Part 1

Invited Speakers – Conférenciers Invités

Flag Vectors of Polytopes: An Overview

Margaret M. Bayer

University of Kansas

A convex polytope is the convex hull of a finite set of points in $\mathbf{R}^d$. A d-dimensional polytope has faces of dimension 0 through $d-1$; each face is itself a convex polytope. The faces (along with $\emptyset$ and P itself), ordered by inclusion, form a lattice. This talk is concerned with a study of the face lattices of convex polytopes.

Of historical importance is the problem of characterizing the face vectors of polytopes; these vectors give the number of faces of each dimension. The characterization of face vectors of 3-dimensional polytopes was done by Steinitz a century ago. For 4-dimensional polytopes the problem is still open. The biggest advance since Steinitz was the characterization of face vectors of simplicial polytopes (where all faces are simplices) by Stanley, and Billera and Lee in 1980.

The face vector is apparently not robust enough for attempts at characterization by combinatorial and algebraic techniques. We turn instead to the *flag vector* of a polytope. For a d-dimensional polytope P, and $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{0, 1, \ldots, d-1\}$, $f_S(P)$ is the number of chains of faces $\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$ with $\dim F_i = s_i$. The *flag vector* of P is the length 2^d vector $(f_S(P))_{S \subset \{0,1,\ldots,d-1\}}$. In the cases of 3-dimensional polytopes and simplicial polytopes the flag vector is determined linearly by the face vector; in general it can be viewed as an extension of the face vector.

Richard Stanley (1979) studied flag vectors of Cohen-Macaulay posets, a class that contains face lattices of convex polytopes. Bayer and Billera (1985) proved the generalized Dehn-Sommerville equations, the complete set of linear equations satisfied by the flag vectors of all convex polytopes. Since then a wide variety of approaches have been used in the study of flag vectors.

A crucial ingredient in the characterization of face vectors of simplicial polytopes is the connection with toric varieties. In the nonsimplicial case, the middle perversity intersection homology of the toric variety gives an h-vector, linearly dependent on the flag vector. Results from algebraic geometry translate into linear inequalities on the flag vector (Stanley 1987). Another main source of linear inequalities is the cd-index of a polytope, discovered by Jonathan Fine (1985). The cd-index is a vector linearly equivalent to the flag vector; it can be viewed as a reduction of the flag vector by the generalized Dehn-Sommerville equations.

Rigidity theory, shellings, and co-algebras have been used to generate inequalities on flag vectors of polytopes. The talk will survey results and highlight techniques. Some results pertain to special classes of polytopes, such as cubical polytopes and zonotopes. Others hold for more general classes of combinatorial objects, such as general graded posets, Eulerian posets, and Gorenstein* lattices.

We are still, apparently, far from a characterization of flag vectors of polytopes. In fact, we do not even know if the closed convex cone of flag vectors is finitely generated. This is an area of active research. It has exposed interesting connections with other areas of combinatorics, algebra and geometry.

The diameter and Laplacian of directed graphs

Fan Chung Graham

University of California, San Diego

We consider Laplacians for directed graphs. The spectral gap of the Laplacian can be used to establish an upper bound for the diameter of a directed graph. In addition, the Laplacian eigenvalues of a directed graph capture various isoperimetric properties of the directed graph. For example, we will discuss several versions of the Cheeger inequalities and derive bounds for mixing time for random walks on directed graphs or non-reversible Markov chains.

As to related links, there are some relavant papers at my homepage: http://www.math.ucsd.edu/ fan

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Recent combinatorial results involving Macdonald polynomials and diagonal harmonics

Jim Hanglund

Ohio State University

We discuss some recent combinatorial results on Macdonald polynomials, including a combinatorial formula of Haiman, Loehr and the speaker for the nonsymmetric Macdonald polynomials, which were introduced by Cherednik in connection with his study of Macdonald's constant term identities. We discuss how these results are connected to various theorems and conjectures, by a variety of authors, on the combinatorics of the space of diagonal harmonics.

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Perfect crystals for quantum affine algebras

Seok-Jin Kang

Seoul National University

In this talk, we present a uniform construction of level 1 perfect crystals for all quantum affine algebras. We also propose a uniform construction of higher level perfect crystals for all classical quantum affine algebras.

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Structure relation and raising/lowering operators for orthogonal polynomials

Tom H. Koornwinder

University of Amsterdam
E-mail address: thk@science.uva.nl

The structure relation for classical orthogonal polynomials (OP's), is traditionally defined as a fixed polynomial times the derivative of the n-th degree OP being equal to some explicit linear combination of the OP's of degree n-1, n and n+1, with coefficients depending on n. By substitution of the three-term recurrence relation, the structure relation gives rise to a relation with a raising of lowering operator. A variant of the structure relation can be obtained, for all OP's in the Askey scheme and the q-Askey scheme, by taking the commutator of the second order operator having the OP's as eigenfunctions and the operator of multiplication by x. The lecture will survey past approaches and results on structure relations etc. for OP's in the (q-)Askey scheme and for multivariable OP's associated with root systems. The so-called string equation also pops up here. Then some new results, in particular in the multivariable case will be presented.

Some references:

(1) W.A. Al-Salam and T.S. Chihara,
Another characterization of the classical orthogonal polynomials,
SIAM J. Math. Anal. 3 (1972), 65-70.
(2) A.S. Zhedanov, "Hidden symmetry" of Askey-Wilson polynomials,
Theoret. and Math. Phys. 89 (1991), 1146-1157.
(3) T.H. Koornwinder, Lowering and raising operators for some special orthogonal polynomials,
arXiv:math.CA/0505378.
(4) T.H. Koornwinder The structure relation for Askey-Wilson polynomials,
arXiv:math.CA/0601303.

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Hopf algebroids: can combinatorialists help?

Nigel Ray

University of Manchester

In this talk I shall attempt to explain certain algebraic concepts which seem ripe for combinatorial modelling. I shall avoid technical details by focusing on the underlying ideas, and will concentrate on a few basic examples that I hope convey the flavour of the challenge to those who may not be algebraic experts. I shall certainly not presuppose any familiarity with algebraic topology!

The study of Hopf algebras was initiated by algebraic topologists in the 1930s, and has been permeating other areas of mathematics and theoretical physics ever since. Thanks to the vision of Gian-Carlo Rota and his associates, the theory entered combinatorics during the 1960s, and their viewpoint has now begun to enjoy modest feedback into topology. During the 1970s, however, topologists had already discovered that certain generalisations of Hopf algebras arise rather naturally in stable homotopy theory, and the resulting structures came of age when their status as cogroupoid objects was properly understood.

I shall describe these ideas in terms of examples of two main types. First are those which are particularly straightforward, and therefore illustrate the basic principles rather well to a general audience, and second are those which are relevant to the study of formal power series, and have close ties with algebraic topology. In some of these cases the structures in question are merely Hopf algebras; in others, the full power of algebroids is required. It would be exciting for topologists if combinatorial models could be constructed in these situations, and I shall outline a couple of situations where some success has been achieved in this direction.

Topics I hope to mention include partitions of finite sets, composition of formal power series, and groupoids as graphs.

The Incidence Algebra of a Composition Poset

Bruce E. Sagan

Michigan State University

A *composition* is just a sequence $w = k_1k_2 \dots k_r$ of positive integers. A number of partial orders on the set of all compositions have been studied recently. For example, Björner and Stanley have defined a poset on compositions which has many of the properties of Young's lattice for partitions. We consider a ordering that was first defined by Bergeron, Bousquet-Mélou, and Dulucq: Given $u = k_1 \dots k_r$ and $w = l_1 \dots l_s$ then we have $u \leq w$ if there is a subsequence $l_{i_1} \dots l_{i_r}$ of w which is componentwise bigger than u, i.e.,

$$k_j \leq l_{i_j} \quad \text{for } 1 \leq j \leq r. \tag{1}$$

Call this poset C. It is interesting, in part, because it is related to the poset of all permutations ordered by pattern containment.

In the first half of the talk, we will study the zeta function, ζ, of the incidence algebra $I(C)$. This is joint work with Anders Björner and full details can be found in the paper at

http://www.math.msu.edu/~sagan/Papers/rmf.pdf

If $w = k_1 \dots k_r$ satisfies $\sum_i k_i = N$, then w is said to be a *composition of* N and we write $|w| = N$. Let c_N be the number of compositions of N. It is well known (and easy to prove) that

$$\sum_{N \geq 0} c_N x^N = \frac{1-x}{1-2x} \tag{2}$$

which is a rational function of x. Now given $u \in C$, consider the generating function

$$Z(u) = \sum_{w \geq u} x^{|w|} = \sum_{w \in C} \zeta(u, w) x^{|w|}.$$

So equation 2 is just $Z(\epsilon)$, where ϵ is the empty composition. We show that $Z(u)$ is always a rational function by using techniques from the theory of formal languages. We also investigate similar generating function for powers of ζ. Surprisingly, to evaluate the sums, hypergeometric series identities are needed.

In the second half of the talk, we will study the Möbius function, $\mu = \zeta^{-1}$, in $I(C)$. This is joint work with Vincent Vatter and full details can be found in the paper at

http://www.math.msu.edu/~sagan/Papers/mfc.pdf

A set of indices $I = \{i_1, \dots, i_r\}$ such that 1 holds is called an *embedding* of u into w. We show that $\mu(u, w)$ gives a signed counting of certain embeddings of u into w. In fact, there are three proofs of this result: one combinatorial via an involution, one topological using discrete Morse theory, and one using the machinery discussed in the previous paragraph (this last being work with Björner). We will present the topological proof, giving an introduction to discrete Morse theory in the process.

The results above have analogues in the work Björner, some of it with Reutenauer, on subword order. We will show that both their results and ours are part of a more general framework. In particular, let P be any poset and consider the *Kleene closure* of all words over P:

$$P^* = \{w = k_1 k_2 \ldots k_r \ : \ k_i \in P \text{ for all } i \text{ and } r \geq 0\}.$$

Then 1 defines a partial order $u \leq w$ on elements of P^*, where the inequalities $k_j \leq l_{i_j}$ are taken in P. With little extra effort, one can prove theorems about P^* which specialize to those about composition order or about subword order by taking P to be a chain or an antichain, respectively.

We will end with a list of intriguing conjectures and open questions concerning these ideas. As another surprise, the Tchebyshev polynomials of the first kind enter into a conjectured formula for the Möbius function of certain intervals of P^* for a particular 3-element poset, P.

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Poset topology and permutation statistics

Michelle Wachs

University of Miami

Various connections between permutation statistics and poset topology have been explored in the literature over the past three decades originating with the work of Stanley. In this talk I will present a connection, recently discovered with John Shareshian. I will discuss how a study of the topology of a certain interesting class of posets has led to results and conjectures on a new q-analog of the Eulerian polynomials. These new q-Eulerian polynomials are the enumerators for the joint distribution of the excedance number and the major index. One of our conjectures is a formula for their q-exponential generating function, which is a nice q-analog of a well-known formula for the exponential generating function of the Eulerian polynomials. A more general version of this conjecture involves an intriguing new class of quasisymmetric functions and a representation of the symmetric group on the cohomology of the toric variety associated with the Coxeter complex of the symmetric group, studied by Procesi, Stanley, and Stembridge.

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