma(x,y)$ (see (2.1)). We also introduce an operator $\mathfrak{D}$ on $\mathbb{Q}[[x_1,\ldots,x_m]]$ defined as follows: For each i and all $j \geq 0$ set $\mathfrak{D}(x_i^j) = x_i(x_i-1)\cdots(x_i-j+1)$, and extend the action of $\mathfrak{D}$ multiplicatively to monomials $x_1^{j_1}\cdots x_m^{j_m}$ and then linearly to all of $\mathbb{Q}[[x_1,\ldots,x_m]]$. Finally, we define polynomials $P_a^b(x_1,\ldots,x_m)$ by setting $P_0^b(x_1,\ldots,x_m)=1$ for all $b\geq 1$ and letting

$$P_a^b(x_1,\ldots,x_m) = \sum_{\substack{\lambda\vdash a\\ \ell(\lambda)\leq b}} \mathfrak{D}\Big(\frac{e_{2\lambda+1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}\Big) \tag{4.2}$$

for $a,b\geq 1$, where $2\lambda+1=[3^{m_1}5^{m_2}\cdots]$ when $\lambda=[1^{m_1}2^{m_2}\cdots]$. Then the main result of [**6**] is the following intriguing formula for $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$.

Theorem 4.1 (Poulalhon-Schaeffer). *Let $\alpha_1,\ldots,\alpha_m \vdash n$ and set $r_i = n-\ell(\alpha_i)$ for all i. Let $g = \frac{1}{2}(\sum_i r_i - n + 1)$. If g is a nonnegative integer, then*

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{2g}\prod_i \mathrm{Aut}(\alpha_i)} \sum P_q^{n-1}(\mathbf{r}-\mathbf{2p}) \prod_{i=1}^{m} (\ell(\alpha_i)+2p_i-1)!\, S_{p_i}(\alpha_i),$$

where $\mathbf{r}-\mathbf{2p} = (r_1-2p_1,\ldots,r_m-2p_m)$ and the sum extends over all tuples $(q,p_1,\ldots,p_m)$ of nonnegative integers with $q+p_1+\cdots+p_m = g$.

Before proceeding to deduce this result from Theorem 2.1, we pause for a few remarks. First, the integer g identified in Theorem 4.1 is called the *genus* of the associated factorizations of $(1\,2\,\cdots\,n)$, and it has well-understood geometric meaning; see [**2**], for example. The primary benefit of the Poulalhon-Schaeffer formula (over Theorem 2.1) is that the dependence on genus is explicit. For instance, when $g=0$ it is immediately clear that Theorem 4.1 reduces to the very simple

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = n^{m-1}\prod_{i=1}^{m}\frac{(\ell(\alpha_i)-1)!}{\mathrm{Aut}(\alpha_i)}.$$

The $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$ in this case are known as *top connection coefficients*, and the above formula was originally given by Goulden and Jackson [**3**].

Secondly, we note that Poulalhon and Schaeffer actually define $P_a(\mathbf{x}) = \sum_{\lambda\vdash a}\mathfrak{D}(e_{2\lambda+1}(\mathbf{x})/\mathrm{Aut}(\lambda))$, and ignore the condition $\ell(\lambda)\leq b$ in our definition of P_a^b. However, replacing P_q^{n-1} with P_q in Theorem 4.1 has nil effect, since for $\mathfrak{D}(e_{2\lambda+1}(\mathbf{x}))|_{\mathbf{x}=\mathbf{r}-\mathbf{2p}}$ to be nonzero some monomial in $e_{2\lambda+1}(\mathbf{x})$ must be of the form $x_1^{j_1}\cdots x_m^{j_m}$ with $j_i \leq r_i - 2p_i$ for all i. This implies $|2\lambda+1| = \sum_i j_i \leq \sum_i (r_i-2p_i) = 2g+n-1-\sum_i 2p_i$, while the conditions $\lambda\vdash q$ and $q+\sum_i p_i = g$ give $|2\lambda+1| = 2q+\ell(\lambda) = 2g-\sum_i 2p_i+\ell(\lambda)$. Thus we require $\ell(\lambda)\leq n-1$ for nonzero contributions to $P_q(\mathbf{r}-\mathbf{2p})$.

Lemma 4.2. *Let $s,t_1,\ldots,t_m$ be nonnegative integers. Set $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{t}=(t_1,\ldots,t_m)$, and let $f(\mathbf{x})$ be a homogeneous polynomial of total degree $t_1+\cdots+t_m-s$. Then*

$$\left[\frac{\mathbf{x}^{\mathbf{t}}}{\mathbf{t}!}\right](x_1+\cdots+x_m)^s f(\mathbf{x}) = s!\,\mathfrak{D}\big(f(\mathbf{x})\big)\big|_{x_1=t_1,\ldots,x_m=t_m}.$$

Proof. Consider the case where $f(\mathbf{x}) = x_1^{j_1}\cdots x_m^{j_m}$ with $\sum_i j_i = \sum_i t_i - s$. Here

$$\begin{aligned}
[\mathbf{x}^{\mathbf{t}}](x_1+\cdots+x_m)^s f(\mathbf{x}) &= [\mathbf{x}^{\mathbf{t}}]\sum_{i_1+\cdots+i_m=s}\frac{s!}{i_1!\cdots i_m!}x_1^{i_1+j_1}\cdots x_m^{i_m+j_m}\\
&= \begin{cases} \dfrac{s!}{\mathbf{t}!}\displaystyle\prod_{i=1}^{m}\frac{t_i!}{(t_i-j_i)!} & \text{if } j_i\leq t_i \text{ for all } i,\\ 0 & \text{otherwise}\end{cases}\\
&= \frac{s!}{\mathbf{t}!}\,\mathfrak{D}(f(\mathbf{x}))|_{\mathbf{x}=\mathbf{t}}.
\end{aligned}$$

The general result now follows by linearity. □

Proof of Theorem 4.1: Comparing (2.1) and (4.1) we find that, for $\gamma = (\gamma_1, \dots, \gamma_l) \vdash n$,

$$R_\gamma(x,y) = 2^{l-1} \prod_{i=1}^{l} \gamma_i \cdot \sum_{p \geq 0} S_p(\gamma) x^{n-l-2p} y^{2p+l-1}.$$

Thus

$$r^{\gamma}_{j,k} = \begin{cases} \dfrac{2^{\ell(\gamma)-1} z_\gamma}{\mathrm{Aut}(\gamma)} S_p(\gamma) & \text{if } (j,k) = (n-\ell(\gamma)-2p, \ell(\gamma)+2p-1), \\ 0 & \text{otherwise.} \end{cases}$$

From this and Theorem 2.1 we immediately have

$$c^{(n)}_{\alpha_1,\dots,\alpha_m} = \frac{n^{m-1}}{2^{2g} \prod_i \mathrm{Aut}(\alpha_i)} \sum_{\mathbf{p}} \left[\frac{\mathbf{x}^{\mathbf{r}-\mathbf{2p}}}{(\mathbf{r}-\mathbf{2p})!} \right] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)} \prod_{i=1}^{m} (\ell(\alpha_i)+2p_i-1)!\, S_{p_i}(\alpha_i),$$

where the outer sum extends over all tuples $\mathbf{p} = (p_1, \dots, p_m)$ of nonnegative integers. Now

$$\left[\frac{\mathbf{x}^{\mathbf{r}-\mathbf{2p}}}{(\mathbf{r}-\mathbf{2p})!} \right] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)} = \sum_{s=0}^{n-1} \sum_{\substack{\lambda \vdash q \\ \ell(\lambda)=n-1-s}} \left[\frac{\mathbf{x}^{\mathbf{r}-\mathbf{2p}}}{(\mathbf{r}-\mathbf{2p})!} \right] \frac{e_1(\mathbf{x})^s}{s!} \frac{e_{2\lambda+1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}, \tag{4.3}$$

where q is chosen to make $e_1(\mathbf{x})^s e_{2\lambda+1}(\mathbf{x})$ of total degree $\sum_i (r_i - 2p_i)$. In particular, if $\lambda \vdash q$ and $\ell(\lambda) = n-1-s$, then $e_1(\mathbf{x})^s e_{2\lambda+1}(\mathbf{x})$ is of degree $|2\lambda+1| + s = 2|\lambda| + \ell(\lambda) + s = 2q+n-1$, so we require

$$2q+n-1 = \textstyle\sum_i (r_i - 2p_i) = (2g+n-1) - \sum_i 2p_i,$$

or simply $q + p_1 + \cdots + p_m = g$. Finally, applying the lemma to the RHS of (4.3) results in

$$\sum_{s=0}^{n-1} \sum_{\substack{\lambda \vdash q \\ \ell(\lambda)=n-1-s}} \mathfrak{D}\left(\frac{e_{2\lambda+1}(\mathbf{x})}{\mathrm{Aut}(\lambda)} \right) \Bigg|_{\mathbf{x}=\mathbf{r}-\mathbf{2p}} = P^{n-1}_q(\mathbf{r}-\mathbf{2p})$$

and this completes the proof. □

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