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Kazhdan-Lusztig immanants and products of matrix minors, II

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Abstract. We show that for each permutation w containing no decreasing subsequence of length k, the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ vanishes on all matrices having k equal columns. We also construct new and simple inequalities satisfied by the minors of totally nonnegative matrices.

Résumé. Nous démontrons que pour chaque permutation w qui ne contient aucune sous-suite décroissante de longeur k, l'immanant de Kazhdan-Lusztig $\mathrm{Imm}_w(x)$ s'annule sur toutes les matrices avec k colonnes identiques. Nous introduisons par ailleurs des inégalités simples et nouvelles satisfaites par les mineurs des matrices complètement non-negatives.

1. Introduction and Preliminaries

The Kazhdan-Lusztig basis $\{C'_w(q) \mid w \in S_n\}$ of the Hecke algebra $H_n(q)$, originally introduced in [**10**], has seen several applications in combinatorics and positivity. In [**14**] Rhoades and Skandera define the Kazhdan-Lusztig immanants via the Kazhdan-Lusztig basis and obtain various positivity results concerning linear combinations of products of matrix minors. Lam, Postnikov, and Pylyavskyy, in turn, use these results in [**11**] to resolve several conjectures in Schur positivity. In this paper, we further develop algebraic properties of the Kazhdan-Lusztig immanants and apply these immanants to obtain additional positivity results.

Fix $n \in \mathbb{N}$ and let $x = (x_{ij})_{1 \leq i,j \leq n}$ be a matrix of n^2 variables. For a pair of subsets $I, J \subseteq [n]$, with $|I| = |J|$, define the (I, J)-minor of x, denoted $\Delta_{I,J}(x)$, to be the determinant of the submatrix of x indexed by rows in I and columns in J. We adopt the convention that the empty minor $\Delta_{\emptyset,\emptyset}(x)$ is equal to 1. An $n \times n$ matrix A is said to be *totally nonnegative* (TNN) if every minor of A is a nonnegative real number. A polynomial $p(x)$ in n^2 variables is called totally nonnegative if whenever $A = (a_{i,j})_{1 \leq i,j \leq n}$ is a totally nonnegative matrix, $p(A) \underset{\mathrm{def}}{=} p(a_{1,1}, \ldots, a_{n,n})$ is a nonnegative real number. [**2**], [**3**], [**9**], [**13**], [**12**], [**19**] give a graph theoretic characterization of totally nonnegative matrices which is used by Rhoades and Skandera in [**15**] and [**14**] to construct several examples of totally nonnegative polynomials.

Let H denote the infinite array $(h_{j-i})_{i,j \geq 1}$, where h_i denotes the complete homogeneous symmetric function of degree i. (see, for example, [**18**]) Here we use the convention that $h_i = 0$ whenever $i < 0$. A polynomial $p(x)$ in n^2 variables is called *Schur nonnegative* (SNN) if whenever K is an $n \times n$ submatrix of H, the symmetric function $p(K)$ is a nonnegative linear combination of Schur functions. By the Jacobi identity, the determinant is a trivial example of a SNN polynomial.

For $i \in [n-1]$, let s_i denote the adjacent transposition in S_n which is written $(i, i+1)$ in cycle notation. For a fixed $w \in S_n$, call an expression $s_{i_1} \cdots s_{i_\ell}$ representing w *reduced* if ℓ is minimal. In this case, define the *length* of w, denoted $\ell(w)$, to be ℓ.

For q a formal indeterminate, define the *Hecke algebra* $H_n(q)$ to be the $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra with generators $T_{s_1}, \ldots, T_{s_{n-1}}$ subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned}$$

For $w \in S_n$, define the Hecke algebra element T_w by

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}},$$

where $s_{i_1} \cdots s_{i_\ell}$ is any reduced expression for w. Specializing at $q = 1$, the map $T_{s_i} \mapsto s_i$ induces an isomorphism between $H_n(1)$ and the symmetric group algebra $\mathbb{C}[S_n]$.

The elements $\{C'_v(q) \,|\, v \in S_n\}$ of the Kazhdan-Lusztig basis of $H_n(q)$ have the form

$$C'_v(q) = \sum_{w \leq v} P_{w,v}(q) q^{-\ell(v)/2} T_w, \tag{1.1}$$

where

$$\{P_{w,v}(q) \,|\, w, v \in S_n\}$$

are polynomials in $\mathbb{N}[q]$ called the Kazhdan-Lusztig polynomials. We recall a couple of elementary properties of the Kazhdan-Lusztig polynomials.

Lemma 1.1. *For $w, v \in S_n$, $P_{w,v}(q) \equiv 0$ if and only if $w \not\leq v$, where $\leq$ is (strong) Bruhat ordering. Also, $P_{w,w}(q) \equiv 1$.*

A polynomial $p(x)$ in n^2 variables is called an *immanant* if it belongs to the $\mathbb{C}$-linear span of $\{x_{1,w(1)} \cdots x_{n,w(n)} \mid w \in S_n\}$. Following [**14**], for $w \in S_n$, define the *w-Kazhdan-Lusztig immanant* by

$$\mathrm{Imm}_w(x) \underset{\mathrm{def}}{=} \sum_{v \in S_n} (-1)^{\ell(w)-\ell(v)} P_{w_0 v, w_0 w}(1) x_{1,v(1)} \cdots x_{n,v(n)}, \tag{1.2}$$

where w_o denotes the long element of S_n, written $n(n-1)\ldots 1$ in one-line notation. Specializing at $w = 1$, we have that $\mathrm{Imm}_1(x) = det(x)$.

It follows from Lemma 1.1 that the expression $(-1)^{\ell(w)-\ell(v)} P_{w_0 v, w_0 w}(1)$ is nonzero if and only if $w \leq v$ in the Bruhat order and that $P_{w_0 w, w_0 w}(1) = 1$. Therefore, the set $\{\mathrm{Imm}_w(x) \mid w \in S_n\}$ forms a basis for the vector space of immanants. The Kazhdan-Lusztig immanants are both TNN and SNN and various examples of TNN and SNN polynomials can be constructed by studying the cone generated by the Kazhdan-Lusztig immanants [**14**]. Moreover, when w is 321-avoiding, the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ is satisfies a natural generalization of Lindström's Lemma [**15**].

2. Main

For $1 \leq k \leq n$, let $\Gamma_{n,k}$ denote the subset of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ consisting of all products of the form $\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x)$, where $I_1, J_1, \ldots, I_k, J_k \subseteq [n]$, $I_1 \uplus \cdots \uplus I_k = J_1 \uplus \cdots \uplus J_k = [n]$, and $|I_j| = |J_j|$ for all $j \in [k]$. Here $\uplus$ denotes disjoint union. Elements of $\Gamma_{n,k}$ are sometimes called complementary products of minors. In [**15**], Kazhdan-Lusztig immanants are used to find that the dimension of span$(\Gamma_{n,2})$ is equal to the n^{th} Catalan number C_n. In this paper we shall relate the dimension of span$(\Gamma_{n,k})$ to pattern avoidence in S_n for arbitrary k.

For $k \in \mathbb{N}$, let $S_{n,k}$ denote the set of permutations in S_n which do not have a decreasing subsequence of length $k+1$. For example, in one-line notation, $S_{3,2} = \{123, 213, 132, 312, 231\}$. Notice that $S_{n,k} = S_n$ for all $k \geq n$. We start by examining the image of $S_{n,k}$ under the Robinson-Schensted correspondence.

Let $\leq_{LR}$ be the preorder on S_n defined in [**10**] and let $s_{[1,k]}$ be the longest element in the subgroup of S_n generated by $s_1, \ldots, s_{k-1}$.

Lemma 2.1. *Suppose $v \notin S_{n,k-1}$. Then we have $v \leq_{LR} s_{[1,k]}$.*

Proof. Given any permutation w, define the pair of tableaux $(P'(w), Q'(w))$ to be the image of w under the Robinson-Schensted column insertion correspondence. Let $\lambda'(w)$ be the shape of these tableaux.

A well-known property of the Robinson-Schensted correspondence implies that $\lambda'(v) \geq \lambda'(s_{[1,k]})$ in the dominance order. This dominance relation in turn is known to be equivalent to the partial order on Kazhdan-Lusztig cells induced by the preorder $\leq_{LR}$. Thus in the preorder $\leq_{LR}$, every permutation in the cell of v precedes every permutation in the cell of $s_{[1,k]}$. (See [**1**], [**6**, Sec. 1], [**8**, Appendix].) $\square$

Proposition 2.2. *Suppose $A \in \mathrm{Mat}_n(\mathbb{C})$ has k equal rows and let $v \in S_{n,k-1}$. Then, $\mathrm{Imm}_v(A) = 0$.*

This result generalizes Proposition 3.14 of Rhoades and Skandera [**15**], which together with [**14**] implies that Proposition 2.2 holds when $k = 2$.

Proof. Define the element $[A]$ of $\mathbb{C}[S_n]$ by

$$[A] = \sum_{w \in S_n} a_{1,w(1)} \cdots a_{n,w(n)} w.$$

Let $i_1 < \cdots < i_k$ be the indices of k rows in A which are equal and let U be the subgroup of S_n which fixes all indices not contained in the set $\{i_1, \ldots, i_k\}$. Then

$$\sum_{u \in U} u$$

factors as $wz_{[1,k]}w'$ for some elements w, w' of S_n. It follows that $[A]$ factors as

$$\begin{aligned} [A] &= \left(\sum_{u \in U} u \right) f(A) \\ &= (wz_{[1,k]}w')f(A) \end{aligned}$$

for some group algebra element $f(A)$.

Let I be the two-sided ideal of $\mathbb{C}[S_n]$ spanned by $\{C'_u(1) \,|\, u \leq_{LR} s_{[1,k]}\}$ and let $\theta : \mathbb{C}[S_n] \to \mathbb{C}[S_n]/I$ be the canonical homomorphism. Clearly we have $\theta([A]) = 0$.

On the other hand, we have

$$\begin{aligned} \theta([A]) &= \theta \left(\sum_{w \in S_n} \mathrm{Imm}_w(A) C'_w(1) \right) \\ &= \sum_{w \in S_n} \mathrm{Imm}_w(A) \theta(C'_w(1)). \end{aligned}$$

Since $\theta(C'_w(1)) = 0$ for all permutations $w \leq_{LR} s_{[1,k]}$, we have

$$0 = \sum_{w} \mathrm{Imm}_w(A) \theta(C'_w(1)),$$

where the sum is over all permutations $w \not\leq_{LR} s_{[1,k]}$, i.e., those permutations having no decreasing subsequence of length k. Since the elements $\theta(C'_w(1))$ in this sum are linearly independent, we must have $\mathrm{Imm}_w(A) = 0$ for each permutation w having no decreasing subsequence of length k. □

Proposition 2.3. *Suppose $\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x) \in \Gamma_{n,k}$. Then, there exist $d_w \in \mathbb{C}$ such that $\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x) = \sum_{w \in S_{n,k}} d_w \mathrm{Imm}_w(x)$.*

Proof. The Kazhdan-Lusztig immanants form a basis for the vector space of immanants, so we may write

$$\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x), \tag{2.1}$$

for some $d_w \in \mathbb{C}$. If $k \geq n$ the claim is trivial, so we assume that $k < n$. We show that $d_w = 0$ whenever $w \notin S_{n,k}$.

Suppose that $C \in Mat_n(\mathbb{C})$ has $k+1$ equal rows. Then, by the pigeonhole principle, there exist two equal rows of C indexed by integers lying in one of $I_1, \ldots, I_k$. Hence, $\Delta_{I_1,J_1}(C) \cdots \Delta_{I_k,J_k}(C) = 0$.

Now let $B = (b_{ij}) \in Mat_n(\mathbb{C})$ be defined by $b_{ij} = 1$ for all i and j. By Proposition 2.2, since $k < n$ we have that $Imm_w(B) = 0$ for every $w \neq w_o$. Also, $Imm_{w_o}(B) = 1$. By the above paragraph, $\Delta_{I_1,J_1}(B) \cdots \Delta_{I_k,J_k}(B) = 0$. Therefore, applying both sides of (2.1) to B, we get that $d_{w_o} = 0$.

For $l \in \mathbb{N}$, define $T_{n,l}$ to be the set difference $S_{n,l} \smallsetminus S_{n,l-1}$. Suppose that $k < m < n$ and suppose that for all p satisfying $m < p \leq n$ we have that $d_w = 0$ for every $w \in T_{n,p}$. Give the elements of $T_{n,m}$ a total order which is an extension of their Bruhat ordering and write $T_{n,m} = \{w_1 < w_2 < \cdots < w_h\}$. Let $t \in [h]$ and suppose by induction that $d_w = 0$ for $w \in \{w_{t+1}, \ldots, w_h\}$. Since $w_t \in T_{n,m}$, there exist $i_1 < i_2 < \cdots < i_m$ such that $w_t(i_1) > w_t(i_2) > \cdots > w_t(i_m)$. Let $D \in Mat_n(\mathbb{C})$ be the matrix obtained by replacing the rows $i_1, \ldots, i_m$ in the permutation matrix for w_t by rows of 1's. By Proposition 2.2, $Imm_w(D) = 0$ for every $w \in S_{n,m-1}$. By (1.1) we also have that $\mathrm{Imm}_w(D) = 0$ for every $w \not\leq w_t$ in the Bruhat order. Since $k < m$, we have that $\Delta_{I_1,J_1}(D)\cdots\Delta_{I_k,J_k}(D)=0$. Thus, applying both sides of (2.1) to D, we get that $d_{w_t}=0$ and the Proposition follows by induction.

□

If $k=2$ in Proposition 2.3, results in [**15**] and [**14**] imply that the d_w must be nonnegative. For k arbitrary, Skandera [**16**] has given an elementary proof that whenever w avoids the patterns 3412 and 4231, (i.e., when the Schubert variety Γ_w corresponding to w is smooth), the coefficient d_w is also nonnegative. Using deeper properties of the dual canonical basis of $\mathcal{O}SL_n\mathbb{C}$, it is possible to show that the coefficients d_w are nonnegative for general k and w.

Proposition 2.4. $\dim(\mathrm{span}_{\mathbb{C}}(\Gamma_{n,k}))=|S_{n,k}|$.

Specializing at $k=2$, $S_{n,2}$ is the set of 321-avoiding permutations, so we have that $|S_{n,2}|=C_n$, the n^{th} Catalan number. Thus, this result is a generalization of Proposition 4.7 of [**15**].

Proof. By Proposition 2.3 we have that $\dim(\mathrm{span}_{\mathbb{C}}(\Gamma_{n,k}))\leq|S_{n,k}|$.

For each collection of sets $I_1,J_1,\ldots,I_k,J_k\subseteq[n]$ with $[n]=I_1\uplus\cdots\uplus I_k=J_1\uplus\cdots\uplus J_k$ and $|I_j|=|J_j|$ for each $j\in k$, let $min(I_1,J_1,\ldots,I_k,J_k)$ denote the unique minimal permutation in the Bruhat order which maps I_i into J_i for each $i\in[k]$. For example, if we set $n=6$, $I_1=\{1,3,6\},I_2=\{2,4\},I_3=\{5\},J_1=\{3,4,6\},J_2=\{1,5\},J_3=\{2\}$, we have that $min(I_1,J_1,I_2,J_2,I_3,J_3)=314526$ in one-line notation.

For $\Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x)\in\Gamma_{n,k}$ it is easy to see that there exist $d_w\in\mathbb{C}$ such that

$$\Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x)=\sum_{w\geq min(I_1,J_1,\ldots,I_k,J_k)}d_w x_{1,w(1)}\cdots x_{n,w(n)},$$

where $d_{min(I_1,J_1,\ldots,I_k,J_k)}=1$. In light of this, it suffices to show that for every permutation $w\in S_{n,k}$, there exists a collection of sets $I_1,J_1,\ldots,I_k,J_k\subseteq[n]$ such that $[n]=I_1\uplus\cdots\uplus I_k=J_1\uplus\cdots\uplus J_k$ and $|I_j|=|J_j|$ for each $j\in k$ and $w=min(I_1,J_1,\ldots,I_k,J_k)$. For then, we have that $dim(span_{\mathbb{C}}(\Gamma_{n,k}))\geq|S_{n,k}|$.

Let $w\in S_{n,k}$. Define a partial order on the set $P=\{(i,w(i))|i\in[n]\}$ by setting $(i,w(i))<(j,w(j))$ if $i<j$ and $w(i)<w(j)$. Now $\{(i_1,w(i_1)),\ldots,(i_m,w(i_m))\}\subseteq P$ with $i_1<\cdots<i_m$ is an antichain in P if and only if $(w(i_1),\ldots,w(i_m))$ is an decreasing subsequence of w. Hence, $\mathrm{width}(P)<k+1$ (see [**17**] for definitions). By Dilworth's Theorem, there exist k disjoint (possibly empty) chains $C_1,\ldots,C_k$ which partition P. Now, for each $j\in[k]$, write $C_j=\{(i_1,w(i_1)),\ldots,(i_{m_j},w(i_{m_j})\}$, with $i_1<\cdots<i_{m_j}$. Since C_j is a chain in P, $(w(i_1),\ldots,w(i_{m_j}))$ is an increasing subsequence of w. Define $I_j=\{i_1,\ldots i_{m_j}\}$ and $J_j=\{w(i_1),\ldots w(i_{m_j})\}$. It is now easy to check that $w=min(I_1,J_1,\ldots,I_k,J_k)$ and we are done.

□

The numbers $|S_{n,k}|$ were studied by Gessel [**7**] who found an expression involving Bessel functions for the generating function $\sum_{n\geq1}|S_{n,k}|t^n$. The authors do not know of a simple form of the polynomial $\sum_{k=1}^{n}|S_{n,k}|t^k$.

Corollary 2.5. *Suppose that* $I_1\uplus I_2=J_1\uplus J_2=[n]$, $|I_1|=|J_1|=n_1$, $|I_2|=|J_2|=n_2$, $w_1\in S_{n_1,k_1}$, *and* $w_2\in S_{n_2,k_2}$. *For* $i=1,2$ *let* x_i *be the submatrix of* x *with row set* I_i *and column set* J_i. *Then, there exist* $d_v\in\mathbb{C}$ *such that* $\mathrm{Imm}_{w_1}(x_1)\mathrm{Imm}_{w_2}(x_2)=\sum_v d_v\mathrm{Imm}_v(x)$, *where the sum is over* v *in* S_{n,k_1+k_2}.

Specializing at $w_1=w_2=1$, we have that the coefficients d_v in the Corollary are in fact nonnegative real numbers. (see [**15**], [**14**]) Again, one may use the properties of the dual canonical basis of $\mathcal{O}SL_n\mathbb{C}$ to show that $\{d_w\,|\,w\in S_n\}$ are nonnegative real numbers.

Proof. For $i=1,2$, by Propositions 2.3 and 2.4 there exist $p_{i,j}(x_i)\in\Gamma_{n_i,k_i}$ and $d_j\in\mathbb{C}$ such that $\mathrm{Imm}_{w_i}(x)=\sum_j d_j p_{i,j}(x_i)$. Since x_1 and x_2 are complementary submatrices of x, for any $p_1(x_1)\in\Gamma_{n_1,k_1}$ and $p_2(x_2)\in\Gamma_{n_2,k_2}$, the product $p_1(x_1)p_2(x_2)$ is contained in Γ_{n,k_1+k_2}. So, the product $\mathrm{Imm}_{w_1}(x_1)\mathrm{Imm}_{w_2}(x_2)$ is a linear combination of elements in Γ_{n,k_1+k_2}. The result now follows from Proposition 2.3.

□

Taken together, Propositions 2.3 and 2.4 imply that for $w\in S_{n,k}$, there exist $p_i(x)\in\Gamma_{n,k}$ and $d_i\in\mathbb{C}$ such that $\mathrm{Imm}_w(x)=\sum_{i=1}^{m}d_ip_i(x)$. Results in [**15**] and [**14**] show that, for $k=2$, we may in fact assume that the $p_i(x)$ are contained in a subset of $\Gamma_{n,2}$ which is in a natural bijective correspondence with the set of Dyck paths of length $2n$. It would be interesting to see if an analogous result holds for general k.

We now investigate when polynomials in $\mathrm{span}(\Gamma_{n,k})$ are TNN or SNN. For any integers n and k satisfying $1 \leq k \leq n$, define the poset $P_{n,k}$ on $\Gamma_{n,k}$ by
$\Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x) \leq \Delta_{I'_1,J'_1}(x)\cdots\Delta_{I'_k,J'_k}(x)$ if and only if the difference
$\Delta_{I'_1,J'_1}(x)\cdots\Delta_{I'_k,J'_k}(x) - \Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x)$ is TNN. In [**15**] the authors develop necessary and sufficient combinatorial conditions for polynomials $p(x) \in \mathrm{span}(\Gamma_{n,2})$ to be TNN. For all positive integers n, $P_{n,2}$ has a unique maximal element given by $\Delta_{I,I}(x)\Delta_{J,J}(x)$, where $I = \{1, 3, 5, \dots\}$ and $J = \{2, 4, 6, \dots\}$. Also, the determinant $\Delta_{[n],[n]}(x)\Delta_{\emptyset,\emptyset}(x)$ is always a minimal element of $P_{n,2}$. In [**14**] the authors show that the combinatorial tests in [**15**] constitute sufficient conditions for polynomial in $\mathrm{span}(\Gamma_{n,2})$ to be SNN. Therefore, whenever $\Delta_{I,J}(x)\Delta_{I',J'}(x) \leq \Delta_{K,L}(x)\Delta_{K',L'}(x)$ in $P_{n,2}$ we also have that $\Delta_{K,L}(x)\Delta_{K',L'}(x) - \Delta_{I,J}(x)\Delta_{I',J'}(x)$ is SNN. It is unknown whether the converse of the last sentence is true.

In [**4**], [**5**], and [**14**] the positivity properties of differences of the form $x_{1,w(1)}\cdots x_{n,w(n)} - x_{1,u(1)}\cdots x_{n,u(n)}$ for $w, u \in S_n$ are studied. The authors prove the following about the subposet $P_{n,n}$ consisting of products of n nonempty minors.

Theorem 2.6. *Let $w, u \in S_n$. Then, the following statements are equivalent.*
1. $w \leq u$ in the Bruhat order.
2. The difference $x_{1,w(1)}\cdots x_{n,w(n)} - x_{1,u(1)}\cdots x_{n,u(n)}$ is TNN.
3. The difference $x_{1,w(1)}\cdots x_{n,w(n)} - x_{1,u(1)}\cdots x_{n,u(n)}$ is SNN.
4. Whenever the difference $x_{1,w(1)}\cdots x_{n,w(n)} - x_{1,u(1)}\cdots x_{n,u(n)}$ is applied to a Jacobi-Trudi matrix, the result is a nonnegative linear combination of monomial symmetric functions.
5. The difference $x_{1,w(1)}\cdots x_{n,w(n)} - x_{1,u(1)}\cdots x_{n,u(n)}$ is a nonnegative linear combination of Kazhdan-Lusztig immanants.

With the above results as motivation, we show that $P_{n,k}$ has a unique maximal element for arbitrary k and that certain comparable elements in $P_{n,k}$ have differences which are SNN as well as TNN.

Lemma 2.7. *Let $(I_1, \dots, I_p)$ and $(I'_1, \dots, I'_p)$ be seqences of sets satisfying*
$I_1 \uplus \cdots \uplus I_p = I'_1 \uplus \cdots \uplus I'_p,$
$|I_i| = |I'_i| \qquad$ *for all i.*
Fix indices $k < \ell$ and define increasing sequences $(\alpha_1, \dots, \alpha_p)$ and $(\alpha'_1, \dots, \alpha'_p)$ by

$$I_k \cup I_\ell = \{\alpha_1, \dots, \alpha_p\},$$
$$I'_k \cup I'_\ell = \{\alpha'_1, \dots, \alpha'_p\}.$$

Define the sequences of sets $(J_1, \dots, J_p)$ and $(J'_1, \dots, J'_p)$ by

$$J_i = \begin{cases} \{\alpha_1, \alpha_3, \dots, \} & \text{if } i = k, \\ \{\alpha_2, \alpha_4, \dots, \} & \text{if } i = \ell, \\ I_i & \text{otherwise.} \end{cases}$$

$$J'_i = \begin{cases} \{\alpha'_1, \alpha'_3, \dots, \} & \text{if } i = k, \\ \{\alpha'_2, \alpha'_4, \dots, \} & \text{if } i = \ell, \\ I'_i & \text{otherwise.} \end{cases}$$

Then the immanant

$$\Delta_{J_1,J'_1}(x)\cdots\Delta_{J_p,J'_p}(x) - \Delta_{I_1,I'_1}(x)\cdots\Delta_{I_p,I'_p}(x)$$

is totally nonnegative and Schur nonnegative.

Proof. This difference is

$$\frac{\Delta_{J_1,J'_1}(x)\cdots\Delta_{J_p,J'_p}(x)}{\Delta_{J_k,J'_k}(x)\Delta_{J_\ell,J'_\ell}(x)}(\Delta_{J_k,J'_k}(x)\Delta_{J_\ell,J'_\ell}(x) - \Delta_{I_k,I'_k}(x)\Delta_{I_\ell,I'_\ell}(x)),$$

which is totally nonnegative and Schur nonnegative by [**15**, Prop. 4.6] and [**14**, Thm. 5.2]. □

Our next result implies that the poset $P_{n,k}$ has a maximal element for any n and k.

Theorem 2.8. *Let $(I_1, \dots, I_p)$ and $(I'_1, \dots, I'_p)$ be two sequences of sets satisfying*

$$I_1 \uplus \cdots \uplus I_p = I'_1 \uplus \cdots \uplus I'_p = [n],$$
$$|I_1| = |I'_i| \qquad \text{for all } i$$

and define sets $J_1, \dots, J_p$ by

$$J_i = \{i \in [n] \,|\, i \equiv j \mod p\}.$$

Then the immanant

$$\Delta_{J_1,J_1}(x) \cdots \Delta_{J_p,J_p}(x) - \Delta_{I_1,I'_1}(x) \cdots \Delta_{I_p,I'_p}(x)$$

is totally nonnegative and Schur nonnegative.

Proof. Applying several iterations of Lemma 2.7 to the sets $I_1, \dots, I_p, I'_1, \dots I'_p$, we obtain the desired result. □

Corollary 2.9. *Let $k < \ell$ and define the sequences of sets $(I_1, \dots, I_k)$ and $(J_1, \dots, J_\ell)$ by*

$$I_j = \{i \in [n] \,|\, i \equiv j \mod k\},$$
$$J_j = \{i \in [n] \,|\, i \equiv j \mod \ell\}.$$

Then the immanant

$$\Delta_{J_1,J_1}(x) \cdots \Delta_{J_p,J_p}(x) - \Delta_{I_1,I'_1}(x) \cdots \Delta_{I_p,I'_p}(x)$$

is totally nonnegative and Schur nonnegative.

Not much is known about the posets $P_{n,k}$ in general. Obviously we have that $P_{n,1} \subset P_{n,2} \subset \cdots \subset P_{n,n}$. By Theorem 2.6 $P_{n,n}$ contains a subposet isomorphic to (the dual of) the Bruhat order on S_n. Also, it is possible to show that any element of span($\Gamma_{3,3}$) is TNN or SNN if and only if it may be expressed as a nonnegative linear combination of Kazhdan-Lusztig immanants. In particular, this allows one to construct the poset $P_{3,3}$ and see that it coincides with the analogous poset constructed by considering SNN differences. It would be interesting to see what $P_{n,k}$ looks like in general.

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