(−1)–ENUMERATION OF SELF–COMPLEMENTARY PLANE PARTITIONS

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Abstract. We prove a product formula for the remaining cases of the weighted enumeration of self–complementary plane partitions contained in a given box where adding one half of an orbit of cubes and removing the other half of the orbit changes the weight by −1. We use nonintersecting lattice path families to express this enumeration as a Pfaffian which can be expressed in terms of the known ordinary enumeration of self–complementary plane partitions.

1. Introduction

A plane partition P can be defined as a finite set of integer points (i,j,k) with $i,j,k>0$ and if $(i,j,k)\in P$ and $1\le i'\le i$, $1\le j'\le j$, $1\le k'\le k$ then $(i',j',k')\in P$. We interpret these points as midpoints of cubes and represent a plane partition by stacks of cubes (see Figure 1). If we have $i\le a$, $j\le b$ and $k\le c$ for all cubes of the plane partition, we say that the plane partition is contained in a box with sidelengths a,b,c.

Plane partitions were first introduced by MacMahon. One of his main results is the following [10, Art. 429, $x\to 1$, proof in Art. 494]:

The number of all plane partitions contained in a box with sidelengths a,b,c equals

$$B(a,b,c)=\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}=\prod_{i=1}^{a}\frac{(c+i)_b}{(i)_b}, \tag{1}$$

where $(a)_n:=a(a+1)(a+2)\dots(a+n-1)$ is the rising factorial.

MacMahon also started the investigation of the number of plane partitions with certain symmetries in a given box. These numbers can also be expressed as product formulas similar to the one given above. In [14], Stanley introduced additional complementation symmetries giving six new combinations of symmetries which led to more conjectures all of which were settled in the 1980's and 90's (see [14, 8, 3, 17]).

Many of these theorems come with q–analogs, that is, weighted versions that record the number of cubes or orbits of cubes by a power of q and give expressions containing q–rising factorials instead of rising factorials (see [1, 2, 11]). For plane partitions with complementation symmetry, it seems to be difficult to find natural q–analogs. However,

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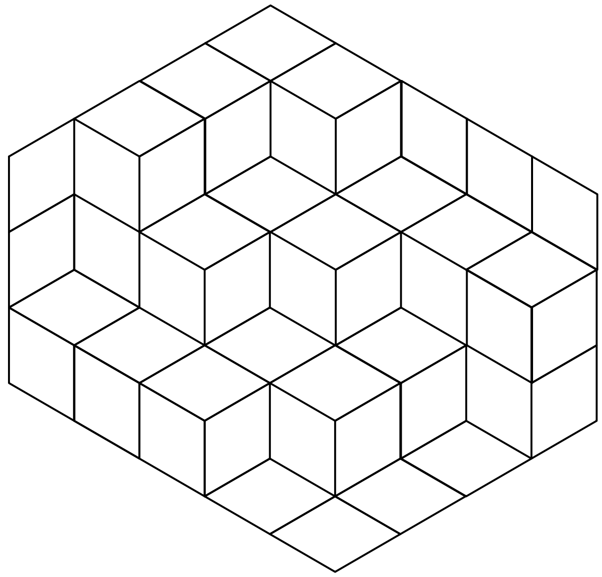

FIGURE 1. A self–complementary plane partition

in Stanley's paper a q–analog for self–complementary plane partitions is given (the weight is not symmetric in the three sidelengths, but the result is). Interestingly, upon setting $q = -1$ in the various q–analogs, one consistently obtains enumerations of other objects, usually with additional symmetry restraints. This observation, dubbed the "'(-1)–phenomenon" has been explained for many but not all cases by Stembridge (see [15] and [16]).

In [7], Kuperberg defines a (-1)–enumeration for all plane partitions with complementation symmetry which admits a nice closed product formula in almost all cases. These conjectures were solved in Kuperberg's own paper and in the paper [4] except for one case without a nice product formula and the case of self-complementary plane partitions in a box with some odd sidelengths which will be the main theorem of this paper. We start with the precise definitions.

A plane partition P contained in the box $a \times b \times c$ is called *self–complementary* if $(i, j, k) \in P \Leftrightarrow (a+1-i, b+1-j, c+1-k) \notin P$ for $1 \le i \le a$, $1 \le j \le b$, $1 \le k \le c$. This means that the box consists exactly of the plane partition and the image obtained from it by symmetry with respect to the central point of the box.

A convenient way to look at a self–complementary plane partition is the projection to the plane along the $(1, 1, 1)$–direction (see Figure 1). A plane partition contained in an $a \times b \times c$–box becomes a rhombus tiling of a hexagon with sidelengths a, b, c, a, b, c. It is easy to see that self-complementary plane partitions correspond exactly to those rhombus tilings with a 180° rotational symmetry.

The (-1)–weight is defined as follows: A self–complementary plane partition contains exactly one half of each orbit under the operation $(i, j, k) \mapsto (a+1-i, b+1-j, c+1-k)$. Let a move consist of removing one half of an orbit and adding the other half. Two plane partitions are connected either by an odd or by an even number of moves, so it is possible to define a relative sign. The sign becomes absolute if we assign weight 1 to the half-full plane partition (see Figure 2).

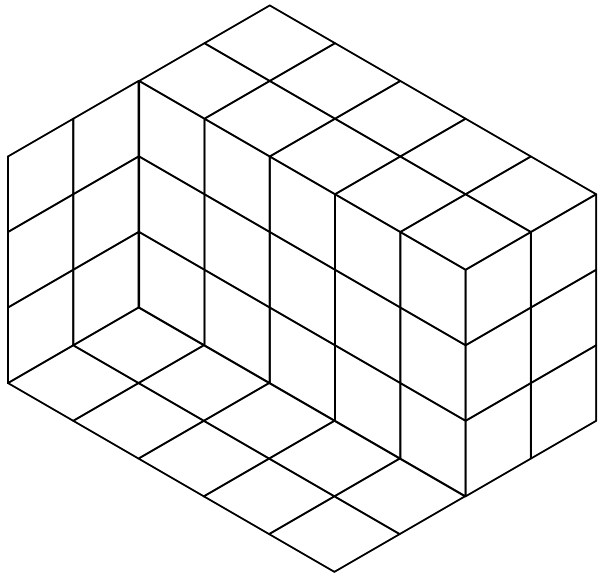

FIGURE 2. A plane partition of weight 1.

Therefore, this weight is $(-1)^{n(P)}$ where $n(P)$ is the number of cubes in the "left" half of the box and we want to evaluate $\sum_P (-1)^{n(P)}$. For example, the plane partition in Figure 1 has weight $(-1)^{10} = 1$.

In order to be able to state the result for the (-1)–enumeration more concisely, Stanley's result on the ordinary enumeration of self–complementary plane partitions is needed. It will also be used as a step in the proof of the (-1)–enumeration.

Theorem 1 (Stanley [14]). *The number $SC(a,b,c)$ of self–complementary plane partitions contained in a box with sidelengths a, b, c can be expressed in terms of $B(a,b,c)$ in the following way:*

$$\begin{array}{rl} B\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)^2 & \textit{for } a, b, c \textit{ even}, \\ B\left(\frac{a}{2}, \frac{b+1}{2}, \frac{c-1}{2}\right) B\left(\frac{a}{2}, \frac{b-1}{2}, \frac{c+1}{2}\right) & \textit{for } a \textit{ even and } b,\ c \textit{ odd}, \\ B\left(\frac{a+1}{2}, \frac{b}{2}, \frac{c}{2}\right) B\left(\frac{a-1}{2}, \frac{b}{2}, \frac{c}{2}\right) & \textit{for } a \textit{ odd and } b,\ c \textit{ even}, \end{array}$$

where $B(a,b,c) = \prod_{i=1}^{a} \frac{(c+i)_b}{(i)_b}$ is the number of all plane partitions in an $a \times b \times c$–box.

Note that a self-complementary plane partition contains exactly half of all cubes in the box. Therefore, there are no self-complementary plane partitions in a box with three odd sidelengths.

Now we can express the (-1)–enumeration of self–complementary plane partitions in terms of $SC(a,b,c)$, the ordinary enumeration of self–complementary plane partitions.

Theorem 2. *The enumeration of self–complementary plane partitions in a box with sidelengths a, b, c counted with weight $(-1)^{n(P)}$ equals*

$$\begin{aligned} B\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right) &\quad \text{for } a,b,c \text{ even,} \\ SC\left(\tfrac{a}{2},\tfrac{b+1}{2},\tfrac{c-1}{2}\right) SC\left(\tfrac{a}{2},\tfrac{b-1}{2},\tfrac{c+1}{2}\right) &\quad \text{for } a \text{ even and } b,\ c \text{ odd} \\ SC\left(\tfrac{a+1}{2},\tfrac{b}{2},\tfrac{c}{2}\right) SC\left(\tfrac{a-1}{2},\tfrac{b}{2},\tfrac{c}{2}\right) &\quad \text{for } a \text{ odd and } b,\ c \text{ even} \end{aligned}$$

where $SC(a, b, c)$ is given in Theorem 1 in terms of the numbers of plane partitions contained in a box and $n(P)$ is the number of cubes in the plane partition P that are not in the half-full plane partition (see Figure 2).

Remark. *Since the sides of the box play symmetric roles this covers all cases. (For three odd sidelengths there are no self-complementary plane partitions.) The case of three even sidelengths has already been proved in* [4].

Note that Theorem 1 and Theorem 2 are very analogous. In the even case, the (-1)–enumeration is the square root of the ordinary enumeration. In the other cases, it is still true that there are half as many linear factors in the (-1)–enumeration (viewed as a polynomial in c, say).

In Stanley's paper [14], the theorem actually gives a q–enumeration of plane partitions. The case $q = -1$ gives the same expression as Theorem 2 above if at least one side has odd length, but this does not give a proof of Theorem 2 because the weights of individual plane partitions are different. In the case of only even sidelengths, Stanley's theorem gives $SC\left(a/2, b/2, c/2\right)^2$ (analogously to Theorem 1) which does not equal the result $B\left(a/2, b/2, c/2\right)$ in Theorem 2.

Stanley's proof uses a special case of the Littlewood-Richardson rule, the expansion of the product of Schur functions $s_\lambda(x_1, \ldots, x_n) \cdot s_\mu(x_1, \ldots, x_n)$ where λ and μ are rectangular partitions whose sidelengths differ by at most one. It is possible to give an alternative proof of Theorem 2 using a modification of Stanley's weight leading to the expansion of $s_\lambda(x_1, \ldots, x_{n+1}) \cdot s_\mu(x_1, \ldots, x_n)$ where λ and μ are rectangular partitions. (Note the different numbers of variables.) See also the remark at the end of the paper.

2. Outline of the proof

For brevity, we just give the proof in the case a even, b, c odd and $(c - b)/2$ even.

Step 1: From plane partitions to families of nonintersecting lattice paths.

We use the projection to the plane along the $(1, 1, 1)$–direction and get immediately that self–complementary plane partitions contained in an $a \times b \times c$–box are equivalent to rhombus tilings of a hexagon with sides a, b, c, a, b, c invariant under $180°$–rotation. A tiling of this kind is clearly determined by one half of the hexagon.

Since the sidelengths a, b, c play a completely symmetric role and two of them must have the same parity we assume without loss of generality that $c - b$ is even and $b \leq c$. The result turns out to be symmetric in b and c, so we can drop the last condition in the statement of Theorem 2. Write x for the positive integer $(c - b)/2$ and divide the hexagon in half with a line parallel to the side of length a (see Figure 3). As shown in the same figure, we find a bijection between these tiled halves and families of nonintersecting lattice paths.

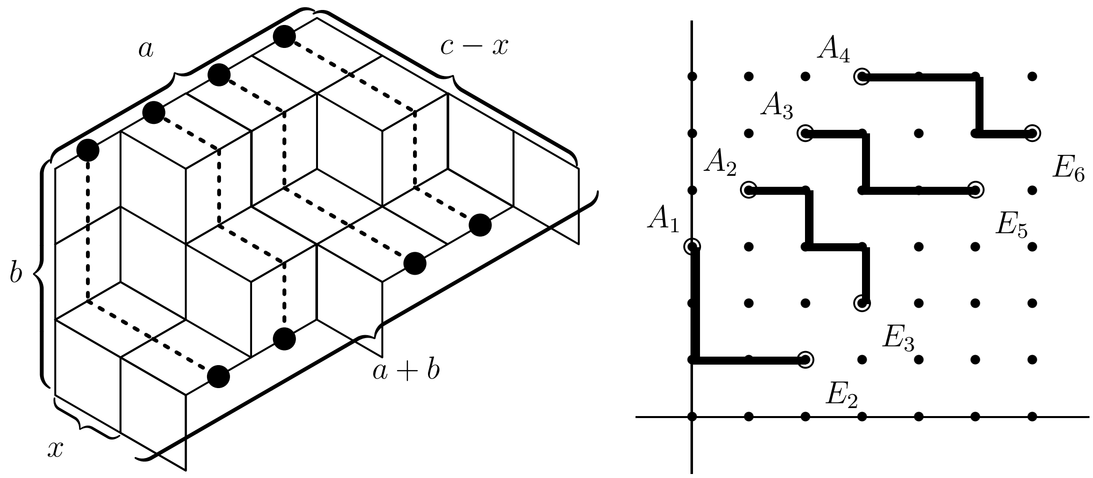

FIGURE 3. The paths for the self–complementary plane partition in Figure 1 and the orthogonal version. ($x = \frac{c-b}{2}$)

The starting points of the lattice paths are the midpoints of the edges on the side of length a. The end points are the midpoints of the edges parallel to a on the opposite boundary. This is a symmetric subset of the midpoints on the cutting line of length $a+b$.

The paths always follow the rhombi of the given tiling by connecting midpoints of parallel rhombus edges. It is easily seen that the resulting paths have no common points (i.e. they are nonintersecting) and the tiling can be recovered from a nonintersecting lattice path family with unit diagonal and down steps and appropriate starting and end points. Of course, the path families will have to be counted with the appropriate (-1)–weight.

After changing to an orthogonal coordinate system (see Figure 3), the paths are composed of unit South and East steps and the coordinates of the starting points are

$$A_i = (i-1, b+i-1) \quad \text{for } i = 1, \dots, a. \tag{2}$$

The end points are a points chosen symmetrically among

$$E_j = (x+j-1, j-1) \quad \text{for } j = 1, \dots, a+b. \tag{3}$$

Here, symmetrically means that if E_j is chosen, then $E_{a+b+1-j}$ must be chosen as well.

Note that the number $a+b$ of potential end points on the cutting line is always odd. Therefore, there is a middle one which is in no path family for even a (see Figure 3).

Now the (-1)–weight has to be defined for the paths. For a path from A_i to E_j we can use the weight $(-1)^{\text{area}(P)}$ where $\text{area}(P)$ is the area between the path and the x–axis and then multiply the weights of all the paths in the family. We have to check that the weight changes sign if we replace a half orbit with the complementary half orbit. If one of the affected cubes is completely inside the half shown in Figure 3, $\sum_P \text{area}(P)$ changes by one. If the two affected cubes are on the border of the figure, two symmetric endpoints, say E_j and $E_{a+b+1-j}$, are changed to E_{j+1} and E_{a+b-j} or vice versa. It is easily checked that in this case $\sum_P \text{area}(P)$ changes by $j + (a+b-j)$ which is odd.

It is straightforward to check that the weight for the "half-full" plane partition (see Figure 2) equals $(-1)^{a(a-2)/8}$ for a even, b, c odd. Therefore, we have to multiply the path enumeration by this global sign.

Step 2: From lattice paths to a sum of determinants

This weight can be expressed as a product of weights on individual steps (the exponent of (-1) is just the height of the step), so the following lemma is applicable. By the main theorem on nonintersecting lattice paths (see [9, Lemma 1] or [5, Theorem 1]) the weighted count of such families of paths can be expressed as a determinant.

Lemma 3. *Let $A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_n$ be integer points meeting the following condition: Any path from A_i to E_l has a common vertex with any path from A_j to E_k for any i, j, k, l with $i < j$ and $k < l$.*

Then we have

$$\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.}) = \det_{1 \le i,j \le n} (\mathcal{P}(A_i \to E_j)), \tag{4}$$

where $\mathcal{P}(A_i \to E_j)$ denotes the weighted enumeration of all paths running from A_i to E_j and $\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.})$ denotes the weighted enumeration of all families of nonintersecting lattice paths running from A_i to E_i for $i = 1, \dots, n$.

The condition on the starting and end points is fulfilled in our case because the points lie on diagonals, so we have to find an expression for $T_{ij} = \mathcal{P}(A_i \to E_j)$, the weighted enumeration of all single paths from A_i to E_j in our problem.

It is well-known that the enumeration of paths of this kind from (x, y) to (x', y') is given by the q–binomial coefficient $\begin{bmatrix} x'-x+y-y' \\ x'-x \end{bmatrix}_q$ if the weight of a path is q^e where e is the area between the path and a horizontal line through its endpoint.

The q–binomial coefficient (see [13, p. 26] for further information) can be defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=n-k+1}^{n} (1 - q^j)}{\prod_{j=1}^{k} (1 - q^j)}.$$

Although it is not obvious from this definition, the q–binomial coefficient is a polynomial in q. So it makes sense to put $q = -1$.

It is easy to verify that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1} = \begin{cases} 0 & n \text{ even, } k \text{ odd,} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{else.} \end{cases} \tag{5}$$

Taking also into account the area between the horizontal line through the endpoint and the x–axis, we obtain

$$T_{ij} = \mathcal{P}(A_i \to E_j) = (-1)^{(x+j-i)(j-1)} \begin{bmatrix} b + x \\ b + i - j \end{bmatrix}_{-1}.$$

Now we apply Lemma 3 to all possible sets of end points. Thus, the (-1)–enumeration can be expressed as a sum of determinants which are minors of the $a \times (a + b)$–matrix T:

Lemma 4. *The (-1)–enumeration can be written as*

$$(-1)^{a(a-2)/8} \sum_{1\le k_1<\cdots<k_{a/2}\le (a+b-1)/2} \det(T_{k_1},\dots,T_{k_{a/2}},T_{a+b+1-k_{a/2}},\dots,T_{a+b+1-k_1})$$

for a even and b,c odd,

where T_{ij} is $(-1)^{(x+j-i)(j-1)}\begin{bmatrix} b+x \\ b+i-j\end{bmatrix}_{-1}$ and T_j denotes the jth column of T which has length a.

Remark. *The same argument works for the ordinary enumeration, we just have to replace T_{ij} by the ordinary path enumeration $\binom{b+x}{b+i-j}$.*

Step 3: The sum of determinants is a single Pfaffian

Recall that the Pfaffian of a skew–symmetric $2n\times 2n$–matrix M is defined as

$$\operatorname{Pf} M = \sum_m \operatorname{sgn} m \prod_{\substack{\{i,j\}\in m\\ i<j}} M_{ij},$$

where the sum runs over all $m = \{\{m_1,m_2\},\{m_3,m_4\},\dots,\{m_{2n-1},m_{2n}\}\}$ with the conditions $\{m_1,\dots,m_{2n}\} = \{1,\dots,2n\}$, $m_{2k-1} < m_{2k}$ and $m_1 < m_3 < \cdots < m_{2n-1}$. The term $\operatorname{sgn} m$ is the sign of the permutation $m_1m_2m_3\dots m_{2n}$.

Specifically, $(\operatorname{Pf} M)^2 = \det M$.

Our sums of determinants can be simplified by a theorem of Ishikawa and Wakayama [6, Theorem 1(1)] which we use to express the sum as a Pfaffian. We use the specialization $A = \begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}$ on the version given in [12, Corollary 3.2] and obtain the following lemma.

Lemma 5. *Let S be a $2m\times 2n$–matrix with $m\le n$ and S^* be the matrix*

$$(S_1,\dots,S_n,S_{2n},\dots,S_{n+1})$$

where S_j denotes the jth column of S. Let A be the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}$. Then the following identity holds:

$$\sum_{1\le k_1<\cdots<k_m\le n} \det(S_{k_1},\dots,S_{k_m},S_{2n+1-k_m},\dots,S_{2n+1-k_1}) = \operatorname{Pf}(S^*A({}^tS^*))$$

$$= \operatorname{Pf}_{1\le i,j\le 2m}\left(\sum_{k=1}^{n}(S_{ik}S_{j,2n+1-k} - S_{jk}S_{i,2n+1-k})\right).$$

Now we apply this lemma to our sums.

Lemma 6. *The Pfaffian for the (-1)–enumeration for $b\le c$ is*

$$(-1)^{a(a-2)/8} \operatorname{Pf}_{1\le i,j\le a}\left(\sum_{k=1}^{\frac{a+b-1}{2}}(T_{ik}T_{j,a+b+1-k} - T_{jk}T_{i,a+b+1-k})\right),$$

for a even and b,c odd,

where $T_{ij} = (-1)^{(x-i)(j-1)} \begin{bmatrix} b+x \\ b+i-j \end{bmatrix}_{-1}$ *(and* $x = (c-b)/2$*).*

Proof. Apply the lemma with $2m = a$, $2n = a+b-1$ and

$$S = (T_1, \dots, T_{\frac{a+b-1}{2}}, T_{\frac{a+b+3}{2}}, \dots, T_{a+b})$$

to obtain

$$\operatorname{Pf}_{1\le i,j\le a}\left(\sum_{k=1}^{\frac{a+b-1}{2}} (T_{ik}T_{j,a+b+1-k} - T_{jk}T_{i,a+b+1-k})\right).$$

□

Lemma 7. *The Pfaffian for the ordinary enumeration* $SC(a,b,c)$ *for* $b \le c$ *is*

$$\operatorname{Pf}_{1\le i,j\le a}\left(\sum_{k=1}^{\lfloor\frac{a+b}{2}\rfloor}\left(\binom{b+x}{b+i-k}\binom{b+x}{j+k-a-1} - \binom{b+x}{b+j-k}\binom{b+x}{i+k-a-1}\right)\right)$$

for a *and* $c-b$ *even.*

Proof. Replace T_{ij} by the ordinary enumeration of the respective paths. This replaces (-1)–binomial coefficients by ordinary ones. (Doing the same thing for the analogous expressions in Section 9 of [4] gives the result for the case of even sidelengths.) □

Remark. *Of course, the closed form of this Pfaffian is known by Stanley's theorem (see Theorem 1). Therefore, we can use them to evaluate the Pfaffian for the* (-1)*–enumeration.*

Step 4: Evaluation of the Pfaffian

Now, the Pfaffian of Lemma 6 can be reduced to products of the known Pfaffians in Lemma 7 corresponding to the ordinary enumeration. The calculations have to be done separately for different parities of the parameters and we present only the case a, x even, b, c odd.

For M_{ij} in $\operatorname{Pf} M$ we can write

$$\sum_{k=1}^{(a+b-1)/2} (-1)^{(k+1)(i+j)}\left(\binom{(b+x-1)/2}{\lfloor(b+i-k)/2\rfloor}\binom{(b+x-1)/2}{\lfloor(j+k-a-1)/2\rfloor} - \binom{(b+x-1)/2}{\lfloor(b+j-k)/2\rfloor}\binom{(b+x-1)/2}{\lfloor(i+k-a-1)/2\rfloor}\right)$$

with $1 \le i, j \le a$.

Splitting the sum into terms $k = 2l$ and $k = 2l-1$ gives

$$\sum_{l=1}^{\lfloor (a+b-1)/4\rfloor} (-1)^{i+j}\left(\binom{(b+x-1)/2}{(b-1)/2+\lfloor (i-1)/2\rfloor-l+1}\binom{(b+x-1)/2}{\lfloor (j-1)/2\rfloor+l-a/2}\right.$$
$$\left.-\binom{(b+x-1)/2}{(b-1)/2+\lfloor (j-1)/2\rfloor-l+1}\binom{(b+x-1)/2}{\lfloor (i-1)/2\rfloor+l-a/2}\right)$$
$$+\sum_{l=1}^{\lceil (a+b-1)/4\rceil}\left(\binom{(b+x-1)/2}{(b-1)/2+\lfloor i/2\rfloor-l+1}\binom{(b+x-1)/2}{\lfloor j/2\rfloor+l-a/2-1}\right.$$
$$\left.-\binom{(b+x-1)/2}{(b-1)/2+\lfloor j/2\rfloor-l+1}\binom{(b+x-1)/2}{\lfloor i/2\rfloor+l-a/2-1}\right) \quad (6)$$

Now we apply some row and column operations to our matrix M. Start with $row(1)$, then write the differences $row(2i+1)-row(2i)$ for $i=1,\dots,a/2-1$, and finally $row(2i-1)+row(2i)$ for $i=1,\dots,a/2$. Now apply the same operations to the columns, so that the resulting matrix is still skew–symmetric. The new matrix has the same Pfaffian only up to sign $(-1)^{(a/2)(a/2-1)/2}$ which cancels with the global sign in Lemma 6.

Computation gives:

$$M_{2i+1,j}-M_{2i,j}=-\sum_{l=1}^{\lfloor (a+b-1)/4\rfloor}(-1)^{j}\left(\binom{(b+x-1)/2+1}{(b-1)/2+i-l+1}\binom{(b+x-1)/2}{\lfloor (j-1)/2\rfloor+l-a/2}\right.$$
$$\left.-\binom{(b+x-1)/2}{(b-1)/2+\lfloor (j-1)/2\rfloor-l+1}\binom{(b+x-1)/2+1}{i+l-a/2}\right)$$

Thus, apart from the first row and column, the left upper corner looks like

$$M_{2i+1,2j+1}-M_{2i,2j+1}-M_{2i+1,2j}+M_{2i,2j}$$
$$=\sum_{l=1}^{\lfloor (a+b-1)/4\rfloor}\left(\binom{(b+x-1)/2+1}{(b-1)/2+i-l+1}\binom{(b+x-1)/2+1}{j+l-a/2}\right.$$
$$\left.-\binom{(b+x-1)/2+1}{(b-1)/2+j-l+1}\binom{(b+x-1)/2+1}{i+l-a/2}\right), \quad (7)$$

where $i,j=1,\dots a/2-1$. Note how similar this is to the original matrix, only the (-1)–binomial coefficients are now replaced with ordinary binomial coefficients. The goal is to identify two blocks in the matrix which correspond to ordinary enumeration of self–complementary plane partitions.

The right upper corner is zero (of size $(a/2-1)\times a/2$).

Furthermore,

$$M_{2i-1,j}+M_{2i,j}=\sum_{l=1}^{\lceil (a+b-1)/4\rceil}\left(\binom{(b+x-1)/2+1}{(b-1)/2+i-l+1}\binom{(b+x-1)/2}{\lfloor j/2\rfloor+l-a/2-1}\right.$$
$$\left.-\binom{(b+x-1)/2}{(b-1)/2+\lfloor j/2\rfloor-l+1}\binom{(b+x-1)/2+1}{i+l-a/2-1}\right)$$

Therefore, we get for the right lower corner of the matrix

$$\begin{aligned} M_{2i-1,2j-1} &+ M_{2i,2j-1} + M_{2i-1,2j} + M_{2i,2j} \\ &= \sum_{l=1}^{\lceil (a+b-1)/4 \rceil} \left(\binom{(b+x-1)/2+1}{(b-1)/2+i-l+1} \binom{(b+x-1)/2+1}{j+l-a/2-1} \right. \\ &\qquad \left. - \binom{(b+x-1)/2+1}{(b-1)/2+j-l+1} \binom{(b+x-1)/2+1}{i+l-a/2-1} \right), \qquad (8) \end{aligned}$$

where $i, j = 1, \ldots, a/2$.

This is almost a block diagonal matrix, only the first row and column spoil the picture.

Example ($a = 8, b = 3, c = 7$):

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 5 & 0 & 0 & -1 & -5 \\ 0 & 0 & 3 & 12 & 0 & 0 & 0 & 0 \\ -1 & -3 & 0 & 9 & 0 & 0 & 0 & 0 \\ -5 & -12 & -9 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & -6 & -15 \\ 0 & 0 & 0 & 0 & 1 & 0 & -9 & -18 \\ 1 & 0 & 0 & 0 & 6 & 9 & 0 & -9 \\ 5 & 0 & 0 & 0 & 15 & 18 & 9 & 0 \end{array} \right)$$

If $\mathbf{(a/2)}$ **is even**, the right lower corner is an $(a/2) \times (a/2)$–matrix with non-zero determinant, as we will see later, thus, we can use the last $a/2$ rows to annihilate the second half of the first row. This potentially changes the entry 0 in position $(1, 1)$, but leaves everything else unchanged. We can use the same linear combination on the last $a/2$ columns to annihilate the second half of the first column. The resulting matrix is again skew–symmetric which means that the entry $(1, 1)$ has returned to the value 0. Since simultaneous row and column manipulations of this kind leave the Pfaffian unchanged, it remains to find out the Pfaffian of the right lower corner $(a/2 \times a/2)$ and the Pfaffian of the left upper corner $(a/2 \times a/2)$.

The right lower block is given by Equation (8). This corresponds exactly to the ordinary enumeration of self–complementary plane partitions in Lemma 7. Therefore, the Pfaffian of this block is $SC(a/2, (b+1)/2, (c+1)/2)$ (which is non-zero as claimed).

The left upper $a/2 \times a/2$ block (including the first row and column) is

$$\left(\begin{array}{c|c} 0 & M_{1,2j+1} - M_{1,2j} \\ \hline M_{2i+1,1} - M_{2i,1} & \displaystyle\sum_{l=1}^{\lfloor \frac{a+b-1}{4} \rfloor} \left(\tbinom{(b+x+1)/2}{(b-1)/2+i-l+1} \tbinom{(b+x+1)/2}{j+l-a/2} - \tbinom{(b+x+1)/2}{(b-1)/2+j-l+1} \tbinom{(b+x+1)/2}{i+l-a/2} \right) \end{array} \right),$$

where i, j run from 0 to $a/2 - 1$ and

$$M_{2i+1,1} - M_{2i,1} = \sum_{l=1}^{\lfloor (a+b-1)/4 \rfloor} \left(\binom{(b+x+1)/2}{(b-1)/2+i-l+1} \binom{(b+x-1)/2}{l-a/2} - \binom{(b+x-1)/2}{(b-1)/2-l+1} \binom{(b+x+1)/2}{i+l-a/2} \right)$$
$$M_{1,2j+1} - M_{1,2j} = \sum_{l=1}^{\lfloor (a+b-1)/4 \rfloor} \left(\binom{(b+x-1)/2}{(b-1)/2-l+1} \binom{(b+x+1)/2}{j+l-a/2} - \binom{(b+x+1)/2}{(b-1)/2+j-l+1} \binom{(b+x-1)/2}{l-a/2} \right).$$

Note that the exceptional row and column almost fit the general pattern. We just have sometimes $(b+x-1)/2$ instead of $(b+x+1)/2$. Replace $row(i)$ with $row(i)-row(i-1)$ for $i = 1, 2, \ldots, a/2-1$ in that order. Then do the same thing for the columns. In the resulting matrix all occurrences of $(b+x+1)/2$ have been replaced with $(b+x-1)/2$.

After shifting the indices by one, we get

$$\sum_{l=1}^{\lfloor \frac{a+b-1}{4} \rfloor} \left(\binom{(b+x-1)/2}{(b-1)/2+i-l} \binom{(b+x-1)/2}{j+l-a/2-1} - \binom{(b+x-1)/2}{(b-1)/2+j-l} \binom{(b+x-1)/2}{i+l-a/2-1} \right),$$

for $i, j = 1, \ldots, a/2$.

The Pfaffian of this matrix can easily be identified as $SC(\frac{a}{2}, \frac{b-1}{2}, \frac{c-1}{2})$ by Lemma 7. Using Theorem 1, we obtain for the (-1)–enumeration

$$SC(\tfrac{a}{2}, \tfrac{b+1}{2}, \tfrac{c+1}{2}) SC(\tfrac{a}{2}, \tfrac{b-1}{2}, \tfrac{c-1}{2}) = SC\left(\tfrac{a}{2}, \tfrac{b+1}{2}, \tfrac{c-1}{2}\right) SC\left(\tfrac{a}{2}, \tfrac{b-1}{2}, \tfrac{c+1}{2}\right),$$

which proves the main theorem in this case.

If $\boldsymbol{(a/2)}$ **is odd**, we move the first row and column to the $(a/2)$th place (which does not change the sign). Now we have an $(a-2)/2 \times (a-2)/2$–block matrix in the left upper corner which has non-zero determinant and thus can be used to annihilate the first half of the exceptional row and column similar to the previous case. By Equation (7) and Lemma 7 this is clearly $SC((a-2)/2, (b+1)/2, (c+1)/2)$.

For the right lower $(a+2)/2 \times (a+2)/2$–block, note that the relevant half of the exceptional column is

$$M_{2i-1,1} + M_{2i,1} = \sum_{l=1}^{\lceil (a+b-1)/4 \rceil} \left(\binom{(b+x+1)/2}{(b+1)/2+i-l} \binom{(b+x-1)/2}{l-a/2-1} - \binom{(b+x-1)/2}{(b+1)/2-l} \binom{(b+x+1)/2}{i+l-a/2-1} \right).$$

We use again row and column operations of the type $row(i)-row(i-1)$. This changes all occurrences of $(b+x+1)/2$ to $(b+x-1)/2$ and the extra row and column now fit the pattern in Equation (8) with $i, j = 0$. After shifting i, j to $i-1, j-1$, we identify this Pfaffian as $SC((a+2)/2, (b-1)/2, (c-1)/2)$. Again, by Theorem 1, the product of the two terms is exactly $SC(a/2, (b-1)/2, (c+1)/2) SC(a/2, (b+1)/2, (c-1)/2)$ as claimed in the theorem. $\square$

Remark. *In the Pfaffian expression for the various enumerations, it is possible to replace the occurrences of x with different variables and still get a nice factorisation. For example, the Pfaffian in Lemma 7 is changed to*

$$\operatorname{Pf}_{1\le i,j\le a}\left(\sum_{k=1}^{\lfloor\frac{a+b}{2}\rfloor}\left(\binom{b+y_1}{b+i-k}\binom{b+y_2}{j+k-a-1}-\binom{b+y_1}{b+j-k}\binom{b+y_2}{i+k-a-1}\right)\right).$$

The Pfaffian of this matrix is still a product of linear factors each involving only one of the two variables y_1 and y_2. Each of these groups of factors corresponds to one of the $B(\cdot,\cdot,\cdot)$–factors in Theorem 1.

This new matrix corresponds to self-complementary plane partitions with certain restrictions which leads to a modification of Stanley's proof giving an alternative proof of Theorem 2 using the expansion of $s_\lambda(x_1,\dots,x_{n+1})\cdot s_\mu(x_1,\dots,x_n)$ where λ and μ are rectangular partitions. (Note the different numbers of variables.)

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