Characterization of Eulerian binomial and Sheffer posets

Richard Ehrenborg and Margaret A. Readdy

ABSTRACT. We completely characterize the factorial functions of Eulerian binomial posets. The factorial function $B(n)$ either coincides with $n!$, the factorial function of the infinite Boolean algebra, or 2^{n-1}, the factorial function of the infinite butterfly poset. We also classify the factorial functions for Eulerian Sheffer posets. An Eulerian Sheffer poset with binomial factorial function $B(n) = n!$ has Sheffer factorial function $D(n)$ identical to that of the infinite Boolean algebra, the infinite Boolean algebra with two new coatoms inserted, or the infinite cubical poset. Moreover, we are able to classify the Sheffer factorial functions of Eulerian Sheffer posets with binomial factorial function $B(n) = 2^{n-1}$ as the doubling of an upside-down tree with ranks 1 and 2 modified.

When we impose the further condition that a given Eulerian binomial or Eulerian Sheffer poset is a lattice, this forces the poset to be the infinite Boolean algebra $\mathbb{B}_X$ or the infinite cubical lattice $\mathbb{C}_X^{<\infty}$. We also include several poset constructions that have the same factorial functions as the infinite cubical poset, demonstrating that classifying Eulerian Sheffer posets is a difficult problem.

RÉSUMÉ. Nous caractérisons complétement les fonctions factorielles des ensembles partiellement ordonnés (posets) binomiaux Eulériens. La fonction factorielle $B(n)$ coincide avec $n!$, la fonction factorielle de l'algèbre de Boole infinie, ou avec 2^{n-1}, la fonction factorielle de l'ensemble partiellement ordonné "papillon" infini. Nous classifions aussi les fonctions factorielles des ensembles partiellement ordonnés (posets) de Sheffer Eulériens. Un poset de Sheffer Eulérien dont la fonction binomiale factorielle est $B(n) = n!$ a la fonction factorielle de Sheffer $D(n)$ indentique avec celle de l'algèbre de Boole infinie, ou avec celle de l'algèbre de Boole infinie avec deux nouveaux coatômes insérés, ou avec celle de l'ensemble partiellement ordonné cubique infini. De plus, nous pouvons classifier les fonctions factorielles de Sheffer des posets de Sheffer Eulériens avec la fonction binomiale factorielle $B(n) = 2^{n-1}$ comme le doublement d'un arbre á l'envers avec les rangs 1 et 2 modifiés.

Quand nous démandons la condition additionnelle qu'un poset binomial Eulérien ou Sheffer Eulérien soit un treillis, celle-ci force l'ensemble à être l'algèbre de Boole infinie $\mathbb{B}_X$ ou le treillis cubique infini $\mathbb{C}_X^{<\infty}$. Plusieures constructions des posets sont inclus qui possèdent les mêmes fonctions factorielles que le poset cubique infini, ce qui demontre que la classification des posets de Sheffer Eulériens est une problème trés difficile.

1. Introduction

Binomial posets were introduced by Doubilet, Rota and Stanley [5] to explain why generating functions naturally occurring in combinatorics have certain forms. They are highly regular posets since the essential requirement is that every two intervals of the same length have the same number of maximal chains. As a result, many poset invariants are determined. For instance, the quintessential Möbius function is described

by the generating function identity

(1.1)
$$\sum_{n \geq 0} \mu(n) \cdot \frac{t^n}{B(n)} = \left(\sum_{n \geq 0} \frac{t^n}{B(n)} \right)^{-1},$$

where $\mu(n)$ is the Möbius function of an n-interval and $B(n)$ is the factorial function, that is, the number of maximal chains in an n-interval. A binomial poset is required to contain an infinite chain so that there are intervals of any length in the poset.

A graded poset is *Eulerian* if its Möbius function is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$ for all $x \leq y$ in the poset. Equivalently, every interval of the poset satisfies the Euler-Poincaré relation: the number of elements of even rank is equal to the number of elements of odd rank in the interval. The major example of Eulerian posets are face lattices of convex polytopes and more generally, the face posets of regular CW-spheres. Hence there is a large geometric and topological interest in understanding them.

A natural question arises: which binomial posets are Eulerian? By equation (1.1) it is clear that the Eulerian property can be determined by knowing the factorial function. In this paper we classify the factorial functions of Eulerian binomial posets. There are two possibilities, namely, for the factorial function to correspond to that of the infinite Boolean algebra or the infinite butterfly poset.

Notice that this classification is on the level of the factorial function, not the poset itself. There are more Eulerian binomial posets than these two essential examples. See Examples 2.9 and 2.10. However, we are able to classify the intervals of Eulerian binomial posets. They are either isomorphic to the finite Boolean algebra or the finite butterfly poset.

Sheffer posets were introduced by Reiner [10] and independently by Ehrenborg and Readdy [6]. A Sheffer poset requires the number of maximal chains of an interval $[x, y]$ of length n to be given by $B(n)$ if $x > \hat{0}$ and $D(n)$ if $x = \hat{0}$. The upper intervals $[x, y]$ where $x > \hat{0}$ have the property of being binomial. Hence the interest is to understand the Sheffer intervals $[\hat{0}, y]$. Just like binomial posets, the Möbius function is completely determined:

(1.2)
$$\sum_{n \geq 1} \overline{\mu}(n) \frac{t^n}{D(n)} = - \left(\sum_{n \geq 1} \frac{t^n}{D(n)} \right) \cdot \left(\sum_{n \geq 0} \frac{t^n}{B(n)} \right)^{-1},$$

where $\overline{\mu}$ is the Möbius function of a Sheffer interval of length n; see [6, 10].

The classic example of a Sheffer poset is the infinite cubical poset (see Example 3.6). In this case, every interval $[x, y]$ of length n, where x is not the minimal element $\hat{0}$, has $n!$ maximal chains. In fact, every such interval is isomorphic to a Boolean algebra. Intervals of the form $[\hat{0}, y]$ have $2^{n-1} \cdot (n-1)!$ maximal chains and are isomorphic to the face lattice of a finite dimensional cube.

In sections 3 and 4 we completely classify the factorial functions of Eulerian Sheffer posets. The factorial function $B(n)$ follows from the classification of binomial posets. The pair of factorial functions $B(n)$ and $D(n)$ fall into three cases (see Theorem 4.1) and one infinite class (Theorem 3.10). Furthermore, for the infinite class we can describe the underlying Sheffer intervals; see Theorem 3.11. For two of the three cases in Theorem 3.11 we can also classify the Sheffer intervals. However for the third case, we construct a multitude of examples of Sheffer intervals. It is a very striking coincidence that this case corresponds to the factorial functions of an infinite cubical lattice. That is, we can find many Sheffer posets having the same factorial functions as the infinite cubical lattice, but the Sheffer intervals are not isomorphic to the finite cubical lattice; see Examples 3.9, 4.2, 4.3 and 4.4. However, if we add the extra requirement that each Sheffer interval is a lattice then we obtain that the Sheffer intervals are isomorphic to cubical lattices.

When we impose the further condition that a given Eulerian binomial or Eulerian Sheffer poset is a lattice, this forces the poset to be the infinite Boolean algebra $\mathbb{B}_X$ or the infinite cubical lattice $\mathbb{C}_X^{<\infty}$. See Examples 2.10 and 4.6.

The classification of the factorial functions hinges on the condition that the posets under consideration contain an infinite chain. In the concluding remarks, we discuss what could happen if this condition is removed. We give examples of posets having their factorial functions behave like the face lattice of the dodecahedron, but themselves are not isomorphic to this lattice.

2. Eulerian binomial posets

DEFINITION 2.1. A locally finite poset P with $\hat{0}$ is called a *binomial poset* if it satisfies the following three conditions:

(i) P contains an infinite chain.

(ii) Every interval $[x, y]$ is graded; hence P has rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval*.

(iii) For all $n \in \mathbb{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* of P.

If P does not satisfy condition (i) and has a unique maximal element then we say P is a *finite binomial poset*.

For standard poset terminology, we refer the reader to [**12**]. The number of elements of rank k in an n-interval is given by $B(n)/(B(k) \cdot B(n-k))$. Especially, an n-interval has $A(n) = B(n)/B(n-1)$ atoms (and coatoms). The function $A(n)$ is called the *atom function* and expresses the factorial function as $B(n) = A(n) \cdot A(n-1) \cdots A(1)$. Directly we have $B(0) = B(1) = A(1) = 1$. Since the atoms of an $(n-1)$-interval are contained among the set of atoms of an n-interval, the inequality $A(n-1) \leq A(n)$ holds. Observe if a finite binomial poset has rank j, the factorial and atom functions are only defined up to j. For further background material on binomial posets, see [**5, 11, 12**].

EXAMPLE 2.2. Let $\mathbb{B}$ be the collection of finite subsets of the positive integers ordered by inclusion. The poset $\mathbb{B}$ is a binomial poset with factorial function $B(n) = n!$ and atom function $A(n) = n$. An n-interval is isomorphic to the Boolean algebra B_n. This example is the *infinite Boolean algebra*.

EXAMPLE 2.3. Let $\mathbb{T}$ be the *infinite butterfly poset*, that is, $\mathbb{T}$ consists of the elements $\{\hat{0}\} \cup \mathbb{P} \times \{1, 2\}$ where $(n, i) \prec (n+1, j)$ for all $i, j \in \{1, 2\}$ and $\hat{0}$ is the unique minimal element. The poset $\mathbb{T}$ is a binomial poset. It has factorial function $B(n) = 2^{n-1}$ for $n \geq 1$ and atom function $A(n) = 2$ for $n \geq 2$. Let T_n denote an n-interval in $\mathbb{T}$.

EXAMPLE 2.4. Given two ranked posets P and Q, define the *rank product* $P * Q$ by

$$P * Q = \{(x, z) \in P \times Q \ : \ \rho_P(x) = \rho_Q(z)\}.$$

Define the order relation by $(x, y) \leq_{P*Q} (z, w)$ if $x \leq_P z$ and $y \leq_Q w$. If P and Q are binomial posets then so is the poset $P * Q$. It has the factorial function $B_{P*Q}(n) = B_P(n) \cdot B_Q(n)$. This example is due to Stanley [**12**, Example 3.15.3 d]. The rank product is also known as the Segre product; see [**4**].

EXAMPLE 2.5. For $q \geq 2$ let P_q be the face poset of an q-gon. Observe that this is a finite binomial poset of rank 3 with the factorial function $B(2) = 2$ and $B(3) = 2q$. Let $q_1, \ldots, q_r$ be a list of integers with each $q_i \geq 2$. Let $P_{q_1, \ldots, q_r}$ be the poset obtained by identifying all the minimal elements of P_{q_1} through P_{q_r} and identifying all the maximal elements. This is also a binomial poset with factorial function $B(2) = 2$ and $B(3) = 2 \cdot (q_1 + \cdots + q_r)$. It is straightforward to see that each rank 3 binomial poset with $B(2) = 2$ is of this form.

The *Euler-Poincaré relation* for a finite graded poset states that it has the same number of elements of even as odd rank. A poset is called *Eulerian* if every non-singleton interval satisfies the Euler-Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$ for all $x \leq y$ in P.

LEMMA 2.6. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*

This lemma is implicit in the two papers [**3, 7**]. We now conclude

PROPOSITION 2.7. *To verify that a poset is Eulerian it is enough to verify that every interval of even rank satisfies the Euler-Poincaré relation.*

For an n-interval of an Eulerian binomial poset the Euler-Poincaré relation states

$$(2.1) \qquad \sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k) \cdot B(n-k)} = 0.$$

Note from Proposition 2.7 this relation will only give information when n is an even integer. Also observe that $B(2) = A(2) = 2$ since every 2-interval is a diamond.

THEOREM 2.8. *Let P be an Eulerian binomial poset with factorial function $B(n)$. Then either*
  (i) *the factorial function $B(n)$ is given by $B(n) = n!$ and every n-interval is isomorphic to the Boolean algebra B_n, or*
  (ii) *the factorial function $B(n)$ is given by $B(n) = 2^{n-1}$ and every n-interval is isomorphic to the butterfly poset T_n.*

It is tempting to state this theorem as, "There are only two Eulerian binomial posets, namely, the infinite Boolean algebra $\mathbb{B}$ and the infinite butterfly poset $\mathbb{T}$." However, this is false. The next two examples demonstrate this.

EXAMPLE 2.9. Let Q be an infinite poset with a minimal element $\hat{0}$ containing an infinite chain such that every interval of the form $[\hat{0}, x]$ is a chain. Observe the poset Q is an infinite tree and, in fact, is a binomial poset with factorial function $B(n) = 1$. Thus we know that both $\mathbb{B} * Q$ and $\mathbb{T} * Q$ are Eulerian binomial posets.

EXAMPLE 2.10. For each infinite cardinal κ there is a Boolean algebra consisting of all finite subsets of a set X of cardinality κ. We denote this poset by $\mathbb{B}_X$. Observe that different cardinals give rise to non-isomorphic Boolean algebras.

We now state a very useful lemma.

LEMMA 2.11. *Let P and P' be two Eulerian binomial posets having atom functions $A(n)$ and $A'(n)$ which agree for $n \leq 2m$, where $m \geq 2$. Then the following equality holds:*

$$(2.2) \qquad \frac{1}{A(2m+1)} \cdot \left(1 - \frac{1}{A(2m+2)}\right) = \frac{1}{A'(2m+1)} \cdot \left(1 - \frac{1}{A'(2m+2)}\right).$$

We will use Lemma 2.11 in the following manner.

COROLLARY 2.12. *Let P and P' be two Eulerian binomial posets satisfying the conditions in Lemma 2.11. Assume furthermore there is a lower and an upper bound for $A'(2m+2)$ of the form $L \leq A'(2m+2) < U$. Let x be the left-hand side of equation (2.2). Then we obtain a lower and an upper bound for $A'(2m+1)$, namely*

$$(2.3) \qquad \frac{1}{x} \cdot \left(1 - \frac{1}{L}\right) \leq A'(2m+1) < \frac{1}{x} \cdot \left(1 - \frac{1}{U}\right).$$

We will see these bounds can be improved by using that $A'(2m+1)$ is in fact an integer.

PROPOSITION 2.13. *Let P' be an Eulerian binomial poset with factorial function $B'(n)$ satisfying $B'(3) = 6$. Then the factorial function is given by $B'(n) = n!$.*

PROOF. Let P be the infinite Boolean algebra $\mathbb{B}$ with atom function $A(n) = n$ and factorial function $B(n) = n!$. We will first prove that the two factorial functions $B(n)$ and $B'(n)$ are identical, equivalently that the two atom functions $A(n)$ and $A'(n)$ are equal.

Assume that the two atom functions A and A' agree up to $2m = j$. Since $A(n) = n$ the left-hand side of equation (2.2) is equal to $1/(j+2)$. We have the following bounds for $A'(j+2)$: $j = A'(j) \leq A'(j+2) < \infty$. Applying Corollary 2.12 we obtain the following bounds on $A'(j+1)$:

$$j + 1 - \frac{2}{j} \leq A'(j+1) < j + 2.$$

Since $A'(j+1)$ is an integer and $j \geq 4$ we conclude that $A'(j+1) = j+1$. This implies that $A'(j+2) = j+2$ and hence we conclude the two atom functions are equal. $\square$

PROPOSITION 2.14. *Let P be a finite binomial poset of rank n with factorial function $B(k) = k!$ for $k \leq n$. Then the poset P is isomorphic to the Boolean algebra B_n.*

PROOF. Directly the result is true for $n \leq 2$. Assume it is true for all posets of rank $n-1$ and consider a poset P of rank n. Since P is a binomial poset with factorial function $B(k) = k!$, we know that the number of elements of rank k in P is given by $\binom{n}{k}$. Especially, the cardinality of P is given by 2^n. Let c be a coatom in the poset. Observe that the interval $[\hat{0}, c]$ is isomorphic to B_{n-1} by the induction hypothesis and hence the coatom c is greater than all but one atom a in the poset P. Similarly, the interval $[a, \hat{1}]$ is also isomorphic to B_{n-1}. Since the two intervals $[a, \hat{1}]$ and $[\hat{0}, c]$ are disjoint and have the same cardinality 2^{n-1}, the poset P is the disjoint union of these two intervals.

Using the binomial property of P, an element z of rank k in the lower interval $[\hat{0}, c]$ is covered by $n - k$ elements in the poset P and by $n - k - 1$ elements in the interval $[\hat{0}, c]$. Thus there is a unique element in $[a, \hat{1}]$ that covers z. Denote this element by $\varphi(z)$. By a similar argument we obtain that φ is a bijective function from $[\hat{0}, c]$ to $[a, \hat{1}]$. Let $z \prec w$ be a cover relation in $[\hat{0}, c]$. Consider the 2-interval $[z, \varphi(w)]$. As every 2-interval is a diamond there is an element v different from w such that $z \prec v \prec \varphi(w)$. Since w is the unique element in $[\hat{0}, c]$ that is covered by $\varphi(w)$, the element v belongs to the upper interval $[a, \hat{1}]$. Also, the element $\varphi(z)$ is the unique element in the upper interval that covers z, we conclude that $v = \varphi(z)$ and especially $\varphi(w)$ covers $\varphi(z)$. Hence the function φ is order-preserving. By the symmetric argument φ^{-1} is also order-preserving. Therefore the poset P is the Cartesian product of $[\hat{0}, c]$ with the two element poset B_1 and we conclude that P is isomorphic to the Boolean algebra B_n. $\square$

PROPOSITION 2.15. *Let P' be an Eulerian binomial poset with factorial function $B'(n)$ satisfying $B'(3) = 4$. Then the factorial function is given by $B'(n) = 2^{n-1}$ for $n \geq 1$.*

PROOF. Let P be the butterfly poset $\mathbb{T}$ and $A(n)$ its atom function, where $A(1) = 1$ and $A(n) = 2$ for $n \geq 2$. Similar to the proof of Proposition 2.13 we consider how $A(n)$ and $A'(n)$ relate.

Assume that the two atom functions agree up to $2m = j$. Now the right-hand side of equation (2.2) is equal to $1/4$. For $A'(j + 2)$ we have the bounds $2 = A'(j) \leq A'(j + 2) < \infty$. Applying Corollary 2.12 we obtain

$$2 \leq A'(j + 1) < 4.$$

Consider now the possibility that $A'(j + 1) = 3$. Let $[x, y]$ be a $(j + 1)$-interval in P'. For $1 \leq k \leq j$ there are $B'(j + 1)/(B'(k) \cdot B'(j + 1 - k)) = 3 \cdot 2^{j-1}/(2^{k-1} \cdot 2^{j-k}) = 3$ elements of rank k in this interval. Let c be a coatom. The interval $[x, c]$ has two atoms, say a_1 and a_2. Moreover, the interval $[x, c]$ has two elements of rank 2, say b_1 and b_2. Moreover we know that each b_j covers each a_i. Let a_3 and b_3 be the third atom, respectively the third rank 2 element, in the interval $[x, y]$. We know that b_3 covers two atoms in $[x, y]$. One of them must be a_1 or a_2, say a_1. But then a_1 is covered by the three elements b_1, b_2 and b_3. But this contradicts the fact that each atom is covered by exactly two elements. Hence this rules out the case $A'(j + 1) = 3$.

The only remaining possibility is $A'(j + 1) = 2$, implying $A'(j + 2) = 2$. Hence the atom functions $A(n)$ and $A'(n)$ are equal. $\square$

LEMMA 2.16. *Let P be a finite binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the butterfly poset T_n.*

PROOF OF THEOREM 2.8: The atom function of an Eulerian binomial poset satisfies $2 = A(2) \leq A(3)$. Hence $B(3) = A(3) \cdot B(2)$ is an even integer greater than or equal to 4. The Euler-Poincaré relation implies that

$$\frac{1}{B(4)} = \frac{1}{B(3)} - \frac{1}{8},$$

implying that $B(3) < 8$. Hence there are only two remaining cases, which are considered in Propositions 2.13 and 2.15. The corresponding structure statements are considered in Proposition 2.14 and Lemma 2.16. $\square$

THEOREM 2.17. *Let L be an Eulerian binomial poset which we furthermore require to be a lattice. Then L is isomorphic to the Boolean algebra $\mathbb{B}_X$ where X is the set of atoms of the poset P.*

PROOF. Since every interval of L is a lattice we can rule out the butterfly factorial function. Hence $B(n) = n!$ and every interval $[\hat{0}, x]$ is a Boolean lattice. Let φ be the map from L to $\mathbb{B}_X$ defined by $\varphi(x) = \{a \in X \ : \ a \leq x\}$. The inverse of φ is given by $\varphi(Y) = \vee_{a \in Y} a$. It is straightforward to see that both φ and φ^{-1} are order-preserving. Hence the two lattices L and $\mathbb{B}_X$ are isomorphic and the result follows. $\square$

We end this section with a result that will be used in Section 4 when we study Eulerian Sheffer posets.

PROPOSITION 2.18. *There is no finite binomial poset P' of rank $j + 1 \geq 4$ with the atom function*

$$A'(n) = \begin{cases} n & \text{if } n \leq j, \\ j + 2 & \text{if } n = j + 1. \end{cases}$$

PROOF. Assume that the poset P' exists. Then it has $j + 2$ atoms and $j + 2$ coatoms. Each atom x lies below exactly j coatoms and each coatom c lies above exactly j atoms. Moreover, by the proof of Proposition 2.13 we know that each of the intervals $[\hat{0}, c]$ and $[x, \hat{1}]$ is isomorphic to B_j.

Define a multigraph G with the $j + 2$ atoms as the vertices. For each coatom c let there be an edge xy between the two unique atoms x and y such that $x, y \not\leq c$. Since each atom is not below exactly two coatoms, each vertex of the graph has degree equal to 2. Hence the graph is a disjoint union of cycles.

Pick a coatom c that corresponds to an edge xy. The coatom c is greater than the j atoms $z_1, \ldots, z_j$. Using that the interval $[\hat{0}, c]$ is a Boolean algebra, let w_i be the unique coatom in the interval $[\hat{0}, c]$ that is not greater than z_i. Let d_i be the atom in the interval $[w_i, \hat{1}] \cong B_2$ distinct from c. Observe for $i \neq k$ we have $z_i < w_k < d_k$. Hence the j coatoms $c, d_1, \ldots, \widehat{d_i}, \ldots, d_j$ are all the coatoms greater than z_i. Moreover, since $j \geq 3$ we conclude that $d_1, \ldots, d_j$ are all distinct.

Consider the j atoms below d_k. They are $z_1, \ldots, \widehat{z_k}, \ldots, z_j$ and exactly one of x and y. Thus the edge e_k corresponding to d_k intersects the edge xy. This holds for all j edges e_k. Hence we obtain the contradiction $4 = \deg(x) + \deg(y) \geq 2 + j$. Thus there is no such finite binomial poset. $\square$

3. Eulerian Sheffer posets

Sheffer posets, also know as upper binomial posets, were first defined by Reiner [**10**] and independently discovered by Ehrenborg and Readdy [**6**].

DEFINITION 3.1. A locally finite poset P with $\hat{0}$ is called a *Sheffer poset* if it satisfies the following four conditions:

(i) P contains an infinite chain.
(ii) Every interval $[x, y]$ is graded; hence P has rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval*.
(iii) Two n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$, such that $y \neq \hat{0}$, $v \neq \hat{0}$, have the same number $D(n)$ of maximal chains.
(iv) Two n-intervals $[x, y]$ and $[u, v]$, such that $x \neq \hat{0}$, $u \neq \hat{0}$, have the same number $B(n)$ of maximal chains.

As in the finite binomial poset case, if P does not satisfy condition (i) and has a unique maximal element then we say P is a *finite Sheffer poset*.

An interval of the form $[\hat{0}, y]$ is called a *Sheffer interval*, whereas an interval $[x, y]$, where $x > \hat{0}$, is called a *binomial interval*. Similarly, the functions $B(n)$ and $D(n)$ are called the *binomial*, respectively, the *Sheffer factorial function*. The number of elements of rank $k \geq 1$ in a Sheffer interval of length n is given by $D(n)/(D(k) \cdot B(n - k))$. Especially, a Sheffer interval $[\hat{0}, y]$ has $C(n) = D(n)/D(n - 1)$ coatoms. The function $C(n)$ is called the *coatom function* and we have $D(n) = C(n) \cdot C(n - 1) \cdots C(1)$. Observe that $D(1) = C(1) = 1$.

We will be using the following two facts to exclude possible factorial functions.

FACT 3.2. (a) The inequality $A(n - 1) \leq C(n) < \infty$ holds since the set of coatoms in a Sheffer interval of rank n, say $[\hat{0}, y]$, contains the set of coatoms in an $(n - 1)$-interval $[x, y]$, and there are a finite number of them.
(b) The value $B(k)$ divides $C(n) \cdots C(n - k + 1)$ for $n > k$, since the number of elements of rank $n - k$ in a Sheffer interval of length n is given by $D(n)/(D(n - k) \cdot B(k)) = C(n) \cdots C(n - k + 1)/B(k)$.

EXAMPLE 3.3. Every binomial poset is a Sheffer poset. The factorial functions are equal, that is, $D(n) = B(n)$ for $n \geq 1$.

EXAMPLE 3.4. The rank product $P * Q$ of two Sheffer posets P and Q is also a Sheffer poset with the factorial functions $B_{P*Q}(n) = B_P(n) \cdot B_Q(n)$ and $D_{P*Q}(n) = D_P(n) \cdot D_Q(n)$.

EXAMPLE 3.5. For a poset P with a unique minimal element $\hat{0}$, let the *dual suspension* $\Sigma^*(P)$ be the poset P with two new elements a_1 and a_2. Let the order relations be as follows: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$ for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and the atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2 \cdot B(n-1)$ for $n \geq 2$.

One may extend the dual suspension Σ^* by inserting k new atoms instead of 2. Yet again it will take a binomial poset to a Sheffer poset. However we have no need of this extension since it does not preserve the Eulerian property.

For a ranked poset P (not necessarily having a unique minimal element) and a set X define the *power poset* P^X as follows. Let the underlying set be given by

$$P^X = \left\{ f : X \to P \ : \ \sum_{x \in X} \rho(f(x)) < \infty \right\}$$

and define the order relation by componentwise comparison, that is, $f \leq_{P^X} g$ if $f(x) \leq g(x)$ for all x in X.

EXAMPLE 3.6. Let P be the three element poset $_0\diagup^{\displaystyle *}\diagdown_1$ and let X be an infinite set. Then the poset $\mathbb{C}_X = P^X \cup \{\hat{0}\}$, that is, the poset P^X with a new minimal element adjoined, is a Sheffer poset. This example is precisely the *infinite cubical poset* with the factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. Similar to Example 2.10, for different infinite cardinalities of X we obtain non-isomorphic cubical posets. Note, however, this poset is not a lattice since the two atoms $(0, 0, \ldots)$ and $(1, 1, \ldots)$ do not have a join.

EXAMPLE 3.7. Let $E_2, E_3, \ldots$ be an infinite sequence of disjoint nonempty finite sets, where E_n has cardinality e_n. Consider the poset

$$U_{e_2, e_3, \ldots} = \{\hat{0}\} \ \cup \ \bigcup_{n \geq 2} \prod_{i \geq n} E_i,$$

where $\prod$ stands for Cartesian product. We make this into a ranked poset by letting $\hat{0}$ be the minimal element, and defining the cover relation by

$$(x_n, x_{n+1}, x_{n+2}, \ldots) \prec (x_{n+1}, x_{n+2}, \ldots),$$

where $x_i \in E_i$. Thus the elements of $\prod_{i \geq n} E_i$ have rank $n-1$. This poset is a Sheffer poset with the atom function $A(n) = 1$ and coatom function is given by $C(n) = e_n$ for $n \geq 2$. We may view this poset as an "upside-down tree" with a minimal element attached.

Naturally, the previous example is not an Eulerian poset. However, we can use it to construct Eulerian Sheffer posets as the next two examples illustrate.

EXAMPLE 3.8. Consider the poset $\mathbb{T} * U_{e_2, e_3, \ldots}$, where $e_2 = e_4 = e_6 = \cdots = 1$. This poset has the factorial functions $B(n) = 2^{n-1}$ and $D(n) = 2^{n-1} \cdot \prod_{i=2}^{n} e_i$. In Theorem 3.10 we will observe that the condition that $e_{2j} = 1$ implies that the poset is Eulerian.

In general the rank product $\mathbb{T} * P$ can be viewed as the "doubling" of the poset P. This notion was introduced by Bayer and Hetyei in [**2**].

EXAMPLE 3.9. Let $\mathbb{B} \cup \{\hat{0}\}$ be the infinite Boolean algebra with a new minimal element adjoined. This is a Sheffer poset with factorial functions $B(n) = n!$ and $D(n) = (n-1)!$. Now consider the rank product $(\mathbb{B} \cup \{\hat{0}\}) * U_{2,2,\ldots}$. It has the factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. This poset has the same factorial functions as the infinite cubical poset and hence it is an Eulerian poset.

For an Eulerian Sheffer poset of rank n, the Euler-Poincaré relation states

$$(3.1) \qquad 1 + \sum_{k=1}^{n} (-1)^k \cdot \frac{D(n)}{D(k) \cdot B(n-k)} = 0.$$

Again by Proposition 2.7 this relation will only give us information for n even. When $n = 2m$ we can write this relation as:

$$(3.2) \qquad \frac{2}{D(2m)} + \sum_{k=1}^{2m-1} (-1)^k \cdot \frac{1}{D(k) \cdot B(2m-k)} = 0.$$

Also note that $D(2) = C(2) = 2$.

THEOREM 3.10. *Let P be an Eulerian Sheffer poset with the binomial factorial function satisfying $B(0) = 1$ and $B(n) = 2^{n-1}$ for $n \geq 1$. Then the coatom function $C(n)$ and the poset P satisfy:*

(i) *$C(3) \geq 2$, and a length 3 Sheffer interval is isomorphic to a poset of the form $P_{q_1,\ldots,q_r}$ described in Example 2.5.*

(ii) *$C(2m) = 2$ for $m \geq 2$ and the two coatoms in a length $2m$ Sheffer interval cover exactly the same elements of rank $2m - 2$.*

(iii) *$C(2m + 1) = h$ is an even positive integer, for $m \geq 2$. Moreover, the set of h coatoms in a Sheffer interval of length $2m + 1$ can be grouped into $h/2$ pairs, $\{c_1, d_1\}$, $\{c_2, d_2\}$, $\ldots$, $\{c_{h/2}, d_{h/2}\}$, such that c_i and d_i cover the same two elements of rank $2m - 1$.*

PROOF. Part (i) is immediate since $A(2) \leq C(3)$. Next we prove (ii). Let $j = 2m$. In this case the Euler-Poincaré relation for a Sheffer j-interval states:

$$(3.3) \qquad \sum_{k=1}^{j} (-1)^k \cdot \frac{1}{D(k) \cdot 2^{j-k-1}} = 0.$$

Use equation (3.3) in the case of a $(j-2)$-interval to eliminate the first $j-2$ terms in the j-interval case of (3.3) gives the equality (ii). Since $D(j)/(D(j-2) \cdot B(2)) = D(j-1)/(D(j-2) \cdot B(1))$, the two coatoms in the Sheffer j-interval cover the same elements of rank $j-2$.

Finally, we consider (iii). Assume that $C(j+1) = h$, where $j = 2m$. Let $[\hat{0}, y]$ be a Sheffer interval of rank $j+1$. The number of elements of rank j and of rank $j-1$ are both given by h. Moreover each element of rank $j-1$ is covered by exactly 2 elements of rank j, and by part (ii), each element of rank j covers 2 elements of rank $j-1$. Hence the order relations between elements of rank $j-1$ and j are those of rank 1 and 2 in the poset $P_{q_1,\ldots,q_r}$ in Example 2.5, where $q_1 + \cdots + q_r = h$.

Let $z_1, \ldots, z_q$ be q coatoms in the Sheffer $(j+1)$-interval $[\hat{0}, y]$ such that z_i covers w_i and w_{i-1}, where we count modulo q in the indices. That is, z_1 through z_q correspond to the edges in a q-gon and w_1 through w_q to the vertices. Consider an element x of rank $j-2$ that is covered by w_1. The interval $[x, y]$ is isomorphic to T_3, that is, the interval has exactly 2 atoms and 2 coatoms. In this interval the element x is covered by one more element of rank $j-1$. Call it v. If the element v does not correspond to the elements $w_2, \ldots, w_q$, we obtain the contradiction that the interval $[x, y]$ has 4 coatoms. If v belongs to the elements $w_2, \ldots, w_q$, say w_i, then the interval $[x, y]$ has the coatoms z_1, z_2, z_i, z_{i+1}. When $q \geq 3$ the set $\{z_1, z_2, z_i, z_{i+1}\}$ has at least 3 members. Hence the only possibility is that $q = 2$ and $v = w_2$. Also the coatoms z_1 and z_2 cover the same elements of rank $j-1$.

We conclude that the only possibility is that all q_i's are equal to 2, that is, $q_1 = \cdots = q_r = 2$. Hence $r = h/2$ and h is an even integer. Moreover, we also obtain a pairing of the coatoms such that the two coatoms in each pair cover the same elements. $\square$

Given a graded poset P of rank n and a subset $S \subseteq \{1, \ldots, n-1\}$, the *rank selected poset P_S* is the graded poset consisting of the elements

$$P_S = \{\hat{0}, \hat{1}\} \cup \{x \in P \ : \ \rho(x) \in S\}.$$

Combining the conclusions of Theorem 3.10, we have

THEOREM 3.11. *Let P be an Eulerian Sheffer poset with the binomial factorial function satisfying $B(0) = 1$ and $B(n) = 2^{n-1}$ for $n \geq 1$ and coatom function $C(n)$. Set $e_2 = e_4 = e_6 = \cdots = 1$ and $e_{2m+1} = C(2m + 1)/2$ for all $m \geq 1$. Let Q be the poset $\mathbb{T} * U_{e_2, e_3, \ldots}$ from Example 3.8. Suppose n is an integer greater than or equal to 3 and $S = \{3, 4, \ldots, n-1\}$. Then the rank selection S of the rank n Sheffer interval $[\hat{0}, y]$ in P is isomorphic to the rank selection S of the rank n Sheffer interval $[\hat{0}, z]$ in Q, that is,*

$$[\hat{0}, y]_S \cong [\hat{0}, z]_S.$$

Furthermore, the poset $[\hat{0}, y]$ is obtained by replacing every length 3 Sheffer interval in $[\hat{0}, z]$ by a rank 3 binomial poset with $C(3)$ coatoms.
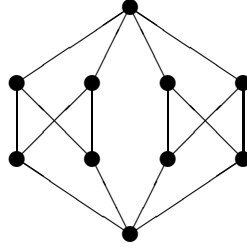
FIGURE 1. A finite Sheffer poset with the same factorial functions as the cubical lattice.

4. Eulerian Sheffer posets with factorial function $B(n) = n!$

In this section we will classify Eulerian Sheffer posets that have the factorial function $B(n) = n!$, that is, every interval $[x, y]$, where $x > \hat{0}$, is a Boolean algebra.

THEOREM 4.1. *Let P be an Eulerian Sheffer poset with binomial factorial function $B(n) = n!$. Then the Sheffer factorial function $D(n)$ satisfies one of the following three alternatives:*

(i) $D(n) = 2 \cdot (n-1)!$. *In this case every Sheffer n-interval is of the form $\Sigma^*(B_{n-1})$.*

(ii) $D(n) = n!$. *In this case the poset is a binomial poset and hence every Sheffer n-interval is isomorphic to the Boolean algebra B_n.*

(iii) $D(n) = 2^{n-1} \cdot (n-1)!$. *If we furthermore assume that a Sheffer n-interval $[\hat{0}, y]$ is a lattice then the interval $[\hat{0}, y]$ is isomorphic to the cubical lattice C_n.*

The cubical posets of Example 3.6 and Example 3.9 demonstrate there is no classification of the non-lattice Sheffer intervals in case (iii) of Theorem 4.1. The following examples further illustrates Sheffer posets (both finite and infinite) having the same factorial functions as the cubical poset.

EXAMPLE 4.2. Let C_n be the finite cubical lattice, that is, the face lattice of an $(n-1)$-dimensional cube. We are going to deform this lattice as follows. The 1-skeleton of the cube is a bipartite graph. Hence the set of atoms A has a natural decomposition as $A_1 \cup A_2$. Every rank 2 element (edge) covers exactly one atom in each A_i. Consider the poset

$$H_n = (C_n - A) \cup A_1 \times \{1, 2\}.$$

That is, we remove all the atoms and add in two copies of each atom from A_1. Define the cover relations for the new elements as follows. If a in A_1 is covered by b then let b cover both copies $(a, 1)$ and $(a, 2)$. The poset H_n is a Sheffer poset with the cubical factorial functions.

The poset in Figure 1 is the atom deformed cubical lattice H_3. This poset is also obtained as length 3 Sheffer interval in Example 3.9.

EXAMPLE 4.3. Let P and Q be two Sheffer posets (finite or infinite) having the cubical factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. Their diamond product, namely $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$, also has the cubical factorial functions.

EXAMPLE 4.4. As an extension of the previous example, let P be a Sheffer poset (finite or infinite) having the cubical factorial functions. Then for a set X the poset $(P - \{\hat{0}\})^X \cup \{\hat{0}\}$ is a Sheffer poset with the cubical factorial functions. The cubical poset (Example 3.6) is an illustration of this.

If we require the extra condition every Sheffer interval is a lattice, we obtain it is in fact the cubical lattice.

PROPOSITION 4.5. *Let P be a finite Sheffer poset of rank n with the cubical factorial functions $B(k) = k!$ for $k \leq n-1$ and $D(k) = 2^{k-1} \cdot (k-1)!$ for $1 \leq k \leq n$. If P is a lattice then P is isomorphic to the cubical lattice C_n.*

PROOF. The proof is by induction on the rank n of P. The induction base $n \leq 2$ is straightforward to verify. Assume true for all posets of rank $n-1$ and consider a rank n poset P. Using the cubical factorial functions, we know that the half open interval $(\hat{0}, \hat{1}]$ contains 3^{n-1} elements. Let c be a coatom

in the poset. The interval $[\hat{0}, c]$ is isomorphic to C_{n-1} by the induction hypothesis. Now define a function $\varphi : (\hat{0}, c] \longrightarrow (\hat{0}, \hat{1}] - (\hat{0}, c]$ as follows. For z in $(\hat{0}, c]$ let $\varphi(z)$ be the unique atom in the interval $[z, \hat{1}]$ that does not belong to the interval $[z, c]$. The existence and uniqueness follows from the fact the atom function satisfies $A(k) - A(k-1) = 1$. Also note that $\varphi(z)$ covers the element z.

We next verify the function φ is injective. If we have $\varphi(z) = \varphi(w)$ then z and w have the same rank. Also observe that $\varphi(z) \not\leq c$ by the definition of the function φ. This contradicts that the interval $[\hat{0}, \hat{1}]$ is a lattice, since z and w have the two upper bounds $\varphi(z)$ and c.

The function φ also preserves the cover relations. If $z \prec w$ the two-interval $[z, \varphi(w)]$ contains two atoms which must be w and $\varphi(z)$. Hence $\varphi(z) \prec \varphi(w)$. Let Φ be the image of the function φ. By a similar argument the inverse function $\varphi^{-1} : \Phi \longrightarrow (\hat{0}, c]$ also preserves the cover relations. Thus as posets $(\hat{0}, c]$ and Φ are isomorphic. Moreover, the disjoint union $(\hat{0}, c] \cup \Phi$ is an upper order ideal of the poset P and has cardinality $2 \cdot 3^{n-2}$.

The poset P has $C(n) = 2n - 2$ coatoms. One of them is the coatom c. Since c covers $2n - 4$ elements there are $2n - 4$ coatoms in Φ. Hence there is a unique coatom d that does not belong to the upper order ideal $(\hat{0}, c] \cup \Phi$. Since the interval $[\hat{0}, d]$ is isomorphic to the cubical lattice C_{n-1} and has $3^{n-2} + 1$ elements, we conclude that the complement of the upper order ideal is the lower order ideal $[\hat{0}, d]$. Thus we have the partition $(\hat{0}, c] \cup \Phi \cup (\hat{0}, d]$ of $P - \{\hat{0}\}$.

It remains to show that there is a bijective function $\psi : (\hat{0}, d] \longrightarrow \Phi$ such that $\psi(z)$ covers z and ψ preserves the cover relation. Define $\psi : (\hat{0}, d] \longrightarrow (\hat{0}, y] - (\hat{0}, d] = (\hat{0}, c] \cup \Phi$ by letting $\psi(z)$ be the unique atom in the interval $[z, \hat{1}]$ that does not belong to the interval $[z, d]$. Observe that if $\psi(z) \in (\hat{0}, c]$ we obtain that $z < \psi(z) \leq c$, contradicting that $(\hat{0}, c]$ and $(\hat{0}, d]$ are disjoint. Hence the image of ψ is Φ. The remaining properties of ψ are proven just like those for the function φ.

Hence $P - \{\hat{0}\}$ is isomorphic to the Cartesian product of the three element poset $\bigwedge$ with $(\hat{0}, c] \cong C_{n-1}$. That is, the poset is isomorphic to the cubical lattice C_n. $\square$

EXAMPLE 4.6. Define $\mathbb{C}_X^{<\infty}$ to be a subposet of the cubical poset $\mathbb{C}_X = P^X \cup \{\hat{0}\}$ in Example 3.6, where P is the three element poset $0 \bullet \!\!\overset{*}{\diagup}\!\!\diagdown\, \bullet 1$ Define

$$\mathbb{C}_X^{<\infty} = \{f \in P^X \;:\; |f^{-1}(1)| < \infty\} \cup \{\hat{0}\}.$$

That is, for each function f only a finite number of elements of X take on non-zero values. Since the union of two finite sets is finite it follows that the join of the two elements is defined. It follows that $\mathbb{C}_X^{<\infty}$ is a lattice. Observe the subposet $\mathbb{C}_X^{<\infty}$ remains a Sheffer poset with the cubical factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. Call this poset the *infinite cubical lattice*.

THEOREM 4.7. *Let L be an Eulerian Sheffer poset that is also a lattice. Then L is either isomorphic to $\mathbb{B}_X$ where X is the set of atoms of L or L is the infinite cubical lattice $\mathbb{C}_X^{<\infty}$ where X is the set of rank 2 elements of L which are greater than some fixed atom a in L.*

PROOF. Using Theorem 2.17 we know that the binomial factorial function is $B(n) = n!$. Since every Sheffer interval is a lattice there are only two choices for the Sheffer factorial function. The case $D(n) = n!$ is indeed the Boolean algebra which is the first alternative of the conclusion of the theorem. Hence let us consider the second choice $D(n) = 2^{n-1} \cdot (n-1)!$. Thus every interval $[\hat{0}, y]$ is a finite cubical lattice.

Let a be an atom of the lattice L and let X be the set of elements of rank 2 which cover a. Define the function $\varphi : L \longrightarrow \mathbb{C}_X^{<\infty}$ as follows. Set $\varphi(\hat{0}) = \hat{0}$. For $x \in L$ and $x > \hat{0}$ let y be the join of a and x. Since the interval $[\hat{0}, y]$ is a finite cubical lattice, the non-minimal elements of this interval can be encoded by functions $g : Y \longrightarrow P$, where is P is the three element poset in Example 4.6. Furthermore we may assume that the set Y is all the elements in the interval $[a, y]$ that cover a. Without loss of generality, we may choose the encoding so that the atom a is the constant function 0.

Encode the element x as such a function $g : Y \longrightarrow P$. Observe that g does not take the value 0, since that would contradict that the join of a and x is y. Now define $f : X \longrightarrow P$ by

$$f(z) = \begin{cases} g(z) & \text{if } z \in Y, \\ 0 & \text{if } z \in X - Y. \end{cases}$$

Observe that since Y is a finite set, we know that f belongs to the lattice $\mathbb{C}_X^{<\infty}$. Hence set $\varphi(x)$ to be the function f.

The inverse of φ is given as follows. For f, a non-zero element of the lattice $\mathbb{C}_X^{<\infty}$ let the set Y be defined as

$$Y = \{z \in X \ : \ f(z) \neq 0\}.$$

In the lattice L let the element y be the join $\bigvee_{z \in Y} z$. Observe that $a \leq y$. Since the interval $[\hat{0}, y]$ is isomorphic to the finite cubical lattice $\mathbb{C}_Y$, let x be the unique element corresponding to the function f restricted to Y. That is, the inverse of φ is given by $\varphi^{-1}(f) = x$. Moreover let $\varphi^{-1}(\hat{0}) = \hat{0}$.

Observe that both φ and φ^{-1} are order preserving, thus proving that the lattices L and $\mathbb{C}_X^{<\infty}$ are isomorphic. $\qquad \square$

Note that it is enough to work with the join operation in this proof, since a locally finite join semi-lattice with unique minimal element is a lattice [**12**, Proposition 3.3.1].

5. Concluding remarks

An interesting research project is to classify the factorial functions of finite Eulerian binomial posets and finite Eulerian Sheffer posets. Two examples of finite Sheffer posets are the face lattices of the dodecahedron and the four-dimensional regular polytope known as the 120-cell. In Theorems 3.10 and 4.1 many finite possibilities for the factorial functions were excluded since there was no possibility to extend the factorial function to higher ranks. A first step in this classification is to consider these cases.

Also note the following lemma, the proof of which follows directly from Proposition 2.7.

LEMMA 5.1. *Let P be an Eulerian finite binomial (Sheffer) poset of odd rank n. Let Q be the poset obtain by taking k disjoint copies of P and identifying the minimal, respectively, maximal elements. Then Q is an Eulerian finite binomial (Sheffer) poset. The only value of the factorial function(s) that changes is the one that enumerates the maximal chains, namely, $B_Q(n) = k \cdot B_P(n)$ in the binomial case, and $D_Q(n) = k \cdot D_P(n)$ in the Sheffer case.*

A larger class of posets to consider are the triangular posets [**5**]. A poset is *triangular* if every interval $[x, y]$, where x has rank n and y has rank m, has $B(n, m)$ maximal chains. Both binomial and Sheffer posets are triangular. A non-trivial Eulerian example of a finite triangular poset is the face lattice of the 4-dimensional regular polytope known as the 24-cell. Can the factorial function $B(n, m)$ be classified for Eulerian triangular posets?

Classifying finite Eulerian Sheffer posets only by their factorial functions seems to be hard as seen from the multitude of examples having the cubical factorial functions. We leave the reader with three examples of Sheffer posets with the same factorial functions as the face lattice of the dodecahedron, each of which is not isomorphic to this face lattice.

EXAMPLE 5.2. *An Eulerian finite Sheffer poset with the same factorial functions as the face lattice of the dodecahedron.* For an n-gon define a CW-complex X_n as follows. First take the antiprism of the n-gon. We then have a CW-complex consisting of two n-gons and $2n$ triangles. Note that at every vertex three triangles and one n-gon meet. Now subdivide each of the two n-gons by placing a vertex in each n-gon and attaching this vertex by n new edges to the n vertices of the n-gon. Let this be the CW-complex X_n.

Observe that X_n consists of $2n + 2$ vertices, $6n$ edges and $4n$ triangles. Moreover, at $2n$ of the vertices 5 triangles meet. At the other two vertices n triangles meet. Label these two vertices a and b. Also note that X_5 is the boundary complex of an icosahedron. Observe for $n \geq 3$ that X_n is a simplicial complex. However, for $n = 2$ it is necessary to view X_2 as a CW-complex.

Construct a CW-complex Y by taking X_2 and X_3 and identifying the vertices labeled a and identifying the vertices labeled b. See Figure 2. The dual of the face poset of Y is an Eulerian Sheffer poset with factorial functions agreeing with the face lattice of a dodecahedron.

EXAMPLE 5.3. For $1 \leq i \leq 3$ let Z_i be the boundary of a 3-dimensional simplex with vertices $z_{i,1}$, $z_{i,2}$, $z_{i,3}$ and $z_{i,4}$. Similarly, for $1 \leq j \leq 4$ let W_j be the spherical CW-complex consisting of two triangles sharing the three edges. Call the vertices $w_{1,j}$, $w_{2,j}$ and $w_{3,j}$. Now identify vertex $z_{i,j}$ with $w_{i,j}$. We then have a CW-complex that has 12 vertices, $3 \cdot 6 + 4 \cdot 3 = 30$ edges and $3 \cdot 4 + 4 \cdot 2 = 20$ triangles. Observe that the vertex figure of every vertex is the disjoint union of a 2-gon and a triangle. Thus the dual of the face poset is Sheffer poset with the same factorial functions as the face lattice of a dodecahedron. In fact, one may obtain several of these CW-complexes by choosing different identifications between the two classes of vertices.
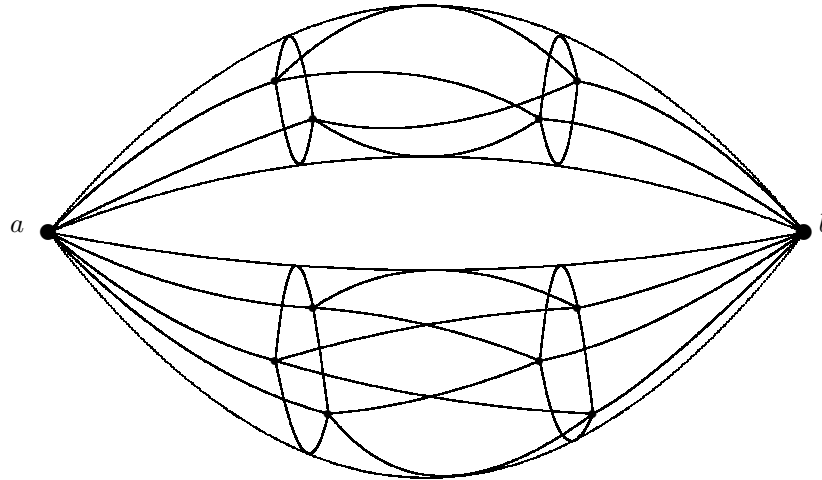
Figure 2. The CW-complex obtained by joining the complexes X_2 and X_3 at the vertices a and b.

Example 5.4. A third example is formed by taking two X_2's from Example 5.2 and the boundary of one 3-dimensional simplex, Z, from Example 5.3 and identifying vertices a_1, a_2, b_1 and b_2 with the vertices of the simplex.

A different proof of Proposition 2.14 may be given using the following result of Stanley. A graded finite poset P is a Boolean algebra if every 3-interval is a Boolean algebra and for every interval $[x, y]$ of rank of least 4 the open interval (x, y) is connected. See [**9**, Lemma 8]. Hence it is natural to ask if one can extend this result to cubical lattices. That is, a graded finite poset P is a cubical lattice if every 3-interval $[x, y]$, where $x > \hat{0}$, is a Boolean algebra, every 3-interval $[\hat{0}, y]$ is the face lattice of a square, and for every interval $[x, y]$ of rank of least 4 the open interval (x, y) is connected.

One may drop the Eulerian condition and ask to characterize Sheffer posets which are lattices. The lattice-theoretic techniques of Farley and Schmidt may be useful [**8**].

Finally, there are long-standing open questions regarding binomial posets. One such question was whether there exist two binomial posets having the same factorial function but non-isomorphic intervals. This question was very recently settled by Jörgen Backelin (personal communication). However, it is still unknown if there is a binomial poset having the atom function $A(n) = F_n$, the nth Fibonacci number. See Exercise 78b, Chapter 3 in [**12**].

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CHARACTERIZATION OF EULERIAN BINOMIAL AND SHEFFER POSETS

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: jrge@ms.uky.edu
URL: http://www.ms.uky.edu/~jrge

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: readdy@ms.uky.edu
URL: http://www.ms.uky.edu/~readdy