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Enumeration of Bruhat intervals between nested involutions in S_n

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Abstract. We build a chain

$$\mathrm{id} = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{\lfloor \frac{n}{2} \rfloor - 1} < \vartheta_{\lfloor \frac{n}{2} \rfloor}$$

of nested involutions in the Bruhat ordering of S_n, with $\vartheta_{\lfloor \frac{n}{2} \rfloor}$ the maximal element for the Bruhat order, and we study the cardinality of the Bruhat intervals $[\vartheta_j, \vartheta_k]$ for all $0 \leq j < k \leq \lfloor \frac{n}{2} \rfloor$, and the number of permutations incomparable with ϑ_t, for all $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$.

Résumé. Nous construisons une chaîne

$$\mathrm{id} = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{\lfloor \frac{n}{2} \rfloor - 1} < \vartheta_{\lfloor \frac{n}{2} \rfloor}$$

des involutions nichées dans l'ordre de Bruhat de S_n, avec $\vartheta_{\lfloor \frac{n}{2} \rfloor}$ l'élément maximal pour l'ordre de Bruhat, et nous étudions la cardinalité des intervalles de Bruhat $[\vartheta_j, \vartheta_k]$ pour tout les $0 \leq j < k \leq \lfloor \frac{n}{2} \rfloor$, et le nombre de permutations incomparables avec ϑ_t, pour tout le $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$.

1. Overview

For any $n \geq 2$, let S_n be the symmetric group of n elements equipped with the Bruhat ordering $\leq$; see e.g. [**3, 6, 8, 21, 22**]. One of the most celebrated combinatorial and algebraic problems is to study its Bruhat graph and its Bruhat intervals $[a, b] = \{z \in S_n : a \leq z \leq b\}$ for $a, b \in S_n$; see e.g. [**1, 7, 12, 15**]. These are intimately related with the Kazhdan–Lusztig polynomials of S_n and the algebraic geometry of Schubert varieties. See e.g. [**9, 10, 13, 14, 18**] and the references therein.

In this work we build a chain

$$\mathrm{id} = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{\lfloor \frac{n}{2} \rfloor - 1} < \vartheta_{\lfloor \frac{n}{2} \rfloor}$$

of nested involutions in the Bruhat ordering of S_n, with $\vartheta_{\lfloor \frac{n}{2} \rfloor}$ the maximal element for the Bruhat order (see Definition 3.1 for the exact definition of ϑ_t, $t = 0, \ldots, \lfloor \frac{n}{2} \rfloor$), and we study the cardinality of the Bruhat intervals $[\vartheta_j, \vartheta_k]$ for all $0 \leq j < k \leq \lfloor \frac{n}{2} \rfloor$. Moreover we study the number of permutations incomparable with ϑ_t, for all $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$. Our results imply and generalize the result of [**27**], where a closed formula for the cardinality of $[\vartheta_0, \vartheta_1]$ is proved. This problem is related to the explicit computation of Kazhdan–Lusztig polynomials for some classes of elements. See e.g. [**24, 25**] and the references therein.

The importance of the set $\{\vartheta_t : t = 0, \ldots, \lfloor \frac{n}{2} \rfloor\}$ lies in the fact that involutions of the symmetric group and, more generally, of Coxeter groups, are elements having nice algebraic properties, see [**28, 29, 30, 31**]. In particular, in [**28**] it is proved that the maximal length element of any conjugacy class in S_n containing involutions is one of the ϑ_t for some $t = 0, \ldots, \lfloor \frac{n}{2} \rfloor$.

2. Preliminaries

In this section we collect together some definitions, notation and results that will be used in the following. We follow [**11, 20, 32**] for combinatorics and poset notation and terminology.

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For $x \in \mathbb{R}$ we let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$; for $n \in \mathbb{N}$ we let $[n] = \{t \in \mathbb{N} : 1 \leq t \leq n\} = \{1, \ldots, n\}$, and $[0] = \emptyset$. For any complex number a, we define the rising factorial as $(a)_0 = 1$ and $(a)_m = \prod_{j=0}^{m-1} (a+j)$ for any $m \in \mathbb{N} \setminus \{0\}$. The cardinality of a set $\mathcal{X}$ will be denoted by $\#\mathcal{X}$.

For any $n \geq 2$, let S_n be the symmetric group of permutations of n objects, viz. the set of all bijections

$$\sigma : [n] \xrightarrow{\sim} [n] .$$

If $\sigma \in S_n$ then we write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(j) = a_j$ for $j \in [n]$. Sometimes we also write σ in *disjoint cycle form* and we usually omit writing the 1–cycles of σ. Given $\sigma, \tau \in S_n$ we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example $(1,2)(2,3) = (1,2,3)$. For any $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we say that a pair $(i,j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$, and we denote the number of inversions of σ by $\operatorname{inv}(\sigma)$.

We set

- : $\mathcal{E}_n = \{(j, j+1) : j \in [n-1]\}$,
- : $T_n = \{(i,j) : 1 \leq i < j \leq n\}$, the set of transpositions in S_n,
- : $D(\sigma) = \{\tau \in \mathcal{E}_n : \operatorname{inv}(\sigma\tau) < \operatorname{inv}(\sigma)\}$, the *descent set* of $\sigma \in S_n$.

We recall the definition of Bruhat order on S_n:

Definition 2.1. Let $n \geq 2$. For any $u, v \in S_n$, $u < v$ in Bruhat order if and only if there exist $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in T_n$ such that

$$\begin{aligned} & v = ut_1 \cdots t_k, \\ & \operatorname{inv}(ut_1 \cdots t_{j+1}) > \operatorname{inv}(ut_1 \cdots t_j) \qquad \text{for any } \ j \in [k-1] . \end{aligned}$$

It is easy to see that $[n, \ldots, 1]$ is the maximum element in S_n for the Bruhat order.

Now we state a criterion for deciding when two permutations are comparable in the Bruhat ordering, which was achieved in [**5**].

Theorem 2.2. *Let $n \geq 2$, and for any $\sigma, \tau \in S_n$, let $\sigma[j,k]$ be the j-th entry in the increasing rearrangement of $\{\sigma(1), \ldots, \sigma(k)\}$ for all $1 \leq j \leq k \leq n-1$, and define $\tau[j,k]$ similarly. Then the following are equivalent:*

1. *$\sigma \leq \tau$ in the Bruhat order,*
2. *$\sigma[j,k] \leq \tau[j,k]$, for all $k \in D(\sigma)$ and $1 \leq j \leq k$,*
3. *$\sigma[j,k] \leq \tau[j,k]$, for all $k \in \{1, \ldots, n-1\} \setminus D(\tau)$ and $1 \leq j \leq k$.*

3. Main Results

Definition 3.1. Let $n \geq 2$. We define

$$\begin{aligned} \vartheta_t &= \prod_{j=0}^{t-1} (j+1, n-j) = (1,n) \cdots (t, n-t+1) \\ &= [n, \ldots, n-t+1, t+1, \ldots, n-t, t, \ldots, 1] \in S_n \end{aligned}$$

for all $0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Obviously ϑ_t is an involution for all $0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor$, and

$$\mathrm{id} = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{\left\lfloor \frac{n}{2} \right\rfloor - 1} < \vartheta_{\left\lfloor \frac{n}{2} \right\rfloor} = \max\{\sigma \in S_n\}$$

in the Bruhat order of S_n.

Definition 3.2. Let $n \geq 2$, and $0 \leq t \leq n-1$. We define

$$\mathcal{F}_t(n) = \begin{cases} \{\sigma \in S_n : \sigma \leq \vartheta_t\} = [\vartheta_0, \vartheta_t] & \text{if } t \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ S_n & \text{if } t \geq \left\lfloor \frac{n}{2} \right\rfloor; \end{cases}$$

and

$$\mathbf{F}_t(n) = \#\mathcal{F}_t(n) ,$$

setting $\mathbf{F}_t(0) = \mathbf{F}_t(1) = 1$.

Lemma 3.3. *Let $n \geq 2$, $0 \leq t \leq n-1$, and $j \in [t+1]$. The number of permutations $\sigma \in \mathcal{F}_t(n)$ with the constraint that there exists a subset $A = \{\alpha_1, \ldots, \alpha_j\} \subset [t+1]$ and an array $B = (\beta_1, \ldots, \beta_j)$ with pairwise distinct coordinates $\beta_k \in [t+1]$ for all $k \in [j]$ such that*

$$\sigma(\alpha_l) = \beta_l \qquad \text{for all } l \in [j]$$

equals

$$j! \binom{t+1}{j}^2 \mathbf{F}_t(n-j).$$

Proof. Of course we can always assume $t \geq 1$ otherwise the result is trivial.
For any fixed $A = \{\alpha_1, \ldots, \alpha_j\}$ and $B = (\beta_1, \ldots, \beta_j)$ with the desired properties, let

$$\mathcal{Z}_t^n[j](A, B) = \{\sigma \in \mathcal{F}_t(n) : \sigma(\alpha_k) = \beta_k \ \text{ for all } k \in [j]\}.$$

We note that from Theorem 2.2 and Definition 3.1 we get that $\mathcal{Z}_t^n[j](A, B) \neq \emptyset$ for all possible choices of A and B.

Consider the order–preserving bijections

$$\begin{aligned} \varphi &: [n] \setminus A \xrightarrow{\sim} [n-t], \\ \psi &: [n] \setminus B \xrightarrow{\sim} [n-t]. \end{aligned}$$

Then from Theorem 2.2 there is a bijection

$$f : \mathcal{Z}_t^n[j](A, B) \xrightarrow{\sim} \mathcal{F}_t(n-j)$$

defined in the following way: we delete $\sigma(k)$ if $k \in A$, whereas for all $k \notin A$

$$\sigma(k) \xrightarrow{f} \psi(\sigma(\varphi(k))).$$

Noticing that there are $\binom{t+1}{j}$ ways for choosing A and $j!\binom{t+1}{j}$ ways for choosing B, the desired result follows. □

Theorem 3.4. *For any $n \geq 2$,*

$$\mathbf{F}_t(n) = \begin{cases} \sum_{j=1}^{t+1} (-1)^{j-1} j! \binom{t+1}{j}^2 \mathbf{F}_t(n-j) & \text{if } \ 0 \leq t \leq n-1, \\ n! & \text{if } \ t \geq n. \end{cases}$$

Proof. Of course we can always assume $t \in [n-1]$ otherwise the result is trivial.
From Theorem 2.2 we see that if $\sigma \in \mathcal{F}_t(n)$ then

$$\{\sigma(k) : k \in [t+1]\} \bigcap [t+1] \neq \emptyset.$$

Let $k \in [t+1]$ and

$$R_k = \{\sigma \in \mathcal{F}_t(n) : \sigma(k) \in [t+1]\}.$$

Then by inclusion–exclusion we have

$$\mathbf{F}_t(n) = \#\left(\bigcup_{k \in [t+1]} R_k\right) = \sum_{j=1}^{t+1} (-1)^{j-1} \sum_{\substack{\mathcal{I} \subset [t+1] \\ \#\mathcal{I} = j}} \#\left(\bigcap_{z \in \mathcal{I}} R_z\right),$$

and the desired result follows from Lemma 3.3. □

The following Corollary is immediate, and it gives a purely combinatorial proof of an identity for the factorial.

Corollary 3.5. *For any $n \geq 2$ and for all $\left\lfloor \frac{n}{2} \right\rfloor \leq k, t \leq n-1$,*

$$\sum_{j=1}^{k+1} (-1)^{j-1} j! \binom{k+1}{j}^2 (n-j)! = \sum_{j=1}^{t+1} (-1)^{j-1} j! \binom{t+1}{j}^2 (n-j)! = n!$$

Proof. From Definition 3.2, $\mathbf{F}_k(n) = \mathbf{F}_t(n) = n!$ for all $\left\lfloor \frac{n}{2} \right\rfloor \leq k, t \leq n-1$. Taking in account Theorem 3.4, the desired result follows. □

We note that this identity can be also proved using the theory of hypergeometric series and applying Chu–Vandermonde summation, see [**2, 17, 19, 23**]. In fact, it is equivalent to

$$\sum_{j=0}^{\infty}(-1)^j j!\binom{t+1}{j}^2 (n-j)! = 0$$

for all $\left\lfloor \frac{n}{2} \right\rfloor \leq t \leq n-1$, and we have that

$$\begin{aligned}\sum_{j=0}^{\infty}(-1)^j j!\binom{t+1}{j}^2 (n-j)! &= \left({}_2F_1\left[\begin{matrix}-t-1,-t-1\\ -n\end{matrix};1\right]\right)\left((1)_n\right)\\ &= \frac{\left((1)_n\right)\left((1-n+t)_{1+t}\right)}{(-n)_{1+t}};\end{aligned}$$

obviously if $\left\lfloor \frac{n}{2} \right\rfloor \leq t \leq n-1$ then $1-n+t \leq 0 \leq 1-n+2t$, therefore $(1-n+t)_{1+t} = 0$.

Now we give an explicit formula for the generating function of the sequence $\{\mathbf{F}_t(n)\}_{n\geq 2t}$ for any $t \geq 1$, and then, using it, we are able to prove a closed formula for the function $\mathbf{F}_t(n)$ for any $t \geq 1$ and any $n \geq 2t$.

THEOREM 3.6. *For any $t \geq 1$,*

$$\sum_{n\geq 2t}\mathbf{F}_t(n)X^n = X^{2t}\frac{\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t+1}(-1)^{j-1}\binom{t+1}{j}^2 j!\,(2t-j+k)!\right)X^k}{\sum_{j=0}^{t+1}(-1)^j\binom{t+1}{j}^2 j!X^j}.$$

PROOF. From Theorem 3.4 we get

$$\begin{aligned}\sum_{n\geq 2t}\mathbf{F}_t(n)X^n &= \sum_{n\geq 2t}\sum_{j=1}^{t+1}(-1)^{j-1}\binom{t+1}{j}^2 j!\mathbf{F}_t(n-j)X^n\\ &= \sum_{j=1}^{t+1}(-1)^{j-1}\binom{t+1}{j}^2 j!X^j\\ &\quad\cdot\left[\sum_{m=2t-j}^{2t-1}\mathbf{F}_t(m)X^m + \sum_{m\geq 2t}\mathbf{F}_t(m)X^m\right].\end{aligned}$$

From Definition 3.2 we have $\mathbf{F}_t(n) = n!$ if $2t+1 \geq n$, thus

$$\sum_{m=2t-j}^{2t-1}\mathbf{F}_t(m)X^m = \sum_{m=2t-j}^{2t-1}m!X^m = \sum_{k=0}^{j-1}(2t-j+k)!X^{2t-j+k},$$

hence

$$\begin{aligned}&\left(\sum_{n\geq 2t}\mathbf{F}_t(n)X^n\right)\left(\sum_{j=0}^{t+1}(-1)^j\binom{t+1}{j}^2 j!X^j\right)\\ &= X^{2t}\left[\sum_{j=1}^{t+1}(-1)^{j-1}\binom{t+1}{j}^2 j!\sum_{k=0}^{j-1}(2t-j+k)!X^k\right]\\ &= X^{2t}\cdot\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t+1}(-1)^{j-1}\binom{t+1}{j}^2 j!\,(2t-j+k)!\right)X^k,\end{aligned}$$

and the desired result follows. □

Theorem 3.7. *For any $t \geq 1$ and any $n \geq 2t$*

$$\mathbf{F}_t(n) = \sum_{z=0}^{\min\{t,n-2t\}} \left[\left(\sum_{j=z+1}^{t+1} (-1)^{j-1} \binom{t+1}{j}^2 j!\,(2t-j+z)! \right) \cdot \left(\sum_{\substack{\alpha \in \mathbb{N}^{t+1} \\ \Omega(\alpha)=n-2t-z}} \frac{(\|\alpha\|)!}{\prod_{k=1}^{t+1}(\alpha_k!)} \left(\prod_{j=1}^{t+1} \left((-1)^{j-1} \binom{t+1}{j}^2 j! \right)^{\alpha_j} \right) \right) \right],$$

where for any multi–index $\alpha = (\alpha_1, \dots \alpha_{t+1}) \in \mathbb{N}^{t+1}$ we set $\|\alpha\| = \sum_{j=1}^{t+1} \alpha_j$ and $\Omega(\alpha) = \sum_{j=1}^{t+1} j \cdot \alpha_j$.

Proof. With an eye on Theorem 3.6, observe first that

$$\begin{aligned} &\frac{1}{\sum_{j=0}^{t+1} (-1)^j \binom{t+1}{j}^2 j! X^j} \\ &= \frac{1}{1 - \sum_{j=1}^{t+1} (-1)^{j-1} \binom{t+1}{j}^2 j! X^j} \\ &= \sum_{l \geq 0} \left(\sum_{j=1}^{t+1} (-1)^{j-1} \binom{t+1}{j}^2 j! X^j \right)^l . \end{aligned} \tag{3.1}$$

Now, for any $r \geq 1$ and any multi–index $\alpha = (\alpha_1, \dots \alpha_r) \in \mathbb{N}^r$, we set $\|\alpha\| = \sum_{j=1}^{r} \alpha_j$ and $\Omega(\alpha) = \sum_{j=1}^{r} j \cdot \alpha_j$, and we recall that for any $r \geq 1$, $s \geq 1$, and $z_1, \dots, z_r \in \mathbb{R}$ we have

$$\left(\sum_{j=1}^{r} z_j \right)^s = \sum_{\substack{\alpha \in \mathbb{N}^r \\ \|\alpha\|=s}} \frac{s!}{\prod_{k=1}^{r}(\alpha_k!)} \left(\prod_{j=1}^{r} z_j^{\alpha_j} \right).$$

Therefore (3.1) equals

$$\begin{aligned} &\sum_{l \geq 0} \left(\sum_{j=1}^{t+1} (-1)^{j-1} \binom{t+1}{j}^2 j! X^j \right)^l \\ &= \sum_{l \geq 0} \sum_{\substack{\alpha \in \mathbb{N}^{t+1} \\ \|\alpha\|=l}} \frac{l!}{\prod_{k=1}^{t+1}(\alpha_k!)} \left[\prod_{j=1}^{t+1} \left((-1)^{j-1} \binom{t+1}{j}^2 j! X^j \right)^{\alpha_j} \right] \\ &= \sum_{\alpha \in \mathbb{N}^{t+1}} \frac{(\|\alpha\|)!}{\prod_{k=1}^{t+1}(\alpha_k!)} \left[\prod_{j=1}^{t+1} \left((-1)^{j-1} \binom{t+1}{j}^2 j! \right)^{\alpha_j} \right] X^{\Omega(\alpha)} \\ &= \sum_{v \geq 0} \left[\sum_{\substack{\alpha \in \mathbb{N}^{t+1} \\ \Omega(\alpha)=v}} \frac{(\|\alpha\|)!}{\prod_{k=1}^{t+1}(\alpha_k!)} \left(\prod_{j=1}^{t+1} \left((-1)^{j-1} \binom{t+1}{j}^2 j! \right)^{\alpha_j} \right) \right] X^v, \end{aligned} \tag{3.2}$$

and combining Theorem 3.6 and (3.2) we get

$$\begin{aligned}
&\sum_{n\geq 2t}\mathbf{F}_t\left(n\right)X^n\\
&=X^{2t}\cdot\left[\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t+1}\left(-1\right)^{j-1}\binom{t+1}{j}^2 j!\left(2t-j+k\right)!\right)X^k\right]\\
&\cdot\sum_{v\geq 0}\left[\sum_{\substack{\alpha\in\mathbb{N}^{t+1}\\ \Omega(\alpha)=v}}\frac{\left(\|\alpha\|\right)!}{\prod_{k=1}^{t+1}\left(\alpha_k!\right)}\left(\prod_{j=1}^{t+1}\left(\left(-1\right)^{j-1}\binom{t+1}{j}^2 j!\right)^{\alpha_j}\right)\right]X^v\\
&=\sum_{l\geq 0}\left[\sum_{z=0}^{\min\{l,t\}}\left(\sum_{j=z+1}^{t+1}\left(-1\right)^{j-1}\binom{t+1}{j}^2 j!\left(2t-j+z\right)!\right)\right.\\
&\left.\cdot\left(\sum_{\substack{\alpha\in\mathbb{N}^{t+1}\\ \Omega(\alpha)=l-z}}\frac{\left(\|\alpha\|\right)!}{\prod_{k=1}^{t+1}\left(\alpha_k!\right)}\left(\prod_{j=1}^{t+1}\left(\left(-1\right)^{j-1}\binom{t+1}{j}^2 j!\right)^{\alpha_j}\right)\right)\right]X^{2t+l}.
\end{aligned}$$

The desired result follows. □

Now we show that knowing the cardinality of Bruhat intervals starting from the identity leads to knowing the cardinality of Bruhat intervals between two general nested involutions.

Theorem 3.8. *Let $n\geq 2$ and $0\leq j<k\leq\left\lfloor\frac{n}{2}\right\rfloor$; then*

$$\#\left[\vartheta_j,\vartheta_k\right]=\mathbf{F}_{k-j}\left(n-2j\right).$$

Proof. In order to prove the statement we exhibit a bijection

$$\begin{array}{rcl} f:\left[\vartheta_j,\vartheta_k\right] & \xrightarrow{\sim} & \mathcal{F}_{k-j}\left(n-2j\right)\\ \sigma & \longmapsto & f_\sigma \end{array}$$

From Theorem 2.2 we see that if $\sigma\in\left[\vartheta_j,\vartheta_k\right]$ then $\sigma\left(l\right)=n+1-l$ for all $l\in[j]\bigcup\left([n]\setminus[n-j]\right)$. We set

$$f_\sigma\left(l\right)=\sigma\left(l+j\right)-j$$

for all $l\in[n-2j]$, and the desired result follows. □

Knowing the cardinality of Bruhat intervals starting from the identity and Bruhat intervals between two nested involutions leads to knowing the number of permutations less or equal than one of the two nested involutions and incomparable with the other one.

Theorem 3.9. *Let $n\geq 2$ and $0\leq j<k\leq\left\lfloor\frac{n}{2}\right\rfloor$; then*

$$\begin{aligned}&\#\{\sigma\in S_n:\sigma\leq\vartheta_k\ \textit{and}\ \sigma\ \textit{is incomparable with}\ \vartheta_j\}\\ &=\mathbf{F}_k\left(n\right)-\mathbf{F}_j\left(n\right)-\mathbf{F}_{k-j}\left(n-2j\right)+1.\end{aligned}$$

Proof. We have

$$\begin{aligned}&\{\sigma\in S_n:\sigma\leq\vartheta_k\ \text{and}\ \sigma\ \text{is incomparable with}\ \vartheta_j\}\\ &=\mathcal{F}_k\left(n\right)\setminus\left(\mathcal{F}_j\left(n\right)\bigcup\left[\vartheta_j,\vartheta_k\right]\right),\end{aligned}$$

and $\mathcal{F}_j\left(n\right)\bigcap\left[\vartheta_j,\vartheta_k\right]=\{\vartheta_j\}$; the desired result follows. □

Corollary 3.10. *Let $n\geq 2$ and $0\leq t\leq\left\lfloor\frac{n}{2}\right\rfloor$; then*

$$\#\{\sigma\in S_n:\sigma\ \textit{is incomparable with}\ \vartheta_t\}=n!+1-\mathbf{F}_t\left(n\right)-\mathbf{F}_{\left\lfloor\frac{n}{2}\right\rfloor-t}\left(n-2t\right).$$

□

4. Remarks

Studying the cardinality of Bruhat intervals between similar nested involutions in different Coxeter systems leads to other challenging questions.

In particular, one can consider the chains

$$\mathrm{id} = \phi_0 < \phi_1 < \cdots < \phi_{n-1} < \phi_n,$$
$$\mathrm{id} = \psi_0 < \psi_1 < \cdots < \psi_{\lfloor \frac{n}{2} \rfloor - 1} < \psi_{\lfloor \frac{n}{2} \rfloor}$$

of nested involutions in the Bruhat ordering of B_n, the *hyperoctahedral group* of rank n (see [**6, 26**]), where

$$\begin{aligned} \phi_r &= \prod_{j=0}^{r-1} (-n+j, n-j)\,, \\ \psi_t &= \prod_{j=0}^{t-1} (j+1, -n+j)\,(-j-1, n-j) \end{aligned}$$

for any $r = 0, \ldots, n-1$ and any $t = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, and to investigate $\#\,[\phi_j, \phi_k]$ for all $0 \le j < k \le n-1$ and $\#\,[\psi_h, \psi_z]$ for all $0 \le h < z \le \lfloor \frac{n}{2} \rfloor$.

We note that in order to study enumeration of Bruhat intervals in a Coxeter system (W, S) (see [**6, 21, 22**] for comprehensive references about Coxeter systems) it is not required that $W < \infty$. In fact, the following fact is well–known, and we refer e.g. to [**6**] for a proof.

Proposition 4.1. *Let (W, S) be a Coxeter system, and $u, v \in W$. Bruhat intervals $[u, v] = \{z \in W : u \le z \le v\}$ are finite (even if $\#S = \infty$). In fact, $\#\,[u, v] \le 2^{l(v)}$, where $l\,(v)$ denotes the length of v.*

Therefore, another tempting choice to investigate the cardinality of Bruhat intervals between suitable involutions would be to consider $\tilde{A}_n$, the affine group of type $\tilde{A}$ and rank n; see [**4, 6, 16**].

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