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Algebraic shifting of cyclic polytopes and stacked polytopes

Satoshi Murai

ABSTRACT. Gil Kalai introduced the shifting-theoretic upper bound relation to characterize the f-vectors of Gorenstein* complexes (or homology spheres) by using algebraic shifting. In the present paper, we study the shifting-theoretic upper bound relation. First, we will study the relation between exterior algebraic shifting and combinatorial shifting. Second, by using the relation above, we will prove that the boundary complex of cyclic polytopes satisfies the shifting theoretic upper bound relation. We also prove that the boundary complex of stacked polytopes satisfies the shifting-theoretic upper bound relation.

RÉSUMÉ. Gil Kalai a défini une relation "shifting-theoretic upper bound" pour caractériser les f-vecteurs des complexes de Gorenstein (sphères d'homologie) en termes de décalages algébriques. Dans cet article, nous étudions cette relation. Premièrement, nous étudions la relation entre le décalage algébrique exterieur et le décalage combinatoire. Ensuite, en utilisant cette relation, nous démontrons que le complexe des frontières des polytopes cycliques satisfait la relation "shifting-theoretic upper bound".

1. Introduction

Let Γ be a simplicial complex on $[n] = \{1, \dots, n\}$. Thus Γ is a collection of subsets of $[n]$ such that (i) $\{j\} \in \Gamma$ for all $j \in [n]$ and (ii) if $\sigma \subset [n]$ and $\tau \in \Gamma$ with $\sigma \subset \tau$, then $\sigma \in \Gamma$. A *k-face* of Γ is an element $\sigma \in \Gamma$ with $|\sigma| = k + 1$. The *k-skeleton* of Γ is a family of $(k+1)$-subset $\Gamma_k = \{\sigma \in \Gamma : |\sigma| = k + 1\}$. Let $f_k(\Gamma) = |\Gamma_k|$ the numbers of k-faces of Γ. The vector $f(\Gamma) = (f_0(\Gamma), f_1(\Gamma), \dots)$ is called the *f-vector* of Γ. If $\sigma = \{s_1, s_2, \dots, s_r\}$ and $\tau = \{t_1, t_2, \dots, t_r\}$ are r-subsets of $[n]$ with $s_j < s_{j+1}$ and $t_j < t_{j+1}$ for $j = 1, 2, \dots, r-1$, write $\sigma \prec_p \tau$ if $s_j \leq t_j$ for all $1 \leq j \leq r$. A simplicial complex Γ is called *shifted* if $\tau \in \Gamma$ and $\sigma \prec_p \tau$ implies $\sigma \in \Gamma$.

The g-theorem gives a complete characterization of the f-vectors of boundary complexes of simplicial polytopes. (see [**10**, pp 75–78].) It has been conjectured that the characterization of g-theorem holds for all Gorenstein* complexes. In the present paper, we call this conjecture the g-conjecture. In [**5**], Kalai introduced the shifting-theoretic upper bound relation to solve the g-conjecture by using algebraic shifting. We recall shifting-theoretic upper bound relation.

Algebraic shifting is an operation which associates with each simplicial complex Γ another shifted simplicial complex $\Delta(\Gamma)$. There are two types of algebraic shifting, i.e., exterior algebraic shifting $\Gamma \to \Delta^e(\Gamma)$ and symmetric algebraic shifting $\Gamma \to \Delta^s(\Gamma)$.

For positive integers $i < j$, we write $[i, j] = \{i, i+1, \dots, j-1, j\}$ and $[i] = \{1, 2, \dots, i\}$. A d-subset σ is called *admissible* if $j \notin \sigma$ implies $[j+1, d-j+2] \subset \sigma$. Let $C(n, d)$ be the boundary complex of the cyclic d-polytope with n vertices. Kalai [**4**] proved that $\Delta^s(C(n,d))$ is pure and $\Delta^s(C(n,d))_{d-1}$ consists of

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all admissible d-subsets of $[n]$, in other words,

$$\begin{aligned}\Delta^s(C(n,d))_{d-1} &= \{[1,\lfloor\frac{d+1}{2}\rfloor]\cup\sigma : \sigma\subset[\lfloor\frac{d+1}{2}\rfloor+1,n],\ |\sigma| = d-\lfloor\frac{d+1}{2}\rfloor\}\\ &\bigcup_{1\le j\le\lfloor\frac{d+1}{2}\rfloor}\{([1,d-j+2]\setminus\{j\})\cup\sigma : \sigma\subset[d-j+3,n],\ |\sigma| = j-1\},\end{aligned}$$

where $\lfloor\frac{d+1}{2}\rfloor$ means the integer part of $\frac{d+1}{2}$. Furthermore, Kalai proved that the boundary complex P of every simplicial d-polytope with n vertices satisfies $\Delta^s(P)\subset\Delta^s(C(n,d))$ by using the Lefschetz property of P (see §1.2 for the Lefschetz property). Furthermore, Kalai noticed that if Γ is a $(d-1)$-dimensional Gorenstein* complex Γ on $[n]$ then the relation $\Delta^s(\Gamma)\subset\Delta^s(C(n,d))$ is equivalent to the Lefschetz property of Γ.

It is not hard to see that if every $(d-1)$-dimensional Gorenstein* complex Γ on $[n]$ satisfies $\Delta^e(\Gamma)\subset\Delta^s(C(n,d))$ then the g-conjecture is true. We say that a $(d-1)$-dimensional complex Γ on $[n]$ satisfies the *shifting-theoretic upper bound relation* if Γ satisfies $\Delta^e(\Gamma)\subset\Delta^s(C(n,d))$. Kalai and Sarkaria conjectured that if Γ is a simplicial complex on $[n]$ whose geometric realization can be embedded in S^{d-1} then Γ satisfies the shifting-theoretic upper bound relation. However, it is not known whether Γ satisfies the shifting-theoretic upper bound relation even if *Gamma* is the boundary complex of a simplicial polytope. In the present paper, we will show that $C(n,d)$ and the boundary complex of stacked polytopes satisfies the shifting-theoretic upper bound relation.

In general, the computation of exterior algebraic shifting is rather difficult. First, we will show that we can use combinatorial shifting to study shifting theoretic upper bound relation. Combinatorial shifting, which was introduced by Erdös, Ko and Rado [**3**], is also an operation which associates with each simplicial complex Γ another shifted simplicial complex $\Delta^c(\Gamma)$. Although combinatorial shifting may not be uniquely determined, it is easily computed by a simple combinatorial method. Regarding the relation between exterior algebraic shifting and combinatorial shifting, we have the following result.

THEOREM 1.6. *Let Γ be a $(d-1)$-dimensional Cohen-Macaulay complex on $[n]$ with $h_d(\Gamma)\neq 0$ and with $h_i(\Gamma)=h_{d-i}(\Gamma)$ for $i=0,1,\dots,\lfloor\frac{d}{2}\rfloor$.*

(i) *If $\Delta^c(\Gamma)_{d-1}\subset\Delta^s(C(n,d))_{d-1}$, then this $\Delta^c(\Gamma)$ is pure.*
(ii) *If there is a combinatorial shifted complex $\Delta^c(\Gamma)$ of Γ with $\Delta^c(\Gamma)_{d-1}\subset\Delta^s(C(n,d))_{d-1}$, then one has $\Delta^e(\Gamma)\subset\Delta^s(C(n,d))$.*

Thus, we can use combinatorial shifting for the shifting-theoretic upper bound relation. Also, since combinatorial shifting is entirely a combinatorial operation, proving $\Delta^c(P)\subset\Delta^s(C(n,d))$ for the boundary complex P of a simplicial d-polytope without using the Lefschetz property would be interesting. By using Theorem 1.6, we compute the exterior algebraic shifted complex of the boundary complex of the cyclic d-polytope.

THEOREM 2.1. *Let $C(n,d)$ be the boundary complex of the cyclic d-polytope with n vertices. Then there is a combinatorial shifted complex $\Delta^c(C(n,d))$ such that $\Delta^c(C(n,d))=\Delta^s(C(n,d))$. Thus, in particular, one has $\Delta^e(C(n,d))=\Delta^s(C(n,d))$.*

We also compute algebraic shifting of the boundary complex of a stacked d-polytope with n vertices.

THEOREM 2.2. *Let $L(n,d)$ be the pure $(d-1)$-dimensional simplicial complex spanned by*

$$\{\{2,\dots,d+1\}\}\cup\{(\{1,\dots,d\}\setminus\{i\})\cup\{j\} : 1<i\le d,\ j>d \text{ or } j=i\}.$$

Let $P(n,d)$ be the boundary complex of a stacked d-polytope with n vertices. Then

(i) *One has $\Delta^e(P(n,d))=\Delta^s(P(n,d))=L(n,d)$.*
(ii) *If Γ is the boundary complex of a simplicial d-polytope with n vertices, then one has*

$$\Delta^s(P(n,d))\subset\Delta^s(\Gamma).$$

Note that $\Delta^s(P(n,d))=L(n,d)$ and (ii) easily follows from the relation $\Delta^s(P(n,d))\subset\Delta^s(C(n,d))$. To prove $\Delta^e(P(n,d))=L(n,d)$, we use the fact that the 1-skeleton of $P(n,d)$ is a chordal graph. However, we are not sure that $L(n,d)$ can be obtained by applying combinatorial shifting to $P(n,d)$, the boundary complex of an arbitrary stacked d-polytopes with n vertices.

1.1. algebraic shifting and combinatorial shifting. To define algebraic shifting, we need the theory of generic initial ideals in the exterior algebra.

Let K be an infinite field, V a vector space over K of dimension n with basis $e_1, \ldots, e_n$ and $E = \bigoplus_{d=0}^{n} \bigwedge^d(V)$ the exterior algebra of V. In other words, E is a K-algebra which satisfies

(i) Each $\bigwedge^d(V)$ is a $\binom{n}{d}$ dimensional K-vector space with the canonical K-basis $\{e_{s_1} \wedge e_{s_2} \wedge \cdots \wedge e_{s_d} : 1 \leq s_1 < s_2 < \cdots < s_d \leq n\}$.

(ii) For any integers $i, j \in [n]$, one has $e_i \wedge e_j = -e_j \wedge e_i$.

For $\sigma = \{s_1, \ldots, s_d\} \subset [n]$ with $s_1 < \cdots < s_d$, we call $e_\sigma = e_{s_1} \wedge \cdots \wedge e_{s_d} \in \bigwedge^d(V)$ a *monomial* of E of degree d. Fix a term order $<$. For every homogeneous element $f = \sum_{|\sigma|=d} \alpha_\sigma e_\sigma \in \bigwedge^d(V)$ with each $\alpha_\sigma \in K$, the monomial $\mathrm{in}_<(f) = \max_<\{e_\sigma : \alpha_\sigma \neq 0\}$ is called the *initial monomial of* f. Also, for every homogeneous ideal $J \subset E$, The *initial ideal of* J is the monomial ideal generated by $\{\mathrm{in}_<(f) : f \in J\}$. A monomial ideal $J \subset E$ is called *strongly stable* if $e_\tau \in J$ and $\tau \prec_p \sigma$ means $e_\sigma \in J$.

Let $GL_n(K)$ denote the general linear group with coefficients in K. Any $\varphi = (a_{ij}) \in GL_n(K)$ induce an automorphism of graded K-algebra E as follows:

$$\varphi(f(e_1, \ldots, e_n)) = f\left(\sum_{i=1}^{n} a_{i1} e_i, \ldots, \sum_{i=1}^{n} a_{in} e_i\right) \text{ for all } f \in E.$$

If $J \subset E$ is a homogeneous ideal, then each $\varphi \in GL_n(K)$ gives another homogeneous ideal $\varphi(J) = \{\varphi(f) : f \in J\}$. Now, we recall the fundamental theorem of generic initial ideals.

LEMMA 1.1 ([**1**, Theorem 1.6]). *Let K be an infinite field. Fix a term order $<$ with $e_1 < \cdots < e_n$. Then, for each homogeneous ideal $J \subset E$, there exists a nonempty Zariski open subset $U \subset GL_n(K)$ such that $\mathrm{in}_<(\varphi(J)) = \mathrm{in}_<(\varphi'(J))$ for all $\varphi, \varphi' \in U$ and this $\mathrm{in}_<(\varphi(J))$ is strongly stable.*

This monomial ideal $\mathrm{in}_<(\varphi(J))$ is called the *generic initial ideal of $J \subset E$ with respect to the term order* $<$ and will be denoted $\mathrm{Gin}_<(J)$. In particular, we write $\mathrm{Gin}(J) = \mathrm{Gin}_{<_{\mathrm{rev}}}(J)$, where $<_{\mathrm{rev}}$ is the degree reverse lexicographic order with $e_1 < e_2 < \cdots < e_n$. In other words, for $\sigma \subset [n]$ and $\tau \subset [n]$ with $\sigma \neq \tau$, define $e_\sigma <_{rev} e_\tau$ if (i) $|\sigma| < |\tau|$ or (ii) $|\sigma| = |\tau|$ and the minimal integer in symmetric difference $(\sigma \setminus \tau) \cup (\tau \setminus \sigma)$ belongs to σ. Also, we define $\sigma <_{rev} \tau$ by the same way.

A *shifting operation* on $[n]$ is an operator which associates with each simplicial complex Γ on $[n]$ a simplicial complex $\Delta(\Gamma)$ on $[n]$ and which satisfies the following conditions:

(S_1) $\Delta(\Gamma)$ is shifted;
(S_2) $\Delta(\Gamma) = \Gamma$ if Γ is shifted;
(S_3) $f(\Gamma) = f(\Delta(\Gamma))$;
(S_4) $\Delta(\Gamma') \subset \Delta(\Gamma)$ if $\Gamma' \subset \Gamma$.

(Exterior algebraic shifting) Let Γ be a simplicial complex on $[n]$. The *exterior face ideal of* Γ is a monomial ideal of E generated by all monomials $e_\sigma \in E$ with $\sigma \notin \Gamma$. The *exterior algebraic shifted complex* of Γ is the simplicial complex $\Delta^e(\Gamma)$ defined by

$$J_{\Delta^e(\Gamma)} = \mathrm{Gin}(J_\Gamma).$$

The shifting operation $\Gamma \mapsto \Delta^e(\Gamma)$ which is in fact a shifting operation ([**6**, Proposition 8.8]), is called *exterior algebraic shifting*.

(Combinatorial shifting) Erdös, Ko and Rado [**3**] introduced combinatorial shifting. Let Γ be a collection of r-subsets of $[n]$, where $r \leq n$. For $1 \leq i < j \leq n$, write $\mathrm{Shift}_{ij}(\Gamma)$ for the collection of r-subsets of $[n]$ whose elements are $C_{ij}(\sigma) \subset [n]$, where $\sigma \in \Gamma$ and where

$$C_{ij}(\sigma) = \begin{cases} (\sigma \setminus \{j\}) \bigcup \{i\}, & \text{if } j \in \sigma,\ i \notin \sigma \text{ and } (\sigma \setminus \{j\}) \bigcup \{i\} \notin \Gamma, \\ \sigma, & \text{otherwise.} \end{cases}$$

We can define $\mathrm{Shift}_{ij}(\Gamma)$ for a simplicial complex Γ by the same way. It follows from, e.g., [**6**, Corollary 8.6] that there exists a finite sequence of pairs of integers $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$ with each $1 \leq i_k < j_k \leq n$ such that

$$\mathrm{Shift}_{i_q j_q}(\mathrm{Shift}_{i_{q-1} j_{q-1}}(\cdots(\mathrm{Shift}_{i_1 j_1}(\Gamma))\cdots))$$

is shifted. Such a shifted complex is called a *combinatorial shifted complex* of Γ and will be denoted by $\Delta^c(\Gamma)$. A combinatorial shifted complex $\Delta^c(\Gamma)$ of Γ is, however, not necessarily unique. The shifting operation $\Gamma \mapsto \Delta^c(\Gamma)$, which is in fact a shifting operation ([**6**, Lemma 8.4]), is called *combinatorial shifting*.

Algebraic shifting behaves nicely. For example, algebraic shifting preserves the Cohen-Macaulay property and preserves the dimension of reduced homology groups. On the other hand, combinatorial shifting does not behave nicely. However, the advantage of combinatorial shifting is that we can easily compute them by purely combinatorial methods. Hence the following problem naturally occurs.

Problem (Kalai [**5**, Problem 24]). What are the relations between combinatorial shifting and algebraic shifting?

We will remark a relation between combinatorial shifting and exterior algebraic shifting. For every $\sigma \subset [n]$ and for every shifted simplicial complex Γ on $[n]$, define

$$m_{\leq\sigma}(\Gamma) = |\{\tau \in \Gamma : \tau \leq_{rev} \sigma \text{ and } |\tau| = |\sigma|\}|.$$

Then we have the following relation between $\Delta^c(\Gamma)$ and $\Delta^e(\Gamma)$.

Lemma 1.2. *Let Γ be a simplicial complex on $[n]$. Then, for any combinatorial shifted complex $\Delta^c(\Gamma)$ and for any subset $\sigma \subset [n]$, one has*

$$m_{\leq\sigma}(\Delta^e(\Gamma)) \geq m_{\leq\sigma}(\Delta^c(\Gamma)).$$

Proof. It is not hard to see that (see [**6**, Lemma 8.3]), for all integers $1 \leq i < j \leq n$, there is $\varphi_{ij} \in GL_n(K)$ such that $\mathrm{in}_{<_{rev}}(\varphi_{ij}(I_\Gamma)) = I_{\mathrm{Shift}_{ij}(\Gamma)}$. For each $\varphi \in GL_n(K)$, define a simplicial complex $\Delta_\varphi(\Gamma)$ by

$$I_{\Delta_\varphi(\Gamma)} = \mathrm{in}_{<_{rev}}(\varphi(I_\Gamma)).$$

Then, it follows from [**8**, Theorem 3.1] that, for every $\sigma \subset [n]$, one has

$$m_{\leq\sigma}(\Delta^e(\Gamma)) \geq m_{\leq\sigma}(\Delta^e(\Delta_\varphi(\Gamma))). \tag{1.1}$$

By the definition of combinatorial shifting, there exists a finite sequence of pairs of integers $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$ such that $\Delta^c(\Gamma) = \Delta_{\varphi_{i_q j_q}}(\Delta_{\varphi_{i_{q-1} j_{q-1}}}(\cdots(\Delta_{\varphi_{i_1 j_1}}(\Gamma))\cdots))$. Also, since $\Delta^c(\Gamma)$ is shifted, the conditions of shifting operation say $\Delta^e(\Delta^c(\Gamma)) = \Delta^c(\Gamma)$. Then, by (1.1), we have

$$\begin{aligned} m_{\leq\sigma}(\Delta^e(\Gamma)) &\geq m_{\leq\sigma}(\Delta^e(\Delta_{\varphi_{i_q j_q}}(\Delta_{\varphi_{i_{q-1} j_{q-1}}}(\cdots(\Delta_{\varphi_{i_1 j_1}}(\Gamma))\cdots)))) \\ &= m_{\leq\sigma}(\Delta^e(\Delta^c(\Gamma))) \\ &= m_{\leq\sigma}(\Delta^c(\Gamma)), \end{aligned}$$

for every $\sigma \subset [n]$, as desired. □

Note that Lemma 1.2 induces some other relations between combinatorial shifting and exterior algebraic shifting. For example, it was used in [**9**] to compare the graded Betti numbers of the Stanley-Reisner ideal of $\Delta^e(\Gamma)$ and $\Delta^c(\Gamma)$.

1.2. The shifting-theoretic upper bound relation. The shifting-theoretic upper bound relation was considered from the viewpoint of symmetric algebraic shifting. Thus, first, we recall symmetric algebraic shifting which was introduced in [**4**]. We refer the reader to [**10**] for the definition of Cohen-Macaulay complexes and Gorenstein* complexes.

(Symmetric algebraic shifting) Let K be a field of characteristic 0 and $R = K[x_1, \ldots, x_n]$ the polynomial ring. Let Γ be a simplicial complex on $[n]$. The *Stanley-Reisner ideal* I_Γ *of* Γ is a monomial ideal generated by all squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ with $\{i_1, i_2, \ldots, i_r\} \notin \Gamma$ and $\{i_1, i_2, \ldots, i_r\} \subset [n]$. The ring $R(\Gamma) = R/I_\Gamma$ is called *the face ring of* Γ.

Let $y_1, y_2, \ldots, y_n$ be generic linear forms in $x_1, x_2, \ldots, x_n$ and M the set of monomials in $y_1, y_2, \ldots, y_n$. For every monomial m in M, denote its image in $R(\Gamma)$ by $\tilde{m}$. Define

$$\mathrm{GIN}(\Gamma) = \{m \in M : \tilde{m} \notin \mathrm{span}\{\tilde{l} : \deg(l) = \deg(m),\ l <_{rev} m\}\}.$$

For every monomial $m \in \mathrm{GIN}(\Gamma)$ with $\deg(m) = r \le n$ which does not involve $y_1, y_2, \dots, y_{r-1}$, write $m = y_{i_1} y_{i_2} \cdots y_{i_r}$ with $i_1 \le i_2 \le \cdots \le i_r$, and define

$$S(m) = \{i_1 - r + 1, i_2 - r + 2, \dots, i_{r-1} - 1, i_r\}.$$

The *symmetric algebraic shifted complex* $\Delta^s(\Gamma)$ of Γ is defined by

$$\Delta^s(\Gamma) = \{S(m) : m \in \mathrm{GIN}(\Gamma),\ \deg(m) = r \le n \text{ and } y_i \text{ does not divides } m \text{ for } i \le r-1\}.$$

The shifting operation $\Gamma \to \Delta^s(\Gamma)$ which is in fact a shifting operation ([**6**, §8]), is called *symmetric algebraic shifting.*

Second, we recall h-vectors. Let Γ be a $(d-1)$-dimensional simplicial complex and $(f_0, f_1, \dots, f_{d-1})$ f-vectors of Γ. The *h-vector* of Γ is defined by the relation

$$\sum_{i=0}^{d} h_i(\Gamma) x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i},$$

where we let $f_{-1} = 1$. This is equivalent to

$$h_i(\Gamma) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1} \text{ and } f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_i(\Gamma).$$

(The Lefschetz property) Let Γ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex and $\vartheta_1, \vartheta_2, \dots, \vartheta_d$ generic linear forms. Then $\vartheta_1, \vartheta_2, \dots, \vartheta_d$ is a system of parameters of $R(\Gamma)$. Let

$$\bigoplus_{i=0}^{d} H_i(\Gamma) = R(\Gamma) / < \vartheta_1, \vartheta_2, \dots, \vartheta_d >,$$

where $H_i(\Gamma)$ is the i-th homogeneous component of $R(\Gamma)/ < \vartheta_1, \vartheta_2, \dots, \vartheta_d >$. It is well known [**10**, pp. 53–58] that

$$h_i(\Gamma) = \dim_K H_i(\Gamma).$$

Let ϑ_{d+1} be an additional general linear form and $s = \max\{k : h_k(\Gamma) \ne 0\}$. A $(d-1)$-dimensional Cohen-Macaulay simplicial complex Γ is called *(strongly) Lefschetz* if, for $0 \le i \le \lfloor \frac{s}{2} \rfloor$, the multiplication

$$\vartheta_{d+1}^{s-2i} : H_i(\Gamma) \to H_{s-i}(\Gamma)$$

is an isomorphism. Note that the boundary complex of every simplicial polytope is Lefschetz. The important aspect of Lefschetz property is that proving the Lefschetz property for all Gorenstein* complexes implies the g-conjecture. See [**10**, pp75–78] for the detail.

Next, we recall some basic property of algebraic shifting.

LEMMA 1.3 ([**6**, Lemma 8.]). *Let Γ be a simplicial complex. The followings are equivalent:*

(i) Γ *is Cohen-Macaulay;*
(ii) $\Delta^e(\Gamma)$ *is Cohen-Macaulay;*
(iii) $\Delta^e(\Gamma)$ *is pure.*

Lemma 1.3 is also true for symmetric algebraic shifting Δ^s. Also, if Γ is Cohen-Macaulay, then h-vectors of Γ appears in $\Delta^e(\Gamma)$ and $\Delta^s(\Gamma)$ by the following way.

LEMMA 1.4 (Kalai [**4**, Lemma 7.1]). *Let Γ be a pure shifted $(d-1)$-dimensional simplicial complex. Let $W_i(\Gamma) = \{\sigma \in \Gamma : |\sigma| = d,\ [d-i] \subset \sigma$ and $d - i + 1 \notin \sigma\}$. Then $h_i(\Gamma) = |W_i(\Gamma)|$.*

PROOF. For every monomial $m \in K[y_1, \dots, y_n]$ with $\deg(u) = i$, denote its image in $R(\Gamma)/ < y_1, y_2, \dots, y_d >$ by $[m]$. Let

$$L_i(\Gamma) = \{m \in \mathrm{GIN}(\Gamma) : \deg(m) = i \text{ and } m \in K[y_{d+1}, \dots, y_n]\}.$$

First, we will show that $\dim_K H_i(\Gamma) = |L_i(\Gamma)|$. If m is a monomial in $K[y_{d+1}, \dots, y_n]$ with $deg(m) = i$ and l is a monomial in $< y_1, \dots, y_d >$ with $deg(l) = i$, then $l <_{rev} m$. Since $\mathrm{GIN}(\Gamma) = \{m \in M : \tilde{m} \notin \mathrm{span}\{\tilde{l} : l <_{rev} m\}\}$, it follows that the set of monomials $\tilde{m}$ with $m \in \mathrm{GIN}(\Gamma)_i \cap < y_1, \dots, y_d >= \{\mathrm{GIN}(\Gamma)_i \setminus L_i(\Gamma)\}$ is a K-basis of $\{R(\Gamma) \cap < y_1, \dots, y_d >\}_i$. Thus $\{[m] : m \in L_i(\Gamma)\}$ is a K-basis of $\{R(\Gamma)/ < y_1, \dots, y_d >\}_i$.

On the other hand, since $y_1, y_2, \ldots, y_n$ are generic linear forms, it follows that $y_1, \ldots, y_d$ are generic system of parameters. Thus $H_i(\Gamma) = \{R(\Gamma)/ < y_1, \ldots, y_d >\}_i$ and, therefore, $\dim_K H_i(\Gamma) = |L_i(\Gamma)|$.

Second, we will show that if $\Delta^s(\Gamma)$ is pure and shifted, then, for all $0 \le i \le d$, we have

$$W_i(\Delta^s(\Gamma)) = \{[d-i] \cup S(m) : m \in L_i(\Gamma)\}. \tag{1.2}$$

For any $m \in L_i(\Gamma)$, we have $\min(S(m)) \ge d-i+2$ and $|S(m)| = i$. Since $\Delta^s(\Gamma)$ is pure and shifted, we have $[d-i] \cup S(m) \in W_i(\Delta^s(\Gamma))$. Conversely, if $[d-i] \cup \sigma \in W_i(\Delta^s(\Gamma))$, then $\sigma \in \Delta^s(\Gamma)$ and $\min(\sigma) \ge d-i+2$. Hence there is $m \in L_i(\Gamma)$ with $S(m) = \sigma$.

Since Γ is shifted, we have $\Delta^s(\Gamma) = \Gamma$. Then the relation (1.2) says $|W_i(\Gamma)| = |L_i(\Gamma)| = \dim_K H_i(\Gamma) = h_i(\Gamma)$. $\square$

Lemma 1.5 (Kalai). *Let Γ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex on $[n]$ with $h_d(\Gamma) \ne 0$. The followings are equivalent:*

(i) *Γ is Lefschetz;*
(ii) *$\Delta^s(\Gamma) \subset \Delta^s(C(n,d))$ and $h_i(\Gamma) = h_{d-i}(\Gamma)$ for all $0 \le i \le d$.*

Proof. ((i) $\Rightarrow$ (ii)) The relation $h_i(\Gamma) = h_{d-i}(\Gamma)$ immediately follows from the definition of Lefschetz property. Note that $\Delta^s(\Gamma)_{d-1} = \bigcup_{j=0}^{d} W_i(\Delta^s(\Gamma))$. We will show $W_i(\Delta^s(\Gamma)) \subset W_i(\Delta^s(C(n,d)))$ for all $0 \le i \le d$. For $0 \le i \le \frac{d}{2}$, the inclusion $W_i(\Sigma) \subset W_i(\Delta^s(C(n,d)))$ is true for an arbitrary simplicial complex Σ. Since $y_1, y_2, \ldots, y_n$ are generic linear forms, it follows that $y_1, \ldots, y_d$ are generic system of parameters and y_{d+1} is an additional generic linear form. Then, by assumption, the multiplication $y_{d+1}^{d-2i} : L_i(\Gamma) \to L_{d-i}(\Gamma)$ is a bijection. Then, for $0 \le i \le \frac{d}{2}$, $L_{d-i}(\Gamma)$ is of the form $L_{d-i}(\Gamma) = \{y_{d+1}^{d-2i} m : m \in L_i(\Gamma)\}$. Also, for every $m \in L_i(\Gamma)$ with $0 \le i \le \frac{d}{2}$, we have

$$S(y_{d+1}^{d-2i} m) = \{i+2, \ldots, d-i+1\} \cup S(m) \tag{1.3}$$

Thus, for $0 \le i \le \frac{d}{2}$, relation (1.2) says that $W_{d-i}(\Delta^s(\Gamma))$ is of the form

$$W_{d-i}(\Delta^s(\Gamma)) = \{[i] \cup \{i+2, \ldots, d-i+1\} \cup S(m) : m \in L_i(\Gamma)\} \subset W_{d-i}(\Delta^s(C(n,d))).$$

((ii) $\Rightarrow$ (i)) If $\Delta^s(\Gamma) \subset \Delta^s(C(n,d))$, then, for $0 \le i \le \frac{d}{2}$, each $W_{d-i}(\Delta^s(\Gamma))$ is of the form

$$W_{d-i}(\Gamma) = \{[i] \cup \{i+2, \ldots, d-i+1\} \cup \sigma \in \Gamma : |\sigma| = i\}.$$

Since $\Delta^s(\Gamma)$ is shifted, there is a natural injection form $W_{d-i}(\Gamma)$ to $W_i(\Gamma)$ as follows:

$$[i] \cup \{i+2, \ldots, d-i+1\} \cup \sigma \mapsto [d-i] \cup \sigma. \tag{1.4}$$

Since $h_i(\Gamma) = h_{d-i}(\Gamma)$, Lemma 1.4 says this injection is a bijection. Then (1.2) and (1.3) implies that the multiplication $y_{d+1}^{d-2i} : L_i(\Gamma) \to L_{d-i}(\Gamma)$ is a bijection. $\square$

Let Γ be a $(d-1)$-dimensional Gorenstein* complex on $[n]$. Since $\Delta^s(\Delta^e(\Gamma)) = \Delta^e(\Gamma)$, Lemma 1.5 says that $\Delta^e(\Gamma)$ is Lefschetz if and only if $\Delta^e(\Gamma) \subset \Delta^s(C(n,d))$ and $h_i(\Gamma) = h_{d-i}(\Gamma)$ for $i = 0, 1, \ldots, \lfloor \frac{d}{2} \rfloor$. Since $h_i(\Gamma) = h_{d-i}(\Gamma)$, where $0 \le i \le \lfloor \frac{d}{2} \rfloor$, are true for arbitrary Gorenstein* complex, if we can prove the relation $\Delta^e(\Gamma) \subset \Delta^s(C(n,d))$ for arbitrary $(d-1)$-dimensional Gorenstein* complex Γ on $[n]$, then we can prove the g-conjecture. However, the relation $\Delta^e(\Gamma) \subset \Delta^s(C(n,d))$ is unknown even for the boundary complex of simplicial polytopes. We say that a $(d-1)$-dimensional simplicial complex Γ on $[n]$ satisfies the shifting-theoretic upper bound relation if Γ satisfies $\Delta^e(\Gamma) \subset \Delta^e(C(n,d))$.

We will show that if $\Delta^c(\Gamma)$ is Lefschetz, then $\Delta^e(\Gamma)$ is also Lefschetz by using Lemma 1.2.

Theorem 1.6. *Let Γ be a $(d-1)$-dimensional Cohen-Macaulay complex on $[n]$ with $h_d(\Gamma) \ne 0$ and with $h_i(\Gamma) = h_{d-i}(\Gamma)$ for $i = 0, 1, \ldots, \lfloor \frac{d}{2} \rfloor$.*

(i) *If $\Delta^c(\Gamma)_{d-1} \subset \Delta^s(C(n,d))_{d-1}$, then this $\Delta^c(\Gamma)$ is pure.*
(ii) *If there is a combinatorial shifted complex $\Delta^c(\Gamma)$ of Γ with $\Delta^c(\Gamma)_{d-1} \subset \Delta^s(C(n,d))_{d-1}$, then one has $\Delta^e(\Gamma) \subset \Delta^s(C(n,d))$.*

Proof. (i) Fix a combinatorial shifted complex $\Delta^c(\Gamma)$ which satisfies the assumption $\Delta^c(\Gamma)_{d-1} \subset \Delta^s(C(n,d))_{d-1}$. We will show $|W_i(\Delta^c(\Gamma))| = h_i(\Gamma)$ for all $0 \le i \le d$.

Let $\sigma(i,n) = [d-i] \cup \{n-i+1,\ldots,n\}$. Then, for every $\sigma \subset [n]$ with $|\sigma| = d$, we have $\sigma \le_{rev} \sigma(i,n)$ if and only if $[d-i] \subset \sigma$. This implies that, for every $(d-1)$-dimensional pure shifted simplicial complex Σ, we have

$$m_{\le \sigma(i,n)}(\Sigma) = \sum_{j=0}^{i} |W_i(\Sigma)|.$$

Then Lemma 1.2 says that , for $0 \le i \le \frac{d}{2}$, we have

$$\sum_{j=0}^{i} |W_j(\Delta^e(\Gamma))| \ge \sum_{j=0}^{i} |W_j(\Delta^c(\Gamma))| \text{ and } \sum_{j=0}^{i} |W_{d-j}(\Delta^e(\Gamma))| \le \sum_{j=0}^{i} |W_{d-j}(\Delta^c(\Gamma))|. \tag{1.5}$$

On the other hand, since $\Delta^c(\Gamma)_{d-1} \subset \Delta^s(C(n,d))_{d-1}$, the injection (1.4) says $|W_i(\Delta^c(\Gamma))| \ge |W_{d-i}(\Delta^c(\Gamma))|$ for $0 \le i \le \frac{d}{2}$. In particular, we have

$$\sum_{j=0}^{i} |W_j(\Delta^c(\Gamma))| \ge \sum_{j=0}^{i} |W_{d-j}(\Delta^c(\Gamma))|. \tag{1.6}$$

Since Γ is Cohen-Macaulay and $h_i(\Gamma) = h_{d-i}(\Gamma)$, Lemmas 1.3 and 1.4 say $|W_i(\Delta^e(\Gamma))| = |W_{d-i}(\Delta^e(\Gamma))| = h_i(\Gamma)$. Thus these inequalities (1.5) and (1.6) are all equal. Inductively, we have $|W_i(\Delta^c(\Gamma))| = |W_i(\Delta^e(\Gamma))| = h_i(\Gamma)$ for all $0 \le i \le d$.

Let L be the pure simplicial complex generated by $\Delta^c(\Gamma)_{d-1}$. Then Lemma 1.4 says L and Γ have the same h-vector, that is, they have the same f-vector. Since $\Delta^c(\Gamma) \supset L$, we have $\Delta^c(\Gamma) = L$. Thus this $\Delta^c(\Gamma)$ is pure.

(ii) We will show $W_i(\Delta^e(\Gamma)) \subset W_i(\Delta^s(C(n,d)))$ for all $0 \le i \le d$. Let $\sigma_0(i) = \max_{<_{rev}}\{W_i(\Delta^s(C(n,d)))\}$, $\sigma_c(i) = \max_{<_{rev}}\{W_i(\Delta^c(\Gamma))\}$ and $\sigma_e(i) = \max_{<_{rev}}\{W_i(\Delta^e(\Gamma))\}$.

Since $\Delta^c(\Gamma) \subset \Delta^s(C(n,d))$, we have $\sigma_0(i) \ge_{rev} \sigma_c(i)$ for all i. On the other hand, since $|W_i(\Delta^c(\Gamma))| = h_i$, we have

$$m_{\le \sigma_c(i)}(\Delta^c(\Gamma)) = \sum_{k=0}^{i} |W_k(\Delta^c(\Gamma))| = \sum_{k=0}^{i} h_k(\Gamma)$$

and

$$m_{\le \sigma_e(i)}(\Delta^e(\Gamma)) = \sum_{k=0}^{i} |W_k(\Delta^e(\Gamma))| = \sum_{k=0}^{i} h_k(\Gamma).$$

Then Lemma 1.2 says $\sigma_c(i) \ge_{rev} \sigma_e(i)$. Thus we have $\sigma_0(i) \ge_{rev} \sigma_e(i)$ for all i.

On the other hand, $W_i(\Delta^s(C(n,d)))$ is the set of smallest $h_i(\Gamma)$ elements w.r.t. $<_{rev}$ which contain $\{1,\ldots,d-i\}$ and which do not contain $\{d-i+1\}$, that is,

$$W_i(\Delta^s(C(n,d))) = \{\sigma \subset [n] : [d-i] \subset \sigma,\ d-i+1 \notin \sigma \text{ and } \sigma \le_{rev} \sigma_0(i)\}.$$

Thus we have $\Delta^e(\Gamma) \subset \Delta^s(C(n,d))$. $\square$

2. Exterior algebraic shifting of Cyclic polytopes and stacked polytopes

2.1. Cyclic polytopes. We recall the definition of cyclic polytopes. We refer the reader to [**2**] for the basic theory of convex polytopes.

Let $\mathbb{R}$ denote the set of real numbers. For any subset M of the d-dimensional Euclidean space $\mathbb{R}^d$, there is a smallest convex set containing M. This convex set is called *convex hull of* M and will be denoted by $\mathrm{conv}(M)$. For $d \ge 2$, the *moment curve* in $\mathbb{R}^d$ is the curve parameterized by

$$t \to x(t) = (t, t^2, \ldots, t^d) \in \mathbb{R}^d.$$

The *cyclic d-polytope* with n vertices is the convex hull P of the form

$$P = \mathrm{conv}(\{x(t_1), x(t_2), \ldots, x(t_n)\}),$$

where $t_1, t_2, \ldots, t_n$ are distinct real numbers.

The main result of this section is the following.

THEOREM 2.1. *Let $C(n,d)$ be the boundary complex of the cyclic d-polytope with n vertices. Then there is a combinatorial shifted complex $\Delta^c(C(n,d))$ such that $\Delta^c(C(n,d)) = \Delta^s(C(n,d))$. Thus, in particular, one has $\Delta^e(C(n,d)) = \Delta^s(C(n,d))$.*

PROOF. (sketch) By virtue of Theorem 1.6, what we have to do is finding a combinatorial shifted complex $\Delta^c(C(n,d))$ which satisfies $\Delta^c(C(n,d)) = \Delta^s(C(n,d))$.

Also, by Gale's evenness condition ([**2**, Theorem 13.6]), we know that $C(n,d)_{d-1}$ is the collection of d-subsets σ of $[n]$ which satisfies, for every $i < j$ with $i, j \notin \sigma$, the number $|\{i, i+1, \ldots, j\} \cap \sigma|$ is even.

Define

$$\mathrm{Shift}_{n\downarrow i}(\Gamma) = \mathrm{Shift}_{ii+1}(\cdots(\mathrm{Shift}_{in-1}(\mathrm{Shift}_{in}(\Gamma)))\cdots)$$

and

$$\mathrm{Shift}_{n\uparrow i}(\Gamma) = \mathrm{Shift}_{in}(\cdots(\mathrm{Shift}_{ii+2}(\mathrm{Shift}_{ii+1}(\Gamma)))\cdots).$$

(i) In case of d is even, then

$$\mathrm{Shift}_{n-1\downarrow n}(\mathrm{Shift}_{n-2\downarrow n}(\cdots(\mathrm{Shift}_{1\downarrow n}(C(n,d))\cdots)) = \Delta^s(C(n,d)).$$

(ii) In case of d is odd, then

$$\mathrm{Shift}_{n-1\uparrow n}(\mathrm{Shift}_{n-2\uparrow n}(\cdots(\mathrm{Shift}_{1\uparrow n}(C(n,d))\cdots)) = \Delta^s(C(n,d)).$$

Since computations of (i) and (ii) are complicated, we omit the proof. □

2.2. Stacked polytopes. We recall the construction of stacked polytopes. Starting with a d-simplex, one can add new vertices by building a shallow pyramids over facets to obtain a simplicial convex d-polytope with n vertices. Such convex polytopes are called *stacked d-polytopes*. Let $P(n,d)$ be the boundary complex of a stacked d-polytope with n vertices. Note that the combinatorial type of $P(n,d)$ is not unique. Then we have the following result for algebraic shifting of stacked polytopes.

THEOREM 2.2. *Let $L(n,d)$ be the pure $(d-1)$-dimensional simplicial complex generated by*

$$\{\{2,\ldots,d+1\}\} \cup \{(\{1,\ldots,d\} \setminus \{i\}) \cup \{j\} : 1 < i \le d,\ j > d \text{ or } j = i\}.$$

Let $P(n,d)$ be the boundary complex of a stacked d-polytope with n vertices. Then

(i) *One has $\Delta^e(P(n,d)) = \Delta^s(P(n,d)) = L(n,d)$.*

(ii) *If Γ is the boundary complex of a simplicial d-polytope with n vertices, then one has*

$$\Delta^s(P(n,d)) \subset \Delta^s(\Gamma).$$

PROOF. (sketch) The equality $\Delta^s(P(n,d)) = L(n,d)$ and (ii) easily follows from the Lefschetz property of the boundary complex of simplicial polytopes.

We will show $\Delta^e(P(n,d)) = \Delta^s(P(n,d))$. The case $d = 2$ is easy. In case of $d \ge 3$, by using Lemma 1.4, it is not hard to show that if $\Delta^e(P(n,d)) \ne L(n,d)$ then $\{d+1, d+2\} \in \Delta^e(P(n,d))$. Note that $\{d+1, d+2\} \notin \Delta^s(P(n,d))$. On the other hand, it is known that 1-skeleton of $P(n,d)$ is a chordal graph if $d \ge 3$. It follows from [**7**, Theorem 4.8] that if G is a chordal graph then $\Delta^e(G) = \Delta^s(G)$. This says that $\{d+1, d+2\} \notin \Delta^e(P(n,d))$ and $\Delta^e(P(n,d)) = \Delta^s(P(n,d)) = L(n,d)$. □

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Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka, 560-0043, Japan,
E-mail address: `s-murai@ist.osaka-u.ac.jp`