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Bijective counting of Kreweras walks and loopless triangulations

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ABSTRACT. We consider lattice walks in the plane starting at the origin, remaining in the first quadrant $i, j \geq 0$ and made of West, South and North-East steps. There are nice formulas for the number of such walks. But, although several proofs of these formulas have been proposed over the years, none of them provides a combinatorial explanation. We give such an explanation. Beside these walks, we enumerate loopless triangulations of the sphere bijectively. Our proofs rely on bijections between walks and triangulations with a distinguished spanning tree. As a by-product, we also enumerate an important class of spanning trees on cubic maps.

RÉSUMÉ. On considère les chemins planaires partant de l'origine, restant dans le quart de plan $i, j \geq 0$ et faits de pas Ouest, Sud et Nord-Est. Il existe de jolies formules énumératives pour ces chemins. Mais, alors que plusieurs démonstrations ont été proposées pour ces formules par le passé, aucune ne fournit d'explication combinatoire. Nous donnons une telle explication. En sus de ces chemins, nous énumérons bijectivement les triangulations sans boucle de la sphère. Nos preuves reposent sur des bijections entre des chemins et des triangulations munies d'un arbre couvrant. Dans le même temps, nous énumérons une famille importante d'arbres couvrants sur les cartes cubiques.

1. Introduction

We consider lattice walks in the plane starting from the origin (0,0), remaining in the first quadrant $i, j \geq 0$ and made of three kind of steps: West, South and North-East. These walks were first studied by Germain Kreweras [**4**] and inherited his name. A Kreweras walk ending at the origin is represented in Figure 1.

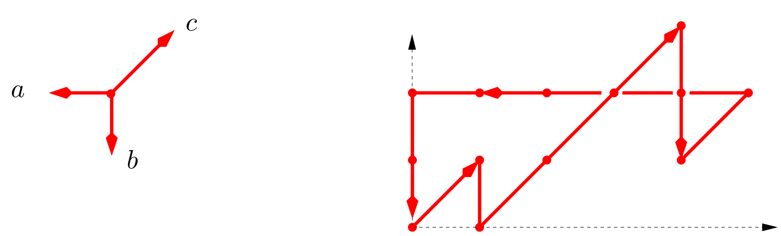

FIGURE 1. The Kreweras walk *cbcccbbcaaaaabb*.

These walks have remarkable enumerative properties. Kreweras proved in 1965 that the number of walks of length $3n$ ending at the origin is:

$$k_n \quad = \quad \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}. \tag{1.1}$$

The original proof of this result is complicated and somewhat unsatisfactory. It was performed by *guessing* the number of walks of size n ending at point (i, j). The conjectured formulas were then checked using the

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recurrence relations between these numbers. The *checking part* involved several hypergeometric identities which were later simplified by Niederhausen [**6**]. In 1986, Gessel gave a different proof in which the *guessing part* was reduced [**3**]. More recently, Bousquet-Mélou proposed a constructive proof (that is, without guessing) of these results and some extensions [**1**]. Still, the simple looking formula (1.1) remained without a direct combinatorial explanation. The problem of finding a combinatorial explanation was raised by Stanley in [**9**]. One of our goals in this paper is to provide such an explanation.

Formula (1.1) for the number of Kreweras walks is to be compared to another formula proved the same year. In 1965, Mullin, following the seminal steps of Tutte, proved via a generating function approach [**5**] that the number of loopless triangulations of size n (see below for precise definitions) is

$$t_n = \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}. \tag{1.2}$$

A bijective proof of Formula (1.2) was outlined by Schaeffer in his Ph.D thesis [**7**]. See also [**8**] for a more general construction concerning non-separable triangulations of a k-gon. We will give an alternative bijective proof for the number of loopless triangulations. Technically speaking, we will work instead on *cubic maps without isthmus* which are the dual of loopless triangulations.

2. How the proofs work

We begin with an account of this paper's content in order to underline the (slightly unusual) logic structure of our proofs.

• In Section 3, we recall some definitions on rooted planar maps. Then, we define a special class of spanning trees called *depth trees*. Depth trees are closely related to the trees that can be obtained by a *depth first search* algorithm.

• In Section 4, we describe a bijection Φ between Kreweras walks ending at the origin and cubic maps without isthmus covered by a depth tree. As an immediate enumerative corollary, we obtain the relation

$$k_n = d_n,$$

between the number k_n of Kreweras walks of size n ending at the origin and the number d_n of cubic maps without isthmus of size n covered by a depth tree.

• In Section 5, we extend the mapping Φ to a larger class of walks called *extended Kreweras walks*. These walks (made of West, South and North-East steps) start from the origin (0,0) and remain in the half-plane $i + j \geq 0$. An extended Kreweras walk ending on the second diagonal (i.e. the line $i + j = 0$) is represented in Figure 2.

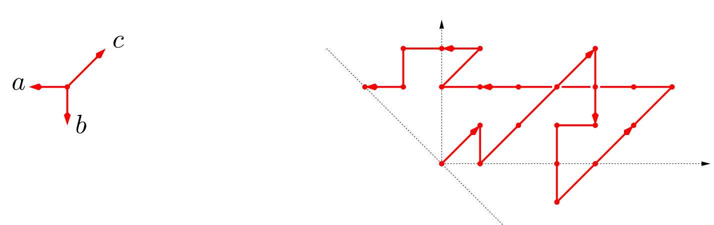

FIGURE 2. An extended Kreweras walk ending on the second diagonal

Unlike the Kreweras walks, the extended Kreweras walks are easy to count. A simple application of the cycle lemma (see Section 5.3 of [**10**]) allows one to prove that the number of extended Kreweras walks of length $3n$ ending on the second diagonal is

$$e_n = \frac{4^n}{2n+1}\binom{3n}{n}.$$

Then, we prove that the mapping Φ can be generalized into a bijection between extended Kreweras walks ending on the second diagonal and cubic maps without isthmus of size n covered by a depth tree with a marked edge not in the tree. Since any cubic map of size n has exactly $n+1$ edges not in the spanning tree, we obtain

$$e_n = (n+1)d_n.$$

As a result, we get

$$d_n \;=\; k_n \;=\; \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n},$$

and recover Equation (1.1).

• In Section 6, we enumerate depth trees on cubic maps. We prove that the number of such trees for a cubic map of size n is 2^n. This result implies that the number of cubic maps of size n is

$$c_n \;=\; \frac{d_n}{2^n} \;=\; \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

Thus, we obtain a combinatorial proof of Formula (1.2).

• In Section 7, we extend the mapping Φ to Kreweras walks ending at $(i,0)$ and discuss some open problems.

3. Definitions and notations

3.1. Kreweras walks. In the following, Kreweras walks are considered as words on the alphabet $\{a,b,c\}$. The letter a (resp. b, c) corresponds to a West (resp. South, North-East) step. For instance, the walk in Figure 1 is *cbcccbbcaaaaabb*. The length of a word w is denoted by $|w|$ and the number of occurrences of a given letter α is denoted by $|w|_\alpha$. Kreweras walks are the words w on the alphabet $\{a,b,c\}$ such that any prefix w' of w satisfies

$$|w'|_a \le |w'|_c \quad \text{and} \quad |w'|_b \le |w'|_c \;. \tag{3.1}$$

Kreweras walks ending at the origin satisfy the additional constraint

$$|w|_a \;=\; |w|_b \;=\; |w|_c. \tag{3.2}$$

These conditions can be interpreted as a ballot problem with three candidates. This is why Kreweras walks sometimes appear under this formulation in the literature [**6**].

Similarly, the extended Kreweras walks (i.e. the walks remaining in the half-plane $i+j \ge 0$) are the words w on $\{a,b,c\}$ such that any prefix w' of w satisfies

$$|w'|_a \;+\; |w'|_b \;\le\; 2|w'|_c \;, \tag{3.3}$$

and walks ending on the second diagonal satisfy the additional constraint

$$|w|_a \;+\; |w|_b \;=\; 2|w|_c. \tag{3.4}$$

Note that the length of any walk ending on the second diagonal is a multiple of 3. The *size* of such a walk of length $3n$ is n. Note also that a walk ending at point $(i,0)$ has a length of the form $l = 3n+2i$ where n is a non-negative integer. A Kreweras walk of length $l = 3n+2i$ ending at $(i,0)$ has *size* n.

3.2. Cubic maps. We recall some definitions about planar maps. A *planar map*, or *map* for short, is an embedding of a connected planar graph in the sphere without intersecting edges, defined up to orientation preserving homeomorphisms of the sphere. Loops and multiple edges are allowed. The *faces* are the connected components of the complement of the graph. Each edge has two *half-edges*, each incident to one of the endpoints. A map is *rooted* if one of its half-edges is distinguished as the *root*. The endpoint of the root is the *root-vertex*. Graphically, the root is indicated by an arrow pointing on the root-vertex (see Figure 3). All the maps considered in this paper are rooted and we shall not further precise it.

Our constructions lead us to consider *legs*, that is, half-edges that are not part of a complete edge. A *growing map* is a map together with some legs, one of them being distinguished as the *head*. We require the

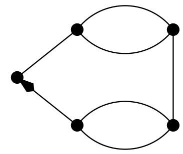

FIGURE 3. A rooted map.

legs to be all in the same face called *head-face.* The endpoint of the head is the *head-vertex.* Graphically, the head is indicated by an arrow pointing away from the *head-vertex.* The root of a growing map can be a leg or a regular half-edge. For instance, the growing map in Figure 4 has 2 legs beside the head, and its root is not a leg.

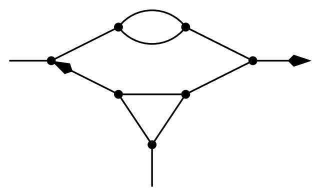

FIGURE 4. A growing map.

A map (or growing map) is *cubic* if any vertex has degree 3. It is k-near-cubic if the root-vertex has degree k and any other vertex has degree 3. For instance, the map in Figure 3 is 2-near-cubic and the growing map in Figure 4 is cubic. Observe that cubic maps are in bijection with 2-near-cubic maps not reduced to a loop by the mapping illustrated in Figure 5.

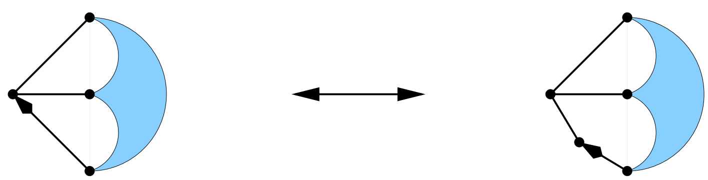

FIGURE 5. Bijection between cubic maps and 2-near-cubic maps.

We will be interested in *non-separable* k-near-cubic maps. A map is *separable* if the edge set can be partitioned into two non-empty parts such that exactly one vertex is incident to some edges in both parts. It is *non-separable* otherwise. In particular, a non-separable map has no loop nor isthmus (i.e. edge whose deletion disconnect the map). For cubic maps and 2-near-cubic maps it is equivalent to be non-separable or without isthmus.

The incidence relation between vertices and edges in cubic maps shows that the number of edges is always a multiple of 3. More generally, if M is a k-near-cubic map with e edges and v vertices, the incidence relation reads: $3(v-1)+k=2e$. Equivalently, $3(v-k+1)=2(e-2k+3)$. It can be shown that $v-k+1$ is non-negative. Hence, the number of edges has the form $e=3n+2k-3$ where n is a non-negative integer. We say that a k-near-cubic map has *size* n if it has $e=3n+2k-3$ edges (and $v=2n+k-1$ vertices). In particular, the mapping of Figure 5 is a bijection between cubic maps of size n ($3n+3$ edges) and 2-near-cubic maps of size $n+1$ ($3n+4$ edges).

The cubic maps without isthmus form an important class of maps because their *duals* are the loopless triangulations. Recall that the dual M^* of a map M is the map obtained by putting a vertex of M^* in each face of M and an edge of M^* across each edge of M.

3.3. Depth trees. A *tree* is a connected graph without cycle. A subgraph T of a connected graph G is a *spanning tree* if it is a tree containing every vertex of G. An edge of G is said to be *internal* if it is in the spanning tree T and *external* otherwise. For any pair of vertices u, v in G, there is a unique path between u and v in the spanning tree T. We call it the T-*path* between u and v.
A map (or growing map) M with a distinguished spanning tree T will be denoted by M_T. Graphically, we shall indicate the spanning tree by thick lines as in Figure 6. A vertex u of M_T is an *ancestor* of another vertex v if it is on the T-path between the root-vertex and v. In this case, v is a *descendant* of u. Two vertices are *comparable* if one is the ancestor of the other. For instance, in Figure 6, the vertices u_1 and v_1 are comparable whereas u_2 and v_2 are not. A *depth tree* is a spanning tree such that any external edge joins comparable vertices. Moreover, we require the edge containing the root to be external. In Figure 6, the tree on the left side is a depth tree but the tree on the right side is not a depth tree since the edge (u_2, v_2) breaks the rule. Finally, a *depth-map* is a map with a distinguished depth tree.

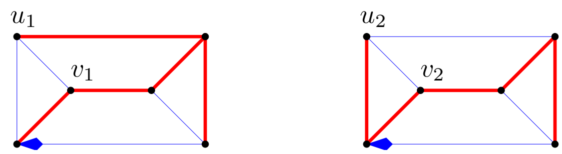

FIGURE 6. A depth tree (left) and a non-depth tree (right).

4. A bijection between Kreweras walks and cubic depth-maps

We define a bijection Φ between Kreweras walks ending at the origin and 2-near-cubic depth-maps (i.e. 2-near-cubic maps with a distinguished depth tree) without isthmus. The general principle of this bijection is to read the walk *from right to left* and interpret each letter as an operation for constructing the map and the tree. We illustrated this step-by-step construction in Figure 8. The intermediary steps are *tree-growing maps*, that is, growing maps together with a distinguished depth tree (indicated by thick lines).

• We start with the tree-growing map M_T^0 consisting of one vertex and two legs. One of the legs is the root, the other is the head (see Figure 7). The spanning tree is reduced to the vertex which is both the root-vertex and the head-vertex.

• We apply successively certain elementary mappings φ_a, φ_b, φ_c (Definition 4.1) corresponding to the letters a, b, c of the Kreweras walk read from right to left.

• When the whole walk is read, we *close* the tree-growing map, that is, we glue the head and the root together as was done in Figure 9.

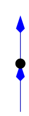

FIGURE 7. The tree-growing map M_T^0.

Let us enter in the details and define the bijection Φ. Consider a growing map M. We *make a tour* of the head-face if we follow its border in counterclockwise direction (i.e. the border of the head-face stays on our left-hand side) starting from the head (see Figure 10). This journey induces a linear order on the legs of M. We shall talk about the *first* and *last* legs of M. Moreover, if the root is a leg, we call *left* (resp. *right*) the legs encountered before (resp. after) the root during the tour of the head-face. For instance, the growing map of Figure 10 has one left leg and two right legs.

We define three mappings φ_a, φ_b, φ_c on tree-growing maps.

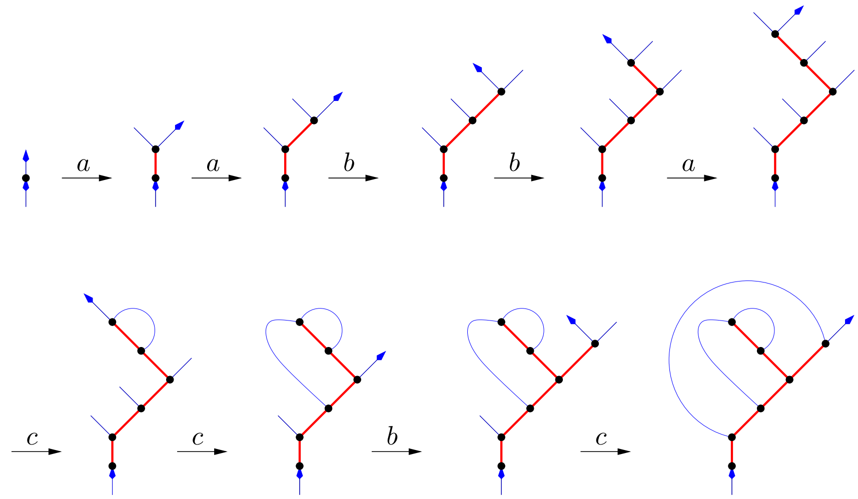

FIGURE 8. Successive applications of the mappings $\varphi_a,\ \varphi_b,\ \varphi_c$ for the walk *cbccabbaa*.

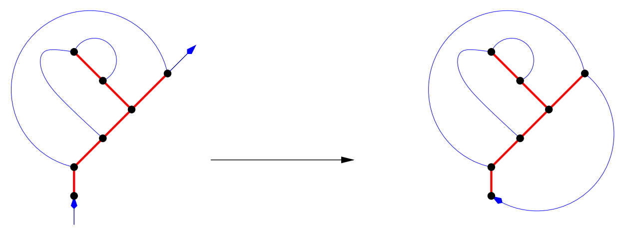

FIGURE 9. Closing the map.

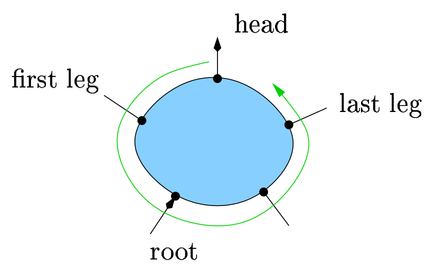

FIGURE 10. Making the tour of the head-face.

DEFINITION 4.1. Let M_T be a tree-growing map (the map is M and the distinguished tree is T).

• The mappings φ_a and φ_b are represented in Figure 11. The tree-growing map $M'_{T'} = \varphi_a(M_T)$ (resp. $\varphi_b(M_T)$) is obtained from M_T by replacing the head by an edge e together with a new vertex v incident with the new head and another leg at its left (resp. right). The tree T' is obtained from T by adding the edge e and the vertex v.

• The tree-growing map $\varphi_c(M_T)$ is only defined if the first and last legs exist (that is, if the head-face contains some legs beside the head) and have distinct and comparable endpoints. We call these legs s and t *with the convention that the endpoint of* s *is an ancestor of the endpoint of* t.
In this case, the tree-growing map $M'_T = \varphi_c(M_T)$ is obtained from M_T by gluing together the head and the leg s while the leg t becomes the new head (see Figure 12). The spanning tree T is unchanged.

• For a word $w = a_1a_2\dots a_n$ on the alphabet $\{a,b,c\}$, we denote by φ_w the mapping $\varphi_{a_1}\circ\varphi_{a_2}\circ\cdots\circ\varphi_{a_n}$.

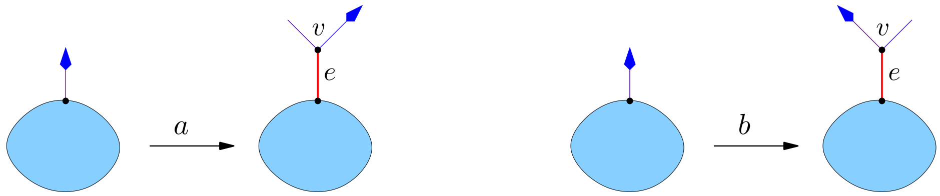

FIGURE 11. The mappings φ_a and φ_b.

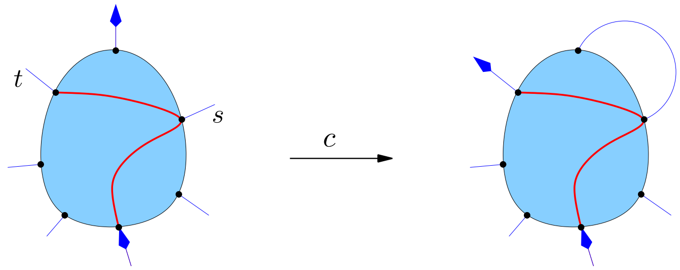

FIGURE 12. The mapping φ_c.

We are now ready to define the mapping Φ on Kreweras walks ending at the origin.

DEFINITION 4.2. Let w be a Kreweras walk ending at the origin. The image of w by the mapping Φ is the map with a distinguished spanning tree obtained by gluing together the root and the head of the tree-growing map $\varphi_w(M_T^0)$.

The mapping Φ has been applied to the walk *cbccabbaa* in Figure 8 and 9. One has to prove that this mapping is well defined. We omit the proof in this extended abstract. However, we highlight one of the key properties: for any suffix w' of w, the tree-growing map $\varphi_{w'}(M_T^0)$ has $|w'|_a - |w'|_c$ left legs and $|w'|_b - |w'|_c$ right legs. (These quantities are non-negative by Equations (3.1) and (3.2).)

We now state the main result of this section.

THEOREM 4.3. *The mapping Φ is a bijection between Kreweras walks of size n (length $3n$) ending at the origin and 2-near-cubic depth-maps without isthmus of size n ($3n+1$ edges).*

COROLLARY 4.4. *The number k_n of Kreweras walks of size n is equal to the number d_n of 2-near-cubic depth-maps without isthmus of size n.*

Observe that d_n is also the number of cubic depth-maps of size $n-1$ without isthmus since the bijection between cubic maps and 2-near-cubic maps represented in Figure 5 can be trivially turned into a bijection between cubic depth-maps and 2-near-cubic depth-maps.

We omit the proof of Theorem 4.3. The general idea is to define the inverse mapping Ψ. This mapping destructs the tree-growing map that Φ constructs and recover the walk. Looking at Figure 8 from bottom-to-top and right-to-left we see how Ψ works.

5. Enumeration of Kreweras walks

Recall that *extended Kreweras walks* are the walks starting from the origin and remaining in the half-plane $i+j \geq 0$. An extended Kreweras walk ending on the *second diagonal* (i.e. the line $i+j=0$) is represented in Figure 2. The counting of extended Kreweras walks reduces to finding the number of 1-dimensional walks with steps $+2$, and -1 starting at 0 and remaining non-negative. This number is easily found by applying the cycle lemma (see Section 5.3 of [**10**]). We obtain the following result:

PROPOSITION 5.1. *There are*

$$e_n = \frac{4^n}{2n+1}\binom{3n}{n} \tag{5.1}$$

extended Kreweras walks of size n ending on the second diagonal.

We now extend the mapping Φ (Definition 4.2) into an injective mapping Φ' on extended Kreweras walks ending on the second diagonal. The mapping Φ' returns a map with a distinguished spanning tree and a marked external edge. In what follows, a map with a distinguished spanning tree is said *marked* if an external edge is marked.

DEFINITION 5.2. Let w be an extended Kreweras walk ending on the second diagonal. The image of w by the mapping Φ' is the map with a distinguished spanning tree obtained from the tree-growing map $\varphi_w(M_T^0)$ by gluing together the head and the unique remaining leg. The (external) edge obtained by gluing these legs is marked.

We applied the mapping Φ' to the extended Kreweras walks *cabccaaaa* in Figure 13. The marked edge is dashed.

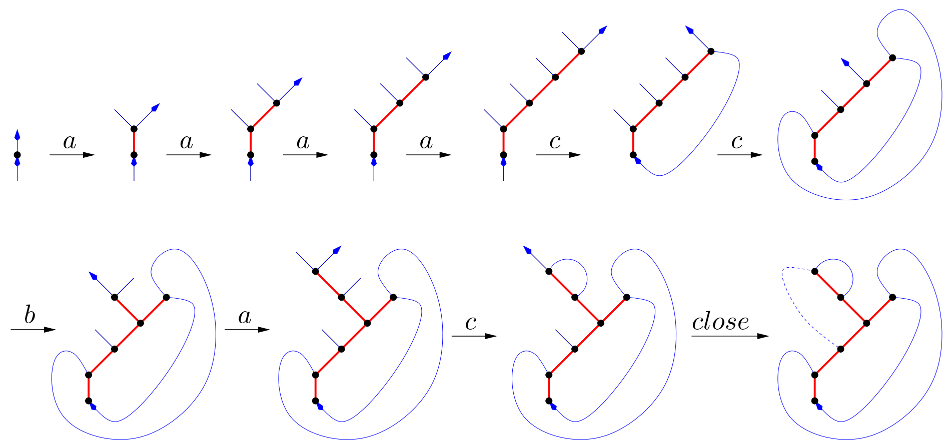

FIGURE 13. The bijection Φ' on the walk *cabccaaaa*.

Observe that the mappings Φ and Φ' coincide on Kreweras walks ending at the origin. In this case, the marked edge is the edge containing the root.

We now state the main result of this section.

THEOREM 5.3. *The mapping Φ' is a bijection between extended Kreweras walks of size n ending on the second diagonal and marked 2-near-cubic depth-maps of size n without isthmus.*

We will not prove Theorem 5.3 but we do explore its consequences. We know from Corollary 4.4 that the number d_n of 2-near-cubic depth-maps without isthmus of size n is equal to the number k_n of Kreweras walks of size n ending at the origin. Consider a 2-near-cubic map M of size n ($3n+1$ edges, $2n+1$ vertices). Since a spanning tree T has $2n$ edges, there are $n+1$ external edges. Therefore, there are $(n+1)d_n = (n+1)k_n$ marked 2-near-cubic depth-map without isthmus. By Theorem 5.3, this is also the number e_n of extended Kreweras walks of size n ending on the second diagonal. We know the number e_n explicitly by Proposition 5.1. Hence, we obtain $(n+1)k_n = e_n = \frac{4^n}{2n+1}\binom{3n}{n}$. This result deserves to be stated as a theorem.

THEOREM 5.4. *There are* $k_n = \dfrac{4^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ *Kreweras walks of size n (length $3n$) ending at the origin.*

6. Enumerating depth trees and cubic maps

In the previous section, we exhibited a bijection Φ' between extended Kreweras walks ending on the second diagonal and marked 2-near-cubic depth-maps without isthmus. As a corollary we obtained the number of 2-near-cubic depth-maps without isthmus of size n: $d_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$. In this section, we prove that any 2-near-cubic map of size n has 2^n depth trees (Corollary 6.5). This implies that the number of 2-near-cubic maps of size n without isthmus is $c_n = \frac{d_n}{2^n} = \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$. Given the bijection between 2-near-cubic maps and cubic maps (see Figure 5), we obtain the following theorem.

THEOREM 6.1. *There are* $c_n = \dfrac{2^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ *cubic maps without isthmus having* $3n$ *edges.*

By duality, c_n is also the number of loopless triangulations with $3n$ edges. Hence, we recover Equation (1.2) announced in the introduction.

The rest of this section is devoted to the counting of depth trees on cubic maps and, more generally, on cubic (potentially non-planar) graphs. We first give an alternative characterization of depth trees. This characterization is based on the depth-first search (DFS) algorithm (see Section 23.3 of [**2**]). We consider the DFS algorithm as an algorithm for constructing a spanning tree T of a graph.

We consider a graph G with a distinguished vertex v_0. In the definition of the DFS algorithm (see below), the subgraph T remains a tree. The vertex v_0 is considered as the root-vertex of the tree. Hence, any vertex in T distinct from v_0 has a *father* in T.

DEFINITION 6.2. Depth-first search (DFS) algorithm.

Initialization: The current vertex is v_0 and the tree T is reduced to v_0.

Core: While the current vertex v is adjacent to a vertex not in T or is distinct from v_0 we do:
If there are some edges linking the current vertex v to a vertex not in T, we choose one of them e at random. We add e and its other endpoint v' to the tree T. The vertex v' becomes the current vertex.
Else, we backtrack, that is, we set the current vertex to be the father of v in T.

End: We return the tree T.

It is well known that the DFS algorithm returns a spanning tree. It is also known [**2**] that the two following properties are equivalent for a spanning tree T of a graph G having a distinguished vertex v_0:
(i) Any external edge joins comparable vertices.
(ii) The tree T can be obtained by a DFS algorithm on the graph G starting from v_0.

Before stating the main result of this section, we need an easy preliminary lemma.

LEMMA 6.3. *Let G be a connected graph with a distinguished vertex v_0 whose deletion does not disconnect the graph. Then, any spanning tree T satisfying conditions $(i)-(ii)$ has at exactly one edge incident to v_0.*

THEOREM 6.4. *Let G be a loopless connected graph with a distinguished vertex v_0 whose deletion does not disconnect the graph. Let e be an edge incident to v_0. If G is a k-near-cubic graph (v_0 has degree k and the other vertices have degree 3) of size n ($3n+2k-3$ edges), then the number of trees containing e and satisfying conditions $(i)-(ii)$ is 2^n.*

Given that the depth trees are the spanning trees satisfying conditions $(i)-(ii)$ and not containing the root, the following corollary is immediate.

COROLLARY 6.5. *For any 2-near-cubic map without isthmus of size n ($3n+1$ edges), there are 2^n depth trees.*

The proof of Theorem 6.4 relies on the intuition that exactly n *real* binary choices have to be made during the execution of a DFS algorithm on a k-near-cubic map of size n. However, making this intuition into a proof requires some work and we shall not do it here.

Remark: Theorem 6.4 shows that any k-near-cubic loopless graph of size n has $k2^n$ trees satisfying the conditions $(i)-(ii)$.

7. Extensions and open problems

7.1. Random generation of triangulations. We introduced the family of extended Kreweras walks ending on the second diagonal. The random generation of such walks of length $3n$ (with uniform distribution) reduces to the random generation of 1-dimensional walks of length $3n$ with steps $+2$, -1 starting and ending at 0 and remaining non-negative. The random generation of these walks is known to be feasible in linear time. (One just needs to generate (with uniform distribution) a word of length $3n+1$ containing n '+2' and $2n+1$ '-1' and to apply the cycle lemma.) Given an extended Kreweras walk w ending on the second diagonal, the construction of the 2-near-cubic depth-map $\Phi'(w)$ can be performed in linear time. Therefore, we have a linear time algorithm for the random generation (with uniform distribution) of 2-near-cubic depth-maps with a marked edge. For any 2-near-cubic map there are 2^n depth trees and then $(n+1)$ possible marked edges. Therefore, if we drop the distinguished edge and the depth tree at the end of the process, we obtain a uniform distribution on 2-near-cubic maps without isthmus. This allows us to generate uniformly cubic maps without isthmus or, dually, loopless triangulations in linear time.

7.2. Kreweras walks ending at $(i,0)$ and $(i+2)$-near-cubic maps. The Kreweras walks ending at $(i,0)$ are the words w on the alphabet $\{a,b,c\}$ with $|w|_a+i=|w|_b=|w|_c$ such that any suffix w' of w satisfies $|w'|_a+i\geq|w'|_c$ and $|w'|_b\geq|w'|_c$.
There is a very nice formula [**4**] counting Kreweras walks of size n (length $3n+2i$) ending at $(i,0)$:

$$k_{n,i} = \frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}. \tag{7.1}$$

There is also a similar formula [**5**] for non-separable $(i+2)$-near-cubic maps of size n ($3n+2i+1$ edges):

$$c_{n,i} = \frac{2^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}. \tag{7.2}$$

In this subsection, we show that the bijection Φ (Definition 4.2) can be extended to walks ending at $(i,0)$. This gives a bijective correspondence explaining why $k_{n,i}=2^nc_{n,i}$.
Consider the tree-growing map M_T^i reduced to a vertex, a root, a head and i left legs (Figure 14). We define the image of a Kreweras walk w ending at $(i,0)$ as the map obtained by closing $\varphi_w(M_T^i)$. We get the following extension of Theorem 4.3.

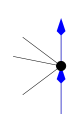

FIGURE 14. The tree-growing map M_T^i when $i=3$.

THEOREM 7.1. *The mapping Φ is a bijection between Kreweras walks of size n (length $3n+2i$) ending at $(i,0)$ and non-separable $(i+2)$-near-cubic maps of size n ($3n+2i+1$ edges) with a depth tree that contains the edge following the root in counterclockwise order around the root-vertex.*

By Theorem 6.4, there are 2^n such trees. Consequently, we obtain the following corollary:

COROLLARY 7.2. *The number $k_{n,i}$ of Kreweras walks of size n ending at $(i,0)$ and the number c_i of non-separable $(i+2)$-near-cubic maps of size n are related by the equation $k_{n,i}=2^nc_{n,i}$.*

One can define the counterpart of extended Kreweras walks in the case of walks ending at $(i, 0)$. These are the words obtained when one chooses an external edge (in a non-separable $(i+2)$-near-cubic depth-map such that the edge following the root is in the tree) and applies the mapping Ψ' which is the inverse of Φ'. We have no simple characterization of this set of words. However, it would be interesting to find a bijective proof that this set has size $\frac{4^n(2i+1)}{(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}$. We were not able to solve this problem yet.

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