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Polynomial realizations of some trialgebras

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Abstract. We realize several combinatorial Hopf algebras based on set compositions, plane trees and segmented compositions in terms of noncommutative polynomials in infinitely many variables. For each of them, we describe a trialgebra structure, an internal product, and several bases.

Résumé. Nous réalisons plusieurs algèbres de Hopf combinatoires dont les bases sont indexées par les partitions d'ensembles ordonnées, les arbres plans et les compositions segmentées en termes de polynômes non-commutatifs en une infinité de variables. Pour chacune d'elles, nous décrivons sa structure de trigèbre, un produit intérieur et plusieurs bases.

1. Introduction

The aim of this note is to construct and analyze several combinatorial Hopf algebras arising in the theory of operads from the point of view of the theory of noncommutative symmetric functions. Our starting point will be the algebra of noncommutative polynomial invariants

$$\mathbf{WQSym}(A) = \mathbb{K}\langle A\rangle^{\mathfrak{S}(A)_{QS}}$$

of Hivert's quasi-symmetrizing action [**8**]. It is known that, when the alphabet A is infinite, $\mathbf{WQSym}(A)$ acquires the structure of a graded Hopf algebra whose bases are parametrized by ordered set partitions (also called set compositions) [**8, 20, 2**]. Set compositions are in one-to-one correspondence with faces of permutohedra, and actually, **WQSym** turns out to be isomorphic to one of the Hopf algebras introduced by Chapoton in [**4**]. From this algebra, Chapoton obtained graded Hopf algebras based on the faces of the associahedra (corresponding to plane trees counted by the little Schröder numbers) and on faces of the hypercubes (counted by powers of 3). Since then, Loday and Ronco have introduced the operads of dendriform trialgebras and of tricubical algebras [**15**], in which the free algebras on one generator are respectively based on faces of associahedras and hypercubes, and are isomorphic (as Hopf algebras) to the corresponding algebras of Chapoton. More recently, we have introduced a Hopf algebra **PQSym**, based on parking functions [**17, 18, 19**], and derived from it a series of Hopf subalgebras or quotients, some of which being isomorphic to the above mentioned ones as associative algebras, but not as Hopf algebras.

In the following, we will show that applying the same techniques, starting from **WQSym** instead of **PQSym**, allows one to recover all of these algebras, together with their original Hopf structure, in a very natural way. This provides in particular for each of them an explicit realization in terms of noncommutative polynomials. The Hopf structures can be analyzed very efficiently by means of Foissy's theory of bidendriform bialgebras [**6**]. A natural embedding of **WQSym** in $\mathbf{PQSym}^*$ implies that **WQSym** is bidendriform, hence, free and self-dual. These properties are inherited by $\mathfrak{TD}$, the free dendriform trialgebra on one generator, and some of them by $\mathfrak{TC}$, the free cubical trialgebra on one generator. A lattice structure on the set of faces of the permutohedron (introduced in [**12**] under the name "pseudo-permutohedron" and rediscovered in [**21**]) leads to the construction of various bases of these algebras. Finally, the natural identification of the homogeneous components of the dual $\mathbf{WQSym}_n^*$ (endowed with the internal product induced by **PQSym**)

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with the Solomon-Tits algebras (that is, the face algebras of the braid arrangements of hyperplanes) implies that all three algebras admit an internal product.

Notations – We assume that the reader is familiar with the standard notations of the theory of noncommutative symmetric functions [**7, 5**] and with the Hopf algebra of parking functions [**17, 18, 19**]. We shall need an infinite totally ordered alphabet $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$, generally assumed to be the set of positive integers. We denote by $\mathbb{K}$ a field of characteristic 0, and by $\mathbb{K}\langle A\rangle$ the free associative algebra over A when A is finite, and the projective limit $\operatorname{proj\,lim}_B \mathbb{K}\langle B\rangle$, where B runs over finite subsets of A, when A is infinite. The *evaluation* of a word w is the sequence whose i-th term is the number of times the letter a_i occurs in w. The *standardized word* $\operatorname{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $\operatorname{Std}(bbacab) = 341624$. For a word w on the alphabet $\{1, 2, \ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter i by the integer $i+k$. If u and v are two words, with u of length k, one defines the *shifted concatenation* $u \bullet v = u \cdot (v[k])$ and the *shifted shuffle* $u \Cup v = u \sqcup\!\sqcup (v[k])$, where $\sqcup\!\sqcup$ is the usual shuffle product.

2. The Hopf algebra WQSym

2.1. Noncommutative quasi-symmetric invariants. The *packed word* $u = \operatorname{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\operatorname{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the polynomial

$$\mathbf{M}_u := \sum_{\operatorname{pack}(w)=u} w\,. \tag{1}$$

For example, restricting A to the first five integers,

$$\mathbf{M}_{13132} = 13132 + 14142 + 14143 + 24243 + 15152 + 15153 + 25253 + 15154 + 25254 + 35354. \tag{2}$$

Under the abelianization $\chi: \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I ($I = (|u|_a)_{a\in A}$ being the evaluation vector of u).

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions [**8**] (and called **NCQSym** in [**2**]), consisting in the invariants of the noncommutative version of Hivert's quasi-symmetrizing action [**9**], which is defined by $\sigma \cdot w = w'$ where w' is such that $\operatorname{Std}(w') = \operatorname{Std}(w)$ and $\chi(w') = \sigma \cdot \chi(w)$. Hence, two words are in the same $\mathfrak{S}(A)$-orbit iff they have the same packed word.

WQSym can be embedded in **MQSym** [**8, 5**], by $\mathbf{M}_u \mapsto \mathbf{MS}_M$, where M is the packed $(0,1)$-matrix whose jth column contains exactly one 1 at row i whenever the jth letter of u is a_i. Since the duality in **MQSym** consists in tranposing the matrices, one can also embed **WQSym*** in **MQSym**. The multiplication formula for the basis $\mathbf{M}_u$ follows from that of $\mathbf{MS}_M$ in **MQSym**:

PROPOSITION 2.1. *The product on* **WQSym** *is given by*

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{u\in u' *_W u''} \mathbf{M}_u\,, \tag{3}$$

where the convolution $u' *_W u''$ *of two packed words is defined as*

$$u' *_W u'' = \sum_{v,w;u=v\cdot w\,\in\,\mathrm{PW},\operatorname{pack}(v)=u',\operatorname{pack}(w)=u''} u\,. \tag{4}$$

For example,

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1121} + \mathbf{M}_{1132} + \mathbf{M}_{2221} + \mathbf{M}_{2231} + \mathbf{M}_{3321}. \tag{5}$$

Similarly, the embedding in **MQSym** implies immediately that **WQSym** is a Hopf subalgebra of **MQSym**. However, the coproduct can also be defined directly by the usual trick of noncommutative symmetric functions, considering the alphabet A as an ordered sum of two mutually commuting alphabets $A'\hat{+}A''$. First, by direct inspection, one finds that

$$\mathbf{M}_u(A'\hat{+}A'') = \sum_{0\le k\le \max(u)} \mathbf{M}_{(u|_{[1,k]})}(A')\mathbf{M}_{\operatorname{pack}(u|_{[k+1,\max(u)]})}(A''), \tag{6}$$

where $u|_B$ denote the subword obtained by restricting u to the subset B of the alphabet, and now, the coproduct Δ defined by

$$\Delta \mathbf{M}_u(A) = \sum_{0\le k\le \max(u)} \mathbf{M}_{(u|_{[1,k]})} \otimes \mathbf{M}_{\mathrm{pack}(u|_{[k+1,\max(u)]})}, \tag{7}$$

is then clearly a morphism for the concatenation product, hence defines a bialgebra structure.

Given two packed words u and v, define the *packed shifted shuffle* $u \Cup_W v$ as the shuffle product of u and $v[\max(u)]$. One then easily sees that

$$\Delta \mathbf{M}_w(A) = \sum_{u,v;\, w\in u\Cup_W v} \mathbf{M}_u \otimes \mathbf{M}_v. \tag{8}$$

For example,

$$\Delta \mathbf{M}_{32121} = 1\otimes \mathbf{M}_{32121} + \mathbf{M}_{11}\otimes \mathbf{M}_{211} + \mathbf{M}_{2121}\otimes \mathbf{M}_{1} + \mathbf{M}_{32121}\otimes 1. \tag{9}$$

Packed words can be naturally identified with *ordered set partitions*, the letter a_i at the jth position meaning that j belongs to block i. For example,

$$u = 313144132 \;\leftrightarrow\; \Pi = (\{2,4,7\},\{9\},\{1,3,8\},\{5,6\})\,. \tag{10}$$

To improve the readability of the formulas, we write instead of Π a *segmented permutation*, that is, the permutation obtained by reading the blocks of Π in increasing order and inserting bars | between blocks.

For example,

$$\Pi = (\{2,4,7\},\{9\},\{1,3,8\},\{5,6\}) \;\leftrightarrow 247|9|138|56. \tag{11}$$

On this representation, the coproduct amounts to deconcatenate the blocks, and then standardize the factors. For example, in terms of segmented permutations, Equation (9) reads

$$\Delta \mathbf{M}_{35|24|1} = 1\otimes \mathbf{M}_{35|24|1} + \mathbf{M}_{12}\otimes \mathbf{M}_{23|1} + \mathbf{M}_{24|13}\otimes \mathbf{M}_{1} + \mathbf{M}_{35|24|1}\otimes 1. \tag{12}$$

The dimensions of the homogeneous components of **WQSym** are the ordered Bell numbers 1, 1, 3, 13, 75, 541, ... (sequence A000670, [**22**]) so that

$$\dim \mathbf{WQSym}_n = \sum_{k=1}^{n} S(n,k)k! = A_n(2)\,, \tag{13}$$

where $A_n(q)$ are the Eulerian polynomials.

2.2. The trialgebra structure of WQSym. A *dendriform trialgebra* [**15**] is an associative algebra whose multiplication $\odot$ splits into three pieces

$$x\odot y = x\prec y + x\circ y + x\succ y\,, \tag{14}$$

where $\circ$ is associative, and

$$(x\prec y)\prec z = x\prec(y\odot z)\,,\quad (x\succ y)\prec z = x\succ(y\prec z)\,,\quad (x\odot y)\succ z = x\succ(y\succ z)\,, \tag{15}$$

$$(x\succ y)\circ z = x\succ(y\circ z)\,,\quad (x\prec y)\circ z = x\circ(y\succ z)\,,\quad (x\circ y)\prec z = x\circ(y\prec z)\,. \tag{16}$$

It has been shown in [**19**] that the augmentation ideal $\mathbb{K}\langle A_n\rangle^+$ has a natural structure of dendriform trialgebra: for two non empty words $u,v\in A^*$, we set

$$u\prec v = \begin{cases} uv & \text{if } \max(u) > \max(v)\\ 0 & \text{otherwise,}\end{cases} \tag{17}$$

$$u\circ v = \begin{cases} uv & \text{if } \max(u) = \max(v)\\ 0 & \text{otherwise,}\end{cases} \tag{18}$$

$$u\succ v = \begin{cases} uv & \text{if } \max(u) < \max(v)\\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

THEOREM 2.2. **WQSym**$^+$ *is a sub-dendriform trialgebra of* $\mathbb{K}\langle A\rangle^+$*, the partial products being given by*

$$\mathbf{M}_{w'} \prec \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'', |u|=|w'|; \max(v)<\max(u)} \mathbf{M}_w, \tag{20}$$

$$\mathbf{M}_{w'} \circ \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'', |u|=|w'|; \max(v)=\max(u)} \mathbf{M}_w, \tag{21}$$

$$\mathbf{M}_{w'} \succ \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'', |u|=|w'|; \max(v)>\max(u)} \mathbf{M}_w, \tag{22}$$

It is known [**15**] that the free dendriform trialgebra on one generator, denoted here by $\mathfrak{TD}$, is a free associative algebra with Hilbert series

$$\sum_{n\geq 0} s_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4t} = 1+t+3t^2+11t^3+45t^4+197t^5+\cdots, \tag{23}$$

the generating function of the *super-Catalan*, or *little Schröder* numbers, counting *plane trees*. The previous considerations allow us to give a simple polynomial realization of $\mathfrak{TD}$. Consider the polynomial

$$\mathbf{M}_1 = \sum_{i\geq 1} a_i \ \in \mathbf{WQSym}, \tag{24}$$

THEOREM 2.3 ([**19**]). *The sub-trialgebra* $\mathfrak{TD}$ *of* **WQSym**$^+$ *generated by* $\mathbf{M}_1$ *is free as a dendriform trialgebra.*

Based on numerical evidence, we conjecture the following result:

CONJECTURE 2.4. **WQSym** *is a free dendriform trialgebra.*

The number g'_n of generators in degree n of **WQSym** as a free dendriform trialgebra would then be

$$\sum_{n\geq 0} g'_n t^n = \frac{OB(t)-1}{2OB(t)^2 - OB(t)} = t + 2\,t^3 + 18\,t^4 + 170\,t^5 + 1\,794\,t^6 + 21\,082\,t^7 + O(t^8). \tag{25}$$

where $OB(t)$ is the generating series of the ordered Bell numbers.

2.3. Bidendriform structure of WQSym. A *dendriform dialgebra*, as defined by Loday [**13**], is an associative algebra D whose multiplication $\odot$ splits into two binary operations

$$x \odot y = x \ll y + x \gg y, \tag{26}$$

called left and right, satisfying the following three compatibility relations for all a, b, and c different from 1 in D:

$$(a \ll b) \ll c = a \ll (b \odot c), \quad (a \gg b) \ll c = a \gg (b \ll c), \quad (a \odot b) \gg c = a \gg (b \gg c). \tag{27}$$

A *codendriform coalgebra* is a coalgebra C whose coproduct Δ splits as $\Delta(c) = \overline{\Delta}(c) + c\otimes 1 + 1\otimes c$ and $\overline{\Delta} = \Delta_{\ll} + \Delta_{\gg}$, such that, for all c in C:

$$(\Delta_{\ll} \otimes Id) \circ \Delta_{\ll}(c) = (Id \otimes \overline{\Delta}) \circ \Delta_{\ll}(c), \tag{28}$$

$$(\Delta_{\gg} \otimes Id) \circ \Delta_{\ll}(c) = (Id \otimes \Delta_{\ll}) \circ \Delta_{\gg}(c), \tag{29}$$

$$(\overline{\Delta} \otimes Id) \circ \Delta_{\gg}(c) = (Id \otimes \Delta_{\gg}) \circ \Delta_{\gg}(c). \tag{30}$$

The Loday-Ronco algebra of planar binary trees introduced in [**14**] arises as the free dendriform dialgebra on one generator. This is moreover a Hopf algebra, which turns out to be self-dual, so that it is also codendriform. There is some compatibility between the dendriform and the codendriform structures, leading to what has been called by Foissy [**6**] a *bidendriform bialgebra*, defined as a bialgebra which is both a dendriform dialgebra and a codendriform coalgebra, satisfying the following four compatibility relations

$$\Delta_{\gg}(a \gg b) = a'b'_{\gg} \otimes a'' \gg b''_{\gg} \ + a' \otimes a'' \gg b \ + b'_{\gg} \otimes a \gg b''_{\gg} \ + ab'_{\gg} \otimes b''_{\gg} \ + a \otimes b, \tag{31}$$

$$\Delta_{\gg}(a \ll b) = a'b'_{\gg} \otimes a'' \ll b''_{\gg} \ + a' \otimes a'' \ll b \ + b'_{\gg} \otimes a \ll b''_{\gg}, \tag{32}$$

$$\Delta_{\ll}(a \gg b) = a'b'_{\ll} \otimes a'' \gg b''_{\ll} + ab'_{\ll} \otimes b''_{\ll} + b'_{\ll} \otimes a \gg b''_{\ll}, \tag{33}$$

$$\Delta_{\ll}(a \ll b) = a'b'_{\ll} \otimes a'' \ll b''_{\ll} + a'b \otimes a'' + b'_{\ll} \otimes a \ll b''_{\ll} + b \otimes a, \tag{34}$$

where the pairs (x', x'') (resp. $(x'_{\ll}, x''_{\ll})$ and $(x'_{\gg}, x''_{\gg})$) correspond to all possible elements occuring in $\overline{\Delta}x$ (resp. $\Delta_{\ll}x$ and $\Delta_{\gg}x$), summation signs being understood (Sweedler's notation).

Foissy has shown [**6**] that a connected bidendriform bialgebra $\mathcal{B}$ is always free as an associative algebra and self-dual as a Hopf algebra. Moreover, its primitive Lie algebra is free, and as a dendriform dialgebra, $\mathcal{B}$ is also free over the space of totally primitive elements (those annihilated by $\Delta_{\ll}$ and $\Delta_{\gg}$). It is also proved in [**6**] that **FQSym** is bidendriform, so that it satisfies all these properties. In [**19**], we have proved that **PQSym**, the Hopf algebra of parking functions, as also bidendriform.

The realization of $\mathbf{PQSym}^*$ given in [**18, 19**] implies that

$$\mathbf{M}_u = \sum_{\mathrm{pack}(\mathbf{a})=u} \mathbf{G}_{\mathbf{a}}. \tag{35}$$

Hence, **WQSym** is a subalgebra of $\mathbf{PQSym}^*$. Since in both cases the coproduct correponds to $A \to A' \hat{+} A''$, it is actually a Hopf subalgebra. It also stable by the tridendriform operations, and by the codendriform half-coproducts. Hence,

THEOREM 2.5. **WQSym** *is a sub-bidendriform bialgebra of* $\mathbf{PQSym}^*$. *More precisely, the product rules are*

$$\mathbf{M}_{w'} \ll \mathbf{M}_{w''} = \sum_{w=u.v \in w' *_W w'', |u|=|w'|; \max(v)<\max(u)} \mathbf{M}_w, \tag{36}$$

$$\mathbf{M}_{w'} \gg \mathbf{M}_{w''} = \sum_{w=u.v \in w' *_W w'', |u|=|w'|; \max(v)\geq\max(u)} \mathbf{M}_w, \tag{37}$$

$$\Delta_{\ll}\mathbf{M}_w = \sum_{w \in u \Cup_W v; \mathrm{last}(w)\leq|u|} \mathbf{M}_u \otimes \mathbf{M}_v, \tag{38}$$

$$\Delta_{\gg}\mathbf{M}_w = \sum_{w \in u \Cup_W v; \mathrm{last}(w)>|u|} \mathbf{M}_u \otimes \mathbf{M}_v. \tag{39}$$

where $|u| \geq 1$ *and* $|v| \geq 1$, *and* $\mathrm{last}(w)$ *means the last letter of* w. *As a consequence,* **WQSym** *is free, cofree, self-dual, and its primitive Lie algebra is free.*

2.4. Duality: embedding $\mathbf{WQSym}^*$ into PQSym. Recall from [**17**] that **PQSym** is the algebra with basis $(\mathbf{F}_{\mathbf{a}})$, the product being given by the shifted shuffle of parking functions, and that $(\mathbf{G}_{\mathbf{a}})$ is the dual basis in $\mathbf{PQSym}^*$.

For a packed word u over the integers, let us define its *maximal unpacking* $\mathrm{mup}(u)$ as the greatest parking function $\mathbf{b}$ for the lexicographic order such that $\mathrm{pack}(\mathbf{b}) = u$. For example, $\mathrm{mup}(321412451) = 641714791$.

Since the basis $(\mathbf{M}_u)$ of **WQSym** can be expressed as the sum of $\mathbf{G}_{\mathbf{a}}$ with a given packed word, the dual basis of $(\mathbf{M}_u)$ in $\mathbf{WQSym}^*$ can be identified with equivalence classes of $(\mathbf{F}_{\mathbf{a}})$ under the relation $\mathbf{F}_{\mathbf{a}} = \mathbf{F}_{\mathbf{a}'}$ iff $\mathrm{pack}(\mathbf{a}) = \mathrm{pack}(\mathbf{a}')$. Since the shifted shuffle of two maximally unpacked parking functions contains only maximally unpacked parking functions, the dual algebra $\mathbf{WQSym}^*$ is in fact a subalgebra of **PQSym**. Finally, since, if $\mathbf{a}$ is maximally unpacked then only maximally unpacked parking functions appear in the coproduct $\Delta\mathbf{F}_{\mathbf{a}}$, one has

THEOREM 2.6. $\mathbf{WQSym}^*$ *is a Hopf subalgebra of* **PQSym**. *Its basis element* $\mathbf{M}^*_u$ *can be identified with* $\mathbf{F}_{\mathbf{b}}$ *where* $\mathbf{b} = \mathrm{mup}(u)$.

So we have

$$\mathbf{F}_{\mathbf{b}'}\mathbf{F}_{\mathbf{b}'} := \sum_{\mathbf{b}\in\mathbf{b}'\Cup\mathbf{b}''} \mathbf{F}_{\mathbf{b}}, \qquad \Delta\mathbf{F}_{\mathbf{b}} = \sum_{u\cdot v=\mathbf{b}} \mathbf{F}_{\mathrm{Park}(u)} \otimes \mathbf{F}_{\mathrm{Park}(v)}, \tag{40}$$

where Park is the parkization algorithm defined in [**19**]. For example,

$$\mathbf{F}_{113}\mathbf{F}_{11} = \mathbf{F}_{11344} + \mathbf{F}_{11434} + \mathbf{F}_{11443} + \mathbf{F}_{14134} + \mathbf{F}_{14143} + \mathbf{F}_{14413} + \mathbf{F}_{41134} + \mathbf{F}_{41143} + \mathbf{F}_{41413} + \mathbf{F}_{44113}. \tag{41}$$

(42) $\Delta\mathbf{F}_{531613} = 1\otimes\mathbf{F}_{531613}+\mathbf{F}_{1}\otimes\mathbf{F}_{31513}+\mathbf{F}_{21}\otimes\mathbf{F}_{1413}+\mathbf{F}_{321}\otimes\mathbf{F}_{312}+\mathbf{F}_{3214}\otimes\mathbf{F}_{12}+\mathbf{F}_{43151}\otimes\mathbf{F}_{1}\mathbf{F}_{531613}\otimes 1.$

2.5. The Solomon-Tits algebra. The above realization of $\mathbf{WQSym}^*$ in $\mathbf{PQSym}$ is stable under the internal product of $\mathbf{PQSym}$ defined in [**18**]. Indeed, by definition of the internal product, if $\mathbf{b}'$ and $\mathbf{b}''$ are maximally unpacked, and $\mathbf{F_b} = \mathbf{F}_{\mathbf{b}'} * \mathbf{F}_{\mathbf{b}''}$, then $\mathbf{b}$ is also maximally unpacked.

Moreover, if one writes $\mathbf{b}' = \{s'_1,\ldots,s'_k\}$ and $\mathbf{b}'' = \{s''_1,\ldots,s''_l\}$ as ordered set partitions, then the parkized word $\mathbf{b} = \mathrm{Park}(\mathbf{b}',\mathbf{b}'')$ corresponds to the ordered set partition obtained from

$$\{s'_1\cap s''_1, s'_1\cap s''_2,\ldots,s'_1\cap s''_l, s'_2\cap s''_1,\ldots,s'_k\cap s''_l\}. \tag{43}$$

This formula was rediscovered in [**2**] and Bergeron and Zabrocki recognized the Solomon-Tits algebra, in the version given by Bidigare [**3**], in terms of the face semigroup of the braid arrangement of hyperplanes. So,

Theorem 2.7. $(\mathbf{WQSym}^*, *)$ *is isomorphic to the Solomon-Tits algebra.*

In particular, the product of the Solomon-Tits algebra is dual to the coproduct $\delta\mathbf{G}(A) = \mathbf{G}(A'A'')$.

2.6. The pseudo-permutohedron. We shall now make use of the lattice of pseudo-permutations, a combinatorial structure defined in [**12**] and rediscovered in [**21**]. *Pseudo-permutations* are nothing but ordered set partitions. However, regarding them as generalized permutations helps uncovering their lattice structure. Indeed, let us say that if i is in a block strictly to the right of j with $i<j$ then we have a full inversion (i,j), and that if i is in the same block as j, then we have a *half* inversion $\frac{1}{2}(i,j)$. The total number of inversions is the sum of these numbers. For example, the table of inversions of $45|13|267|8$ is

$$\left\{\frac{1}{2}(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ \frac{1}{2}(2,6),\ \frac{1}{2}(2,7),\ (3,4),\ (3,5),\ \frac{1}{2}(4,5),\ \frac{1}{2}(6,7)\right\}, \tag{44}$$

and it has 9.5 inversions.

One can now define a partial order $\preceq$ on pseudo-permutations by setting $p_1 \preceq p_2$ if the value of the inversion (i,j) in the table of inversions of p_1 is smaller than or equal to its value in the table of inversions of p_2, for all (i,j). This partial order is a lattice [**12**]. In terms of packed words, the covering relation reads as follows. The successors of a packed word u are the packed words v such that

- if all the $i-1$ are to the left of all the i in u then u has as successor the element where all letters j greater than or equal to i are replaced by $j-1$.
- if there are k letters i in u, then one can choose an integer j in the interval $[1,k-1]$ and change the j righmost letters i into $i+1$ and the letters l greater than i into $l+1$.

For example, $w = 44253313$ has five successors,

$$33242212,\ 44243313,\ 55264313,\ 55264413,\ 54263313. \tag{45}$$

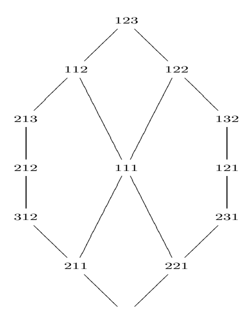

Figure 1. The pseudo-permutohedron of degree 3.

Theorem 2.8 ([**21**]). *Let u and v be two packed words. Then $\mathbf{M}_u\mathbf{M}_v$ is an interval of the pseudo-permutohedron lattice. The minimum of the interval is given by $u\cdot v[\max(u)]$ and its maximum by $u[\max(v)]\cdot v$.*

For example,

$$\mathbf{M}_{13214}\mathbf{M}_{212} = \sum_{u\in[13214656,35436212]}\mathbf{M}_u. \tag{46}$$

2.7. Other bases of WQSym and WQSym*. Since there is a lattice structure on packed words and since we know that the product $\mathbf{M}_u\mathbf{M}_v$ is an interval of this lattice, we can define several interesting bases, depending on the way we use the lattice.

As in the case of the permutohedron, one can take sums of $\mathbf{M}_u$, over all the elements upper or lower than u in the lattice, or restricted to elements belonging to the same "class" as u (see [**5, 1**] for examples of such bases). In the case of the permutohedron, the classes are the descent classes of permutations. In our case, the classes are the intervals of the pseudo-permutohedron composed of words with the same standardization.

Summing over all elements upper (or lower) than a word u naturally yields multiplicative bases on **WQSym**. Summing over all elements upper (or lower) than u inside its standardization class leads to analogs of the usual bases of *QSym*.

2.7.1. *Multiplicative bases.* Let

$$\mathcal{S}_u := \sum_{v\preceq u} \mathbf{M}_v \quad \text{and} \quad \mathcal{E}_u := \sum_{u\preceq v} \mathbf{M}_v. \tag{47}$$

For example,

$$\mathcal{S}_{212} = \mathbf{M}_{212} + \mathbf{M}_{213} + \mathbf{M}_{112} + \mathbf{M}_{123}. \tag{48}$$

$$\mathcal{E}_{212} = \mathbf{M}_{212} + \mathbf{M}_{312} + \mathbf{M}_{211} + \mathbf{M}_{321}. \tag{49}$$

$$\mathcal{S}_{1122} = \mathbf{M}_{1122} + \mathbf{M}_{1123} + \mathbf{M}_{1233} + \mathbf{M}_{1234}. \tag{50}$$

Since both $\mathcal{S}$ and $\mathcal{E}$ are triangular over the basis $\mathbf{M}_u$ of **WQSym**, we know that these are bases of **WQSym**.

THEOREM 2.9. *The sets* $(\mathcal{S}_u)$ *and* $(\mathcal{E}_u)$ *where* u *runs over packed words are bases of* **WQSym**. *Moreover, their product is given by*

$$\mathcal{S}_{u'}\mathcal{S}_{u''} = \mathcal{S}_{u'[\max(u'')]\cdot u''}. \tag{51}$$

$$\mathcal{E}_{u'}\mathcal{E}_{u''} = \mathcal{E}_{u'\cdot u''[\max(u')]}. \tag{52}$$

For example,

$$\mathcal{S}_{1122}\mathcal{S}_{132} = \mathcal{S}_{4455132}. \tag{53}$$

$$\mathcal{E}_{1122}\mathcal{E}_{132} = \mathcal{E}_{1122354}. \tag{54}$$

2.7.2. *Quasi-ribbon basis of* **WQSym**. Let us first mention that a basis of **WQSym** has been defined in [**2**] by summing over intervals restricted to standardization classes of packed words.

We will now consider similar sums but taken the other way round, in order to build the analogs of **WQSym** of Gessel's fundamental basis F_I of *QSym*. Indeed, as already mentioned, the $\mathbf{M}_u$ are mapped to the M_I of *QSym* under the abelianization $\mathbb{K}\langle A\rangle \to \mathbb{K}[X]$ of **WQSym**. Since the pair of dual bases (F_I, R_I) of $(QSym, \mathbf{Sym})$ is of fundamental importance, it is natural to ask whether one can find an analogous pair for $(\mathbf{WQSym}, \mathbf{WQSym}^*)$. To avoid confusion in the notations, we will denote the analog of F_I by Φ_u instead of $\mathbf{F}_u$ since this notation is already used in the dual algebra $\mathbf{WQSym}^* \subset \mathbf{PQSym}$, with a different meaning. The analog of R basis in $\mathbf{WQSym}^*$ will still be denoted by R. The representation of packed words by segmented permutations is more suited for the next statements since one easily checks that two words u and v having the same standardized word satisfy $v \preceq u$ iff v is obtained as a segmented permutation from the segmented permutation of u by inserting any number of bars. Let

$$\Phi_\sigma := \sum_{\sigma'} \mathbf{M}_{\sigma'} \tag{55}$$

where σ' runs ver the set of segmented permutations obtained from σ by inserting any number of bars. For example,

$$\Phi_{14|6|23|5} = \mathbf{M}_{14|6|23|5} + \mathbf{M}_{14|6|2|3|5} + \mathbf{M}_{1|4|6|23|5} + \mathbf{M}_{1|4|6|2|3|5}. \tag{56}$$

Since (Φ_u) is triangular over $(\mathbf{M}_u)$, it is a basis of **WQSym**. By construction, it satisfies a product formula similar to that of Gessel's basis F_I of *QSym* (whence the choice of notation). To state it, we need an analogue of the shifted shuffle, defined on the special class of segmented permutations encoding set compositions.

The *shifted shuffle* $\alpha \Cup \beta$ of two such segmented permutations is obtained from the usual shifted shuffle $\sigma \Cup \tau$ of the underlying permutations σ and τ by inserting bars

- between each pairs of letters coming from the same word if they were separated by a bar in this word,
- after each element of β followed by an element of α.

For example,

$$2|1 \Cup 12 = 2|134 + 23|14 + 234|1 + 3|2|14 + 3|24|1 + 34|2|1. \tag{57}$$

THEOREM 2.10. *The product and coproduct in the basis* Φ *are given by*

$$\Phi_{\sigma'}\Phi_{\sigma''} = \sum_{\sigma \in \sigma' \Cup \sigma''} \Phi_\sigma. \tag{58}$$

$$\Delta\Phi_\sigma = \sum_{\sigma'|\sigma''=\sigma \ \text{or}\ \sigma'\cdot\sigma''=\sigma} \Phi_{\mathrm{Std}(\sigma')} \otimes \Phi_{\mathrm{Std}(\sigma'')}. \tag{59}$$

For example, we have

$$\Phi_1\Phi_{13|2} = \Phi_{124|3} + \Phi_{2|14|3} + \Phi_{24|13} + \Phi_{24|3|1}. \tag{60}$$

$$\Delta\Phi_{35|14|2} = 1 \otimes \Phi_{35|14|2} + \Phi_1 \otimes \Phi_{4|13|2} + \Phi_{12} \otimes \Phi_{13|2} + \Phi_{23|1} \otimes \Phi_{2|1} + \Phi_{24|13} \otimes \Phi_1 + \Phi_{35|14|2} \otimes 1. \tag{61}$$

Note that under abelianization, $\chi(\Phi_u) = F_I$ where I is the evaluation of u.

2.7.3. *Ribbon basis of* **WQSym***. Let us now consider the dual basis of Φ. We have seen that it should be regarded as an analog of the ribbon basis of **Sym**. By duality, one can state:

THEOREM 2.11. *Let* R_σ *be the dual basis of* Φ_σ. *Then the product and coproduct in this basis are given by*

$$R_{\sigma'}R_{\sigma''} = \sum_{\sigma=\tau|\nu \ \text{or}\ \sigma=\tau\nu; \mathrm{Std}(\tau)=\sigma', \mathrm{Std}(\nu)=\sigma''} R_\sigma. \tag{62}$$

$$\Delta R_\sigma = \sum_{\sigma'.\sigma''=\sigma} R_{\mathrm{Std}(\sigma')} \otimes R_{\mathrm{Std}(\sigma'')}. \tag{63}$$

Note that there are more elements coming from $\tau|\nu$ than from $\tau\nu$ since the permutation σ has to be *increasing* between two bars.

For example,

$$R_{21}R_1 = R_{212} + R_{221} + R_{213} + R_{231} + R_{321}. \tag{64}$$

3. Hopf algebras based on Schröder sets

In Section 2.2, we recalled that the little Schröder numbers build up the Hilbert series of the free dendriform trialgebra on one generator $\mathfrak{TD}$. We will see that our relization of $\mathfrak{TD}$ endows it with a natural structure of bidendriform bialgebra. In particular, this will prove that there is a natural self-dual Hopf structure on $\mathfrak{TD}$. But there are other ways to arrive at the little Schröder numbers from the other Hopf algebras **WQSym** and **PQSym**. Indeed, the number of classes of packed words of size n under the sylvester congruence is s_n, and the number of classes of parking functions of size n under the hypoplactic congruence is also s_n. The hypoplactic quotient of **PQSym*** has been studied in [**19**]. It is not isomorphic to $\mathfrak{TD}$ nor to the sylvester quotient of **WQSym** since it is a non self-dual Hopf algebra whereas the last two are self-dual, and furthemore isomorphic as bidendriform bialgebras and as dendriform trialgebras.

3.1. The free dendriform trialgebra again. Recall that we realized the free dendriform trialgebra in Section 2.2 as the subtrialgebra of **WQSym** generated by $\mathbf{M}_1$, the sum of all letters. It is immediate that $\mathfrak{TD}$ is stable by the codendriform half-coproducts of **WQSym***. Hence,

THEOREM 3.1. $\mathfrak{TD}$ *is a sub-bidendriform bialgebra, and hence a Hopf subalgebra of* **WQSym***. *In particular,* $\mathfrak{TD}$ *is free, self-dual and its primitive Lie algebra is free.*

3.2. Lattice structure on plane trees. Given a plane tree T, define its *canonical word* as the maximal packed word w in the pseudo-permutohedron such that $\mathcal{T}(w) = T$.

For example, the canonical words up to $n = 3$ are

$$\{1\}, \quad \{11, 12, 21\}, \quad \{111,\ 112,\ 211,\ 122,\ 212,\ 221,\ 123,\ 213,\ 231,\ 312,\ 321\} \tag{65}$$

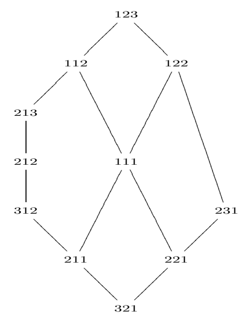

FIGURE 2. The lattice of plane trees represented by their canonical words for $n = 3$.

Define the *second canonical word* of each tree T as the minimal packed word w in the pseudo-permutohedron such that $\mathcal{T}(w) = T$.

A packed word $u = u_1 \cdots u_n$ is said to *avoid the pattern* $w = w_1 \cdots w_k$ if there is no sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $u' = u_{i_1} \cdots u_{i_k}$ has same inversions and same half-inversions as w.

For example, 41352312 avoids the patterns 111 and 1122, but not 2311 since 3522 has the same (half)-inversions.

THEOREM 3.2. *The canonical words of trees are the packed words avoiding the patterns 121 and 132. The second canonical words of trees are the packed words avoiding the patterns 121 and 231.*

Set $u \sim_T v$ iff $\mathcal{T}(u) = \mathcal{T}(v)$. We now define two orders $\sim_T$-classes of packed words

1. A class S is smaller than a class S' if the canonical word of S is smaller than the canonical word of S' in the pseudo-permutohedron.
2. A class S is smaller than a class S' if there is a pair (w, w') in $S \times S'$ such that w is smaller than w' in the pseudo-permutohedron.

THEOREM 3.3. *These two orders coincide and are also identical with the one defined in* [**21**]. *Moreover, the restriction of the pseudo-permutohedron to the canonical words of trees is a lattice.*

3.3. Some bases of $\mathfrak{TD}$.

3.3.1. *The basis $\mathcal{M}_T$.* Let us start with the already defined basis $\mathcal{M}_T$. First note that $\mathcal{M}_T$ expressed as a sum of $\mathbf{M}_u$ in **WQSym** is an interval of the pseudo-permutohedron. From the above description of the lattice, we obtain easily:

THEOREM 3.4 ([**21**]). *The product $\mathcal{M}_{T'}\mathcal{M}_{T''}$ is an interval of the lattice of plane trees. On trees, the minimum $T' \wedge T''$ is obtained by gluing the root of T'' at the end of the leftmost branch of T', whereas the maximum $T' \vee T''$ is obtained by gluing the root of T' at the end of the rightmost branch of T''.*

On the canonical words w' and w'', the minimum is the canonical word associated with $w' \cdot w''[\max(w')]$ and the maximum is $w'[\max(w'')] \cdot w''$.

3.3.2. *Complete and elementary bases of $\mathfrak{TD}$.* We can also build two multiplicative bases as in **WQSym**.

THEOREM 3.5. *The set $(\mathcal{S}_w)$ (resp. $(\mathcal{E}_w)$) where w runs over canonical (resp. second canonical) words are multiplicative bases of $\mathfrak{TD}$.*

3.4. Internal product on $\mathfrak{TD}$. If one defines $\mathfrak{TD}$ as the Hopf subalgebra of **WQSym** defined by

$$\mathcal{M}_T = \sum_{\mathcal{T}(u)=T} \mathbf{M}_u\,, \tag{66}$$

then $\mathfrak{TD}^*$ is the quotient of $\mathbf{WQSym}^*$ by the relation $\mathbf{F}_u \equiv \mathbf{F}_v$ iff $\mathcal{T}(u) = \mathcal{T}(v)$. We denote by S_T the dual basis of $\mathcal{M}_T$.

Theorem 3.6. *The internal product of* $\mathbf{WQSym}_n^*$ *induces an internal product on the homogeneous components* $\mathfrak{TD}_n^*$ *of the dual algebra. More precisely, one has*

$$S_{T'} * S_{T''} = S_T, \tag{67}$$

where T *is the tree obtained by applying* $\mathcal{T}$ *to the biword of the canonical words of the trees* T' *and* T''.

For example, representing trees as their canonical words, one has

$$S_{221} * S_{122} = S_{231}; \qquad S_{221} * S_{321} = S_{321}; \tag{68}$$

$$S_{453223515} * S_{433442214} = S_{674223518}. \tag{69}$$

3.5. Sylvester quotient of WQSym. One can check by direct calculation that the sylvester quotient [**10**] of **WQSym** is also stable by the tridendriform operations, and by the codendriform half-coproducts since the elements of a sylvester class have the same last letter. Hence,

Theorem 3.7. *The sylvester quotient of* **WQSym** *is a dendriform trialgebra, a bidendriform bialgebra, and hence a Hopf algebra. It is isomorphic to* $\mathfrak{TD}$ *as a dendriform trialgebra, as a bidendriform bialgebra and as a Hopf algebra.*

4. A Hopf algebra of segmented compositions

In [**19**], we have built a Hopf subalgebra $\mathbf{SCQSym}^*$ of the hypoplactic quotient $\mathbf{SQSym}^*$ of $\mathbf{PQSym}^*$, whose Hilbert series is given by

$$1 + \sum_{n\geq 1} 3^{n-1} t^n. \tag{70}$$

This Hopf algebra is not self-dual, but admits lifts of Gessel's fundamental basis F_I of $QSym$ and its dual basis. Since the elements of $\mathbf{SCQSym}^*$ are obtained by summing up hypoplactic classes having the same packed word, thanks to the following diagram, it is obvious that $\mathbf{SCQSym}^*$ is also the quotient of **WQSym** by the hypoplactic congruence.

$$\begin{array}{ccc} \mathbf{PQSym}^* & \xrightarrow{hypo} & \mathbf{SQSym}^* \\ {\scriptstyle (pack)}\uparrow & & \uparrow{\scriptstyle (pack)} \\ \mathbf{WQSym} & \xrightarrow{hypo} & \mathbf{SCQSym}^* \end{array} \tag{71}$$

4.1. Segmented compositions. Define a *segmented composition* as a finite sequence of integers, separated by vertical bars or commas, *e.g.*, $(2,1\,|\,2\,|\,1,2)$.

The number of segmented compositions having the same underlying composition is obviously 2^{l-1} where l is the length of the composition, so that the total number of segmented compositions of sum n is 3^{n-1}. There is a natural bijection between segmented compositions of n and sequences of length $n-1$ over three symbols $<, =, >$: start with a segmented composition $\mathbf{I}$. If the i-th position is not a descent of the underlying ribbon diagram, write $<$; otherwise, if i is followed by a comma, write $=$; if i is followed by a bar, write $>$.

Now, with each word w of length n, associate a segmented composition $S(w) = s_1 \cdots s_{n-1}$ where s_i is the correct comparison sign between w_i and w_{i+1}. For example, given $w = 1615116244543$, one gets the sequence (and the segmented composition):

$$<><>=<><=<>>\Longleftrightarrow (2|2|1,2|2,2|1|1). \tag{72}$$

4.2. A subalgebra of $\mathfrak{TD}$. Given a segmented composition $\mathbf{I}$, define

$$M_{\mathbf{I}} = \sum_{S(T)=\mathbf{I}} \mathcal{M}_T = \sum_{S(u)=\mathbf{I}} \mathbf{M}_u\,. \tag{73}$$

For example,

$$M_{12|1} = \mathcal{M}_{2231} \qquad M_{1|3} = \mathcal{M}_{2123} + \mathcal{M}_{2134} + \mathcal{M}_{3123} + \mathcal{M}_{3124} + \mathcal{M}_{4123}. \tag{74}$$

THEOREM 4.1. *The $M_{\mathbf{I}}$ generate a subalgebra $\mathfrak{TC}$ of $\mathfrak{TD}$. Their product is given by*

$$M_{\mathbf{I}'}M_{\mathbf{I}''} = M_{\mathbf{I}'\triangleright\mathbf{I}''} + M_{\mathbf{I}',\mathbf{I}''} + M_{\mathbf{I}'|\mathbf{I}''}. \tag{75}$$

where $\mathbf{I}'\triangleright\mathbf{I}''$ is obtained by gluing the last part of $\mathbf{I}'$ and the first part of $\mathbf{I}''$, so that $\mathfrak{TC}$ is the free cubical trialgebra on one generator [**15**].

For example,

$$M_{1|21}M_{31} = M_{1|241} + M_{1|2131} + M_{1|21|31}. \tag{76}$$

4.3. A lattice structure on segmented compositions. Given a segmented composition $\mathbf{I}$, define its *canonical word* as the maximal packed word w in the pseudo-permutohedron such that $S(w) = \mathbf{I}$.

For example, the canonical words up to $n = 3$ are

$$\{1\}, \qquad \{11, 12, 21\}, \qquad \{111,\ 112,\ 211,\ 122,\ 221,\ 123,\ 231,\ 312,\ 321\} \tag{77}$$

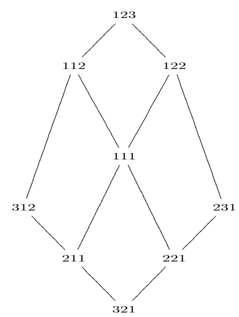

FIGURE 3. The lattice of segmented compositions represented by their canonical words at $n = 3$.

Define the *second canonical word* of a segmented composition $\mathbf{I}$ as the minimal packed word w in the pseudo-permutohedron such that $S(w) = \mathbf{I}$.

THEOREM 4.2. *The canonical words of segmented compositions are the packed words avoiding the patterns 121, 132, 212, and 213. The second canonical words of segmented compositions are the packed words avoiding the patterns 121, 231, 212, and 312.*

Let $u \sim_S v$ iff $S(u) = S(v)$. We define two orders on $\sim_S$-equivalence classes of words.

1. A class S is smaller than a class S' if the canonical word of S is smaller than the canonical word of S' in the pseudo-permutohedron.
2. A class S is smaller than a class S' if there exists two elements (w, w') in $S \times S'$ such that w is smaller than w' in the pseudo-permutohedron.

PROPOSITION 4.3. *The two orders coincide. Moreover, the restriction of the pseudo-permutohedron to the canonical segmented words is a lattice.*

4.4. Multiplicative bases. We can build two multiplicative bases, as in **WQSym**. They are particularly simple:

THEOREM 4.4. *The set $(\mathcal{S}_w)$ where w runs into the set of canonical segmented words is a basis of $\mathfrak{TC}$. The set $(\mathcal{E}_w)$ where w runs into the set of second canonical segmented words is a basis of $\mathfrak{TC}$.*

4.5. Internal product on $\mathfrak{TC}$. If one defines $\mathfrak{TC}$ as the Hopf subalgebra of **WQSym** as in Equation (73), then $\mathfrak{TC}^*$ is the quotient of $\mathbf{WQSym}^*$ by the relation $\mathbf{F}_u \equiv \mathbf{F}_v$ iff $S(u) = S(v)$. We denote by $S_{\mathbf{I}}$ the dual basis of $M_{\mathbf{I}}$.

THEOREM 4.5. *The internal product of $\mathbf{WQSym}^*$ induces an internal product on the homogeneous components $\mathfrak{TC}_n^*$ of $\mathfrak{TC}^*$. More precisely, one has*

$$S_{\mathbf{I}'} * S_{\mathbf{I}''} = S_{\mathbf{I}}, \tag{78}$$

where $\mathbf{I}$ is the segmented composition obtained by applying S to the biword of the canonical words of the segmented compositions $\mathbf{I}'$ and $\mathbf{I}''$.

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