SQUARE q, t-LATTICE PATHS AND $\nabla(p_n)$

NICHOLAS A. LOEHR AND GREGORY S. WARRINGTON

ABSTRACT. The *combinatorial q, t-Catalan numbers* are weighted sums of Dyck paths introduced by J. Haglund and studied extensively by Haglund, Haiman, Garsia, Loehr, and others. The q, t-Catalan numbers, besides having many subtle combinatorial properties, are intimately connected to symmetric functions, algebraic geometry, and Macdonald polynomials. In particular, the n'th q, t-Catalan number is the Hilbert series for the module of diagonal harmonic alternants in $2n$ variables; it is also the coefficient of s_{1^n} in the Schur expansion of $\nabla(e_n)$. Using q, t-analogues of labelled Dyck paths, Haglund et al. have proposed combinatorial conjectures for the monomial expansion of $\nabla(e_n)$ and the Hilbert series of the diagonal harmonics modules.

This article extends the combinatorial constructions of Haglund et al. to the case of lattice paths contained in squares. We define and study several q, t-analogues of these lattice paths, proving combinatorial facts that closely parallel corresponding results for the q, t-Catalan polynomials. We also conjecture an interpretation of our combinatorial polynomials in terms of the nabla operator. In particular, we conjecture combinatorial formulas for the monomial expansion of $\nabla(p_n)$, the "Hilbert series" $\langle \nabla(p_n), h_{1^n} \rangle$, and the sign character $\langle \nabla(p_n), s_{1^n} \rangle$.

1. INTRODUCTION

In 1996, A. Garsia and M. Haiman introduced a two-variable analogue of the Catalan numbers called the *q, t-Catalan numbers* [7]. Garsia and Haiman's definition of the q, t-Catalan, which arose from their study of Macdonald polynomials and diagonal harmonics, was quite complicated. Several years later, J. Haglund [8] conjectured an elementary combinatorial definition of the q, t-Catalan numbers as weighted sums of Dyck paths relative to two statistics called area and bounce. Shortly thereafter, Haiman proposed an equivalent combinatorial interpretation involving area and a third statistic called dinv. Garsia and Haglund eventually proved that the two combinatorial definitions were equivalent to the original definition of Garsia and Haiman [5, 6]. Haiman proved many of the conjectures relating the q, t-Catalan numbers to the representation theory of diagonal harmonics modules and the algebraic geometry of the Hilbert scheme [17, 18]. Meanwhile, various authors studied the subtle combinatorial properties of the combinatorial q, t-Catalan numbers and their generalizations [4, 9, 13, 14, 19, 20, 21, 22, 23, 24]. Surveys of different aspects of this research can be found in [15, 16, 19], and especially [11].

This article discusses a generalization of the combinatorial q, t-Catalan numbers in which Dyck paths are replaced by lattice paths inside squares. We develop the combinatorial theory of these "square q, t-lattice paths," which closely parallels the corresponding theory for the q, t-Catalan numbers. We also conjecture algebraic interpretations for our combinatorial generating functions in terms of the nabla operator introduced by F. Bergeron and Garsia [1, 2, 3]. In particular, we conjecture a combinatorial formula for the monomial expansion of $\nabla(p_n)$ that is quite similar to a formula for $\nabla(e_n)$ conjectured in [13].

To motivate and organize our work on lattice paths inside squares, we begin by quickly reviewing the combinatorial and algebraic results associated with the combinatorial q, t-Catalan numbers.

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The main body of the paper discusses the corresponding results and conjectures for our square q,t-lattice paths.

1.1. **Combinatorial Aspects of the q,t-Catalan Numbers.** This section reviews the essential definitions and combinatorial results involving the q,t-Catalan numbers. More details can be found in [11, 19] and in various papers listed in the bibliography.

(1) *Lattice Paths and Dyck Paths.* A *lattice path* in a $c \times d$ rectangle is a path from $(0,0)$ to (c,d) consisting of c east steps and d north steps of length 1. Such a path can be represented as a word $w = w_1 \cdots w_{c+d}$ with d zeroes (encoding north steps) and c ones (encoding east steps). Let $\mathcal{R}_{c,d}$ be the set of lattice paths from $(0,0)$ to (c,d). A *Dyck path of order n* is a lattice path in an $n \times n$ rectangle that never visits any point (x,y) with $y < x$. Let $\mathcal{D}_n$ be the set of Dyck paths of order n.

(2) *Statistics on Paths.* In addition to a classical area statistic on Dyck paths, there are two main statistics relevant to this paper: a dinv statistic introduced by Haiman and a bounce statistic introduced by Haglund. We will refer to these three statistics throughout this section, but omit their definitions as they arise as special cases of the corresponding statistics introduced in §2.1 for square q,t-lattice paths.

(3) *Combinatorial q,t-Catalan Numbers.* Haglund [8] defined the combinatorial q,t-Catalan numbers by the formula

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}.$$

There exists a bijection $\alpha : \mathcal{D}_n \to \mathcal{D}_n$ that maps the ordered pair of statistics $(\mathrm{area}, \mathrm{bounce})$ to $(\mathrm{dinv}, \mathrm{area})$. Therefore, we have

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)},$$

which is Haiman's formula for the combinatorial q,t-Catalan numbers.

(4) *Univariate and Joint Symmetry.* The existence of the bijection α implies that

$$\sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} = \sum_{D \in \mathcal{D}_n} q^{\mathrm{dinv}(D)} = \sum_{D \in \mathcal{D}_n} q^{\mathrm{bounce}(D)}$$

and that $C_n(q,1) = C_n(1,q)$. This fact is called *univariate symmetry* of the q,t-Catalan numbers. A stronger result called *joint symmetry* states that $C_n(q,t) = C_n(t,q)$. This result is a corollary of Garsia and Haglund's long proof linking $C_n(q,t)$ to the nabla operator [5, 6]; there is no known direct bijective proof of joint symmetry.

(5) *Recursion.* For $0 \le k \le n$, let $\mathcal{D}_{n,k}$ consist of all Dyck paths $D \in \mathcal{D}_n$ ending with exactly k east steps. It is equivalent to require that $h_0(D) = k$. Set

$$C_{n,k}(q,t) = \sum_{D \in \mathcal{D}_{n,k}} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}. \tag{1}$$

By considering the length $r = h_1(D)$ of $H_1(D)$ for paths $D \in \mathcal{D}_{n,k}$, Haglund [8] proved the recursion

$$C_{n,k}(q,t) = q^{k(k-1)/2} t^{n-k} \sum_{r=0}^{n-k} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q C_{n-k,r}(q,t) \text{ for } 1 \le k \le n \tag{2}$$

with initial conditions $C_{n,0}(q,t) = \chi(n=0)$ for all $n \ge 0$. Since $C_n(q,t) = t^{-n} C_{n+1,1}(q,t)$, this recursion uniquely determines the q,t-Catalan numbers.

(6) *Fermionic Formula.* By iterating the recursion, Haglund [8] derived an explicit "fermionic" formula for $C_n(q,t)$ as a sum over compositions of n:

$$C_n(q,t) = \sum_{w_0+\cdots+w_s=n,\ w_i>0} q^{\sum_i \binom{w_i}{2}} t^{\sum_i i w_i} \prod_{i=0}^{s-1} \begin{bmatrix} w_{i+1}+w_i-1 \\ w_{i+1}, w_i-1 \end{bmatrix}_q. \tag{3}$$

The summand indexed by $(w_0,\ldots,w_s)$ counts those Dyck paths D such that $v_i(D)=w_i$ for $0\leq i\leq s$.

(7) *Specialization at $t=1/q$.* Using the recursion for $C_{n,k}(q,t)$, Haglund [8] showed that

$$q^{\binom{n}{2}} C_{n,k}(q,1/q) = \frac{[k]_q}{[n]_q} \begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q q^{(k-1)n}.$$

Later, Loehr [19, 24] gave algebraic and bijective proofs of the equivalent formula

$$q^{\binom{n+1}{2}-nk} C_{n,k}(q,1/q) \quad = \quad \begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q - q^k \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q. \tag{4}$$

Using $C_n(q,t)=t^{-n}C_{n+1,1}(q,t)$ in these formulas, we obtain

$$q^{\binom{n}{2}} C_n(q,1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_q. \tag{5}$$

Garsia and Haiman proved (5) for their original definition of the q,t-Catalan numbers, using completely different methods [7].

(8) *Statistics for Labelled Paths.* The area and dinv statistics for Dyck paths extend naturally to *labelled Dyck paths*, but we omit their definitions. A *labelled Dyck path* of order n is a path $D\in\mathcal{D}_n$ in which each vertical step is assigned a label between 1 and n. We require that the labels of vertical steps in the same column strictly increase from bottom to top. Let $\mathcal{P}_n$ denote the set of all such objects with distinct labels; let $\mathcal{Q}_n$ denote the set of all such objects where labels may be repeated (subject to the increasing-column condition). We can represent a labelled Dyck path $Q\in\mathcal{Q}_n$ by a pair of vectors $(g(Q),r(Q))$, where $g(Q)=(g_0(Q),\ldots,g_{n-1}(Q))$ is the area vector of the path (ignoring labels), and $r(Q)=(r_0(Q),\ldots,r_{n-1}(Q))$ is the sequence of labels in Q from bottom to top. We call $r(Q)$ the *label vector* of Q. Define the *content function* for Q by letting $c_Q(j)$ be the number of j's in the label vector $r(Q)$.

area and dinv are easily extended to labelled Dyck paths. Haglund and Loehr [14] studied the generating function

$$H_n(q,t) = \sum_{P\in\mathcal{P}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)},$$

obtaining a fermionic formula and other results. Later, Loehr [21] defined a third statistic pmaj on $\mathcal{P}_n$, which generalizes Haglund's bounce statistic to labelled paths. The pmaj statistic was used to derive other results about $H_n(q,t)$, such as univariate symmetry, a recursion for $H_n(q,t)$, and the specialization $q^{n(n-1)/2}H_n(q,1/q)=[n+1]_q^{n-1}$. The larger collection of objects $\mathcal{Q}_n$ was first introduced in [13]; its significance is discussed below.

(9) *Combinatorial Extensions.* Haglund's basic idea of studying area, bounce, and dinv statistics on Dyck paths has many fruitful combinatorial generalizations. Besides the labelled Dyck paths just mentioned, one can introduce statistics for Schröder paths, lattice paths inside triangles of different shapes, lattice paths inside trapezoids, labelled versions of these paths, etc. These generalizations share many important properties, like univariate symmetry, joint symmetry (often conjectural), and nice specializations when $t=1/q$. Most of the generalizations also have conjectured algebraic interpretations involving Macdonald

polynomials. We will not enter into any details here but merely refer the reader to the papers in the bibliography. Of course, the goal of the present paper is to introduce yet another extension of the basic combinatorial setup to lattice paths inside squares.

1.2. **Algebraic Aspects of the q,t-Catalan Numbers.** This section reviews the principal theorems and conjectures connecting Haglund's combinatorial q,t-Catalan numbers (and their extensions) to the theory of Macdonald polynomials, diagonal harmonics modules, and symmetric functions. We assume the reader is familiar with the basic definitions and results in symmetric function theory and representation theory; see [25, 26] for more details. We begin by recalling some necessary notation and definitions.

(1) *Partitions.* Let $\mu = (\mu_1 \geq \cdots \geq \mu_s > 0)$ be a partition. Define $|\mu| = \mu_1 + \cdots + \mu_s$, $\mu \vdash n$ iff $|\mu| = n$, $\ell(\mu) = s$, $n(\mu) = \sum_{i=1}^{s}(i-1)\mu_i$, $\mu_i = 0$ for all $i > s$, and write μ' for the transpose of μ. If $\lambda, \mu \vdash n$, write $\lambda \geq \mu$ iff λ dominates μ iff $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. We draw the Ferrers diagram D of μ in the first quadrant of the xy-plane, left-justified, with the longest row appearing at the bottom. With this convention, for each cell $c \in D$ we define the *arm*, *coarm*, *leg*, and *coleg* of c (denoted $a(c)$, $a'(c)$, $l(c)$, and $l'(c)$) to be the number of cells strictly east, strictly west, strictly north, and strictly south of c in D. Also define the following elements in the polynomial ring $\mathbb{Q}[q,t]$:

$$\begin{aligned}
M &= (1-q)(1-t) \\
B_\mu &= \sum_{c\in\mu} q^{a'(c)}t^{l'(c)} \\
\Pi_\mu &= \prod_{\substack{c\in\mu \\ (a'(c),l'(c))\neq(0,0)}} (1-q^{a'(c)}t^{l'(c)}) \\
T_\mu &= q^{\sum_{c\in\mu} a(c)}t^{\sum_{c\in\mu} l(c)} = q^{n(\mu')}t^{n(\mu)} \\
w_\mu &= \prod_{c\in\mu}\left[(q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1}).\right]
\end{aligned}$$

For example, if $\mu = (3,2)$, then $|\mu| = 5$, $\ell(\mu) = 2$, $n(\mu) = 2$, $\mu' = (2,2,1)$, $n(\mu') = 4$, $B_\mu = 1 + q + q^2 + t + qt$, $\Pi_\mu = (1-q)(1-q^2)(1-t)(1-qt)$, $T_\mu = q^4t^2$, and

$$w_\mu = (q^2-t^2)(q-t^2)(1-t)(q-t)(1-t)(t-q^3)(t-q^2)(1-q)(1-q^2)(1-q).$$

(2) *Symmetric Functions.* Let F be the field $\mathbb{Q}(q,t)$. Let Λ_F^n denote the set of symmetric functions homogeneous of degree n in the variables $z_1, \ldots, z_N$ (where $N \geq n$) with coefficients in F. Λ_F^n is an F-vector space whose bases are indexed by partitions of n. We will use the five classical bases for Λ_F^n [25, 26]: the *monomial basis* $\{m_\mu : \mu \vdash n\}$, the *homogeneous basis* $\{h_\mu : \mu \vdash n\}$, the *elementary basis* $\{e_\mu : \mu \vdash n\}$, the *power-sum basis* $\{p_\mu : \mu \vdash n\}$, and the *Schur basis* $\{s_\mu : \mu \vdash n\}$. The *Hall scalar product* is defined on Λ_F^n by requiring that the Schur basis be orthonormal. Then the power-sum basis is orthogonal relative to this scalar product, and the monomial basis is dual to the homogeneous basis.

(3) *Macdonald Polynomials.* Besides the five classical bases, there are also five "Macdonald-type" bases for Λ_F^n [25, 16, 15]: the *Macdonald polynomials* $\{P_\mu : \mu \vdash n\}$, the *dual Macdonald polynomials* $\{Q_\mu : \mu \vdash n\}$, the *integral Macdonald polynomials* $\{J_\mu : \mu \vdash n\}$, the *transformed integral Macdonald polynomials* $\{H_\mu : \mu \vdash n\}$, and the *modified Macdonald polynomials* $\{\tilde{H}_\mu : \mu \vdash n\}$. We will only use the modified Macdonald polynomials, which can be defined quickly as follows. Let ϕ_q be the unique F-linear map on Λ_F^n defined on the basis $\{p_\mu\}$ by $\phi_q(p_\mu) = (\prod_{i=1}^{\ell(\mu)}[1-q^{\mu_i}])p_\mu$. Similarly, define a linear map ϕ_t by requiring that $\phi_t(p_\mu) =$

$(\prod_{i=1}^{\ell(\mu)}[1-t^{\mu_i}])p_\mu$. Then there exists a unique basis $\{\tilde{H}_\mu : \mu \vdash n\}$ of Λ_F^n characterized by the axioms:

(1) $\phi_q(\tilde{H}_\mu) = \sum_{\lambda \geq \mu} a_{\lambda,\mu} s_\lambda$ for some $a_{\lambda,\mu} \in F$

(2) $\phi_t(\tilde{H}_\mu) = \sum_{\lambda \geq \mu'} b_{\lambda,\mu} s_\lambda$ for some $b_{\lambda,\mu} \in F$

(3) $\langle \tilde{H}_\mu, s_n \rangle = 1$.

Haglund recently conjectured an explicit combinatorial formula for $\tilde{H}_\mu$ [10], and this conjecture was proved by Haglund, Haiman, and Loehr by verifying the three axioms [12].

(4) *Nabla Operator.* The nabla operator, introduced by F. Bergeron and Garsia [1, 2, 3], is the unique F-linear map on Λ_F^n defined on the basis $\{\tilde{H}_\mu\}$ by $\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu$.

(5) *Diagonal Harmonics.* For $n \geq 1$, define $R_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Define the diagonal action of the symmetric group S_n on R_n by $\pi \cdot x_i = x_{\pi(i)}$ and $\pi \cdot y_i = y_{\pi(i)}$ for $\pi \in S_n$. The S_n-module R_n is doubly graded by total degree in the x-variables and total degree in the y-variables. Define the module of *diagonal harmonics* to be

$$DH_n = \left\{ f \in R_n : \sum_{i=1}^{n} \partial x_i^h \partial y_i^k f = 0 \text{ for all } h,k \text{ with } h+k \geq 1 \right\}.$$

Define the module of *diagonal harmonic alternants* to be

$$DHA_n = \{f \in DH_n : \pi \cdot f = \mathrm{sgn}(\pi) f \text{ for all } \pi \in S_n\}.$$

Both DH_n and DHA_n are bihomogeneous submodules of R_n. For any bihomogeneous submodule V of R_n, let $V^{h,k}$ denote the elements of V that are homogenous of degree h in the x-variables and homogeneous of degree k in the y-variables. The *Hilbert series of* V is defined to be

$$\mathrm{Hilb}(V) = \sum_{h,k \geq 0} q^h t^k \dim(V^{h,k}).$$

Writing each $V^{h,k}$ as a direct sum of irreducible S_n-submodules and replacing each such submodule by the associated Schur function, we obtain the *Frobenius series of* V, denoted $\mathrm{Frob}(V)$, which is an element of Λ_F^n. In symbols,

$$\mathrm{Frob}(V) = \sum_{h,k \geq 0} q^h t^k \mathrm{Frob}(V^{h,k}),$$

where $\mathrm{Frob}(V^{h,k})$ is the image of the character of $V^{h,k}$ under the classical Frobenius map.

We can now state some of the theorems and conjectures giving the algebraic significance of the q,t-Catalan numbers and their extensions.

(1) *Master Theorem for the q,t-Catalan Numbers:* For all $n \geq 1$, the following five elements of $\mathbb{Q}(q,t)$ are all equal (and are, therefore, elements of $\mathbb{N}[q,t]$):

(a) $\sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}$ (Haglund's combinatorial formula)

(b) $\sum_{D \in \mathcal{D}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}$ (Haiman's combinatorial formula)

(c) $\langle \nabla(e_n), s_{1^n} \rangle$ (nabla formula)

(d) $\sum_{\mu \vdash n} T_\mu^2 M B_\mu \Pi_\mu / w_\mu$ (Garsia-Haiman's rational-function formula)

(e) $\mathrm{Hilb}(DHA_n)$ (representation-theoretical formula)

The equality of (a) and (b) follows from the bijection ϕ mentioned earlier. Formula (d) arises from the expansion of e_n in terms of the basis $\{\tilde{H}_\mu\}$, namely

$$e_n = \sum_{\mu \vdash n} (M B_\mu \Pi_\mu / w_\mu) \tilde{H}_\mu \tag{6}$$

(Theorem 2.4 in [7]). Applying the definition of ∇ and the fact that $\langle \tilde{H}_\mu, s_{1^n} \rangle = T_\mu$, it follows that (c) equals (d). The equality of (d) and (e) follows from a difficult theorem of Mark Haiman giving a complete character formula for DH_n [17, 18]. The equality of (a) and (d) is also a hard result, due to Garsia and Haglund, whose proof uses intricate symmetric function identities and plethystic machinery [5, 6].

(2) *Hilbert Series Conjecture:* For all $n \geq 1$, the following six elements of $\mathbb{Q}(q,t)$ are all equal (and are, therefore, elements of $\mathbb{N}[q,t]$):
   (a) $\sum_{P \in \mathcal{P}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}$ (first combinatorial formula)
   (b) $\sum_{P \in \mathcal{P}_n} q^{\mathrm{area}(P)} t^{\mathrm{pmaj}(P)}$ (second combinatorial formula)
   (c) $\langle \nabla(e_n), h_{1^n} \rangle$ (nabla formula)
   (d) $\sum_{\mu \vdash n} \langle \tilde{H}_\mu, h_{1^n} \rangle T_\mu M B_\mu \Pi_\mu / w_\mu$ (first rational-function formula)
   (e) $\sum_{\mu \vdash n+1} M \Pi_\mu B_\mu^{n+1} / w_\mu$ (second rational-function formula)
   (f) $\mathrm{Hilb}(DH_n)$ (representation-theoretical formula)

   At the time of this writing, the following equalities have been proved: (a)=(b) holds by a bijection in [19, 21]; (c)=(d) follows easily from (6); (c)=(e) was proved by Haglund in [9]; (d)=(f) follows from Haiman's results on the character of DH_n [17, 18]. The open conjecture states that the combinatorial formulas (a) and (b) equal the algebraic formulas (c) through (f).

(3) *Shuffle Conjecture* [13]*:* For all $n \geq 1$, the following five elements of Λ_F^n are all equal (and are, therefore, Schur-positive):
   (a) $\sum_{Q \in \mathcal{Q}_n} q^{\mathrm{dinv}(Q)} t^{\mathrm{area}(Q)} z_1^{c_Q(1)} \cdots z_n^{c_Q(n)}$ (combinatorial formula)
   (b) $\nabla(e_n)$ (nabla formula)
   (c) $\sum_{\mu \vdash n} \tilde{H}_\mu T_\mu M B_\mu \Pi_\mu / w_\mu$ (Macdonald polynomial formula)
   (d) $\sum_{\lambda \vdash n} s_\lambda \left(\sum_{\mu \vdash n+1} M \Pi_\mu B_\mu s_{\lambda'}[B_\mu] / w_\mu \right)$ (rational-function Schur expansion)
   (e) $\mathrm{Frob}(DH_n)$ (representation-theoretical formula)

   Here, $s_{\lambda'}[B_\mu]$ denotes the value of the Schur function $s_{\lambda'}(z_1, \ldots, z_n)$ at $z_i = q^{a'(c_i)} t^{l'(c_i)}$, where $c_1, \ldots, c_n$ are the cells of μ in any order. It is known that (b)=(c) by (6), (b)=(d) by a result of Haglund [9], and (c)=(e) by results of Haiman [17, 18]. The conjecture states that these four algebraic quantities are given by the combinatorial formula (a). If the conjecture is true, then (a) gives the monomial expansion of $\nabla(e_n)$. It has been proved that (a) is a symmetric function in the variables $z_1, \ldots, z_n$ [13].

(4) *Extensions of these Conjectures.* Many of the algebraic conjectures and results extend to the more general combinatorial objects mentioned earlier. For example, the generating functions for q,t-Schröder paths are related to the polynomials $\langle \nabla(e_n), e_d h_{n-d} \rangle$ [4, 9]. Generating functions for unlabelled and labelled m-Dyck paths (paths from $(0,0)$ to (mn, n) never going below the line $x = my$) are conjectured to give information about the sign character and monomial expansion of $\nabla^m(e_n)$ [13, 19, 22, 23]. We again refer the reader to [11] and the papers in the bibliography for more information.

2. Combinatorics of Square q,t-Lattice Paths

2.1. Statistics for Square Paths. A *square lattice path of order* n is a lattice path in an $n \times n$ square. Let $\mathcal{SQ}_n$ denote the set of square lattice paths of order n. We now define three statistics on paths in $\mathcal{SQ}_n$ generalizing the area, dinv, and bounce statistics defined in §1.1.

(1) *Square* area *Statistic.* Let $S \in \mathcal{SQ}_n$. Set $\ell = \ell(S)$ to be the minimum possible value such that S stays weakly above the line $y = x - \ell$. We call ℓ the *deviation* of the path S. Since S begins at the origin and ends at (n,n), we see that $0 \leq \ell \leq n$. Define the *area vector* $g(S) = (g_0(S), \ldots, g_{n-1}(S))$ by requiring that $g_i(S) + n - i$ be the number of complete boxes in the i'th row from the bottom between S and the line $x = n$. Note that the entries of this vector can be negative, but that this area vector reduces to the area vector in §1.1 when

S is a Dyck path. We define $\text{area}(S) = \sum_{i=0}^{n-1}(\ell + g_i(S))$. This can be interpreted as the number of complete boxes to the right of S and to the left of the line $y = x - \ell$.

(2) *Square* dinv *Statistic.* Suppose $S \in \mathcal{SQ}_n$ has $(g_0(S), \ldots, g_{n-1}(S))$ as its area vector. Define

$$\text{dinv}(S) = \sum_{i<j} \chi(g_i(S) - g_j(S) \in \{0,1\}) + \sum_i \chi(g_i(S) < -1).$$

If S is a Dyck path, then the condition $g_i(S) < -1$ never holds, and this formula for $\text{dinv}(S)$ reduces to the formula given in §1.1.

(3) *Square* bounce *Statistic.* Let $S \in \mathcal{SQ}_n$ have deviation ℓ. The *break point* of S, $(\ell_x(S), \ell_y(S))$, is the leftmost point along the path S lying on the line $y = x - \ell$.

We now proceed to define a bounce path $\text{bpath}(S)$ in analogy with the bounce paths defined for Dyck paths. The bounce path for S consists of two pieces: a *positive part* located northeast of the break point, and a *negative part* located southwest of the break point. First consider the positive part. A ball starts at (n,n) and makes an initial vertical move V_{-1} of length $v_{-1} = \ell$ ending at $(n, n-\ell)$. The ball then makes alternating horizontal and vertical moves $H_0, V_0, H_1, V_1, \ldots, H_s, V_s$ until it reaches the break point. We let h_i and v_i denote the length of the moves H_i and V_i, respectively. We determine h_i and v_i for each $i \geq 0$ as follows. First, the ball moves west h_i units until it is blocked by the north step of S ending at the horizontal level occupied by the ball. Second, the ball moves south $v_i = h_i$ units to return to the line $y = x - \ell$. As before, the steps that block the ball's westward motion are called *blocking north steps.*

The negative part of the bounce path traces the motion of a second bouncing ball that starts at the origin and moves northeast towards the break point. This ball makes an initial horizontal move H_{-1} of length $h_{-1} = \ell$ from $(0,0)$ to $(\ell, 0)$. It then makes alternating vertical and horizontal moves $V_{-2}, H_{-2}, V_{-3}, H_{-3}, \ldots, V_u, H_u$ until it reaches the break point. For each $i < -1$, the ball moves north v_i units until it is blocked by the east step of S *ending* at the vertical line occupied by the ball. (Note that this is *not* just a reflected version of the bounce algorithm in the positive part.) The ball then moves east $h_i = v_i$ units to return to the line $y = x - \ell$. The east steps that block the ball's northward motion are called *blocking east steps.*

Finally, we define the *bounce statistic* for any path $S \in \mathcal{SQ}_n$. Let $V_u, \ldots, V_s$ be the nonzero vertical moves in $\text{bpath}(S)$, where $u \leq 0 \leq s$. Set $\text{bounce}(S) = \sum_{i=u}^{s}(i-u)v_i$. Also set $\text{bmin}(S) = u$ and $\text{bmax}(S) = s$.

For a Dyck path D, the deviation ℓ is 0, the break point is $(0,0)$, the positive part of the bounce path coincides with the bounce path described in §1.1, and the negative part of the bounce path is empty. In this case, we have $\text{bmin}(D) = 0$ (we ignore the zero moves V_{-1} and H_{-1}), and the bounce statistic just defined reduces to the formula used in §1.1.

For example, Figure 1.1 illustrates a path $S \in \mathcal{SQ}_{15}$ and its bounce path. For this path, $\ell(S) = 3$, the break point is $(8,5)$,

$$g(S) = (0,-1,-2,-1,-1,-3,-2,-2,-2,-3,-2,-1,-1,0,1),$$

$\text{area}(S) = 25$, $\text{dinv}(S) = 52$, $\text{bmin}(S) = -4$, $\text{bmax}(S) = 2$, $(v_{-4}, \ldots, v_2) = (h_{-4}, \ldots, h_2) = (1,2,2,3,3,2,2)$, and $\text{bounce}(S) = 49$.

2.2. **Comparison of the Statistics.**

Theorem 1. *There exists a bijection $\phi : \mathcal{SQ}_n \to \mathcal{SQ}_n$ such that* $\text{area}(\phi(S)) = \text{dinv}(S)$ *and* $\text{bounce}(\phi(S)) = \text{area}(S)$. *The deviation of $\phi(S)$ is the number of -1's in $g(S)$; the break point of $\phi(S)$ is*

$$(\ell_x(\phi(S)), \ell_y(\phi(S))) = (|\{j : g_j(S) < 0\}|, |\{j : g_j(S) < -1\}|);$$

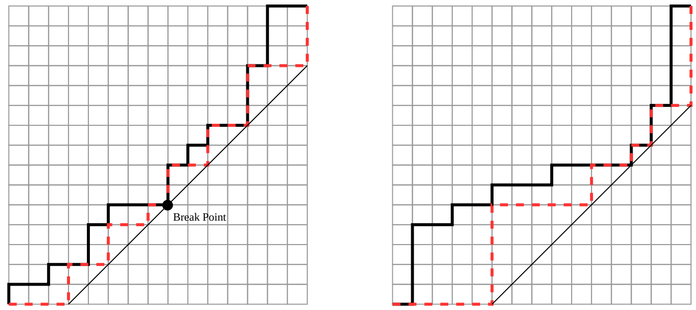

FIGURE 1. The lefthand picture is a path S in $\mathcal{SQ}_{15}$ (solid path) along with bpath(S) (dotted path). The righthand picture is the image of S under ϕ.

$\mathrm{bmin}(\phi(S)) = \min_j g_j(S)$*; and* $\mathrm{bmax}(\phi(S)) = \max_j g_j(S)$*. Moreover,* $\phi(S)$ *ends with an east step iff* S *begins with a north step, and* $\phi|_{\mathcal{D}_n} : \mathcal{D}_n \to \mathcal{D}_n$ *is the inverse of the bijection* α *from* §*1.1.*

For example, let S be the path from Figure 1.1. Figure 1.2 shows $\phi(S)$ along with its bounce path. Note that $g(S)$ has two -3's, five -2's, five -1's, two 0's, and one 1, so that

$$(v_{-3}(\phi(S)), \ldots, v_1(\phi(S))) = (2, 5, 5, 2, 1).$$

Furthermore, $\mathrm{area}(S) = 25 = \mathrm{bounce}(\phi(S))$ and $\mathrm{dinv}(S) = 52 = \mathrm{area}(\phi(S))$.

2.3. **Symmetry Properties.** For all $n \geq 1$, define

$$S_n(q,t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)} = S_n(q,t) = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.$$

(The second equality follows from the bijection ϕ.) Letting $q = 1$ or $t = 1$ here, we obtain the following *univariate symmetry* properties.

Corollary 2.

$$\sum_{S \in \mathcal{SQ}_n} q^{\mathrm{dinv}(S)} = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{bounce}(S)} = \sum_{S \in \mathcal{SQ}_n} q^{\mathrm{area}(S)}.$$

Conjecture 3. The *joint symmetry* property $S_n(q,t) = S_n(t,q)$ holds for all n.

This conjecture has been confirmed by computer for $1 \leq n \leq 11$.

Computing $S_n(q,t)$ for small values of n, one sees that the polynomial $S_n(q,t)$ is always divisible by 2. Our next goal is to explain this property. Define $\mathcal{SQ}_n^N$, $\mathcal{SQ}_n^E$, ${}^N\mathcal{SQ}_n$, and ${}^E\mathcal{SQ}_n$ to be the paths in $\mathcal{SQ}_n$ that end with a north step, end with an east step, begin with a north step, and begin with an east step, respectively. Set

$$S_n^\gamma(q,t) = \sum_{S \in \mathcal{SQ}_n^\gamma} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)}; \qquad {}^\gamma S_n(q,t) = \sum_{S \in {}^\gamma\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}.$$

for $\gamma \in \{E, N\}$. We will show that $S_n^N(q,t) = S_n^E(q,t) = S_n(q,t)/2$. Since ϕ sends paths with initial north steps to paths with terminal east steps and vice versa, it also follows that ${}^N S_n(q,t) = {}^E S_n(q,t) = S_n(q,t)/2$. We call these identities *pair-symmetries*.

To prove the pair-symmetries, it suffices to construct a bijection $\psi : \mathcal{SQ}_n^E \to \mathcal{SQ}_n^N$ preserving area and bounce. We begin by introducing a cyclic shift map $\mathrm{cyc} : \mathcal{SQ}_n \to \mathcal{SQ}_n$. Let $S \in \mathcal{SQ}_n$ be encoded by the word $w_1 w_2 \cdots w_{2n} \in \{0,1\}^{2n}$. Define cyc$(S)$ to be the path encoded by the word $w_{2n} w_1 w_2 \cdots w_{2n-1}$.

Lemma 4. *For* $S \in \mathcal{SQ}_n$*,* $\mathrm{area}(S) = \mathrm{area}(\mathrm{cyc}^i(S))$ *for all integers* i*.*

Theorem 5. *There is a bijection* $\psi : \mathcal{SQ}_n^E \to \mathcal{SQ}_n^N$ *preserving* area *and* bounce. *Consequently,*

$$S_n^N(q,t) = S_n^E(q,t) = S_n(q,t)/2 = {}^N S_n(q,t) = {}^E S_n(q,t).$$

We close this section with an alternate formula for ${}^E S_n(q,t) = S_n(q,t)/2$.

Theorem 6. *Let* $\operatorname{dinv}_0(S) = \sum_{i<j} \chi(g_i(S) - g_j(S) \in \{0,1\}) + \sum_i \chi(g_i(S) < 0)$. *Then*

$${}^E S_n(q,t) = \sum_{S \in {}^E\mathcal{SQ}_n} q^{\operatorname{dinv}(S)} t^{\operatorname{area}(S)} = \sum_{U \in \mathcal{SQ}_n^E} q^{\operatorname{dinv}_0(U)} t^{\operatorname{area}(U)}.$$

2.4. **Recursion for Square Paths.** In §1.1, we saw that the generating functions $C_{n,k}(q,t)$ for Dyck paths of order n with $h_0 = k$ satisfied the recursion (2). Now we prove a similar recursion for square q,t-lattice paths. The idea is to remove the "earliest" bounce in a square q,t-lattice path, namely the negative bounce arriving at the break point.

Formally, for $n > 0$ and $1 \le k \le n$, we set

$$R_{n,k}(q,t) = \sum_{S \in \mathcal{SQ}_n} q^{\operatorname{area}(S)} t^{\operatorname{bounce}(S)} \chi(h_{\operatorname{bmin}(S)} = k,\ \ell(S) > 0).$$

The condition $\ell(S) > 0$ means that S is not a Dyck path, while $h_{\operatorname{bmin}(S)} = k$ means that the last horizontal move in the negative part of the bounce path (arriving at the break point) has length k. To take care of the Dyck paths in $\mathcal{SQ}_n$, we define $R_{n,0}(q,t) = C_n(q,t) = t^{-n} C_{n+1,1}(q,t)$ for $n \ge 0$. For $k = n \ge 0$, we have $R_{n,n}(q,t) = q^{\binom{n}{2}}$ since the only path that contributes is the one that goes east n steps and then north n steps. Clearly, $S_{n,k}(q,t) = \sum_{k=0}^n R_{n,k}(q,t)$.

Theorem 7. *For* $0 < k < n$,

$$R_{n,k}(q,t) = q^{\binom{k}{2}} t^{n-k} \sum_{r=1}^{n-k} \begin{bmatrix} r+k \\ r,k \end{bmatrix}_q C_{n-k,r}(q,t) + q^{\binom{k}{2}} t^{n-k} \sum_{r=1}^{n-k} q^k \begin{bmatrix} r+k-1 \\ r-1,k \end{bmatrix}_q R_{n-k,r}(q,t). \tag{7}$$

Note that recursions (2) and (7), and the initial conditions, uniquely determine the quantities $R_{n,k}(q,t)$ and $S_n(q,t)$ and provide an efficient method for computing them.

2.5. **Fermionic Formula.** We now obtain a fermionic formula for $S_n(q,t)$ in analogy with (3).

Theorem 8. *For* $n \ge 1$,

$$\begin{aligned} S_n(q,t) = 2q^{\binom{n}{2}} + \sum_{\substack{w_0+\cdots+w_s=n \\ w_i>0;\, s\ge 1}} \Bigg(& q^{\operatorname{pow}_1} t^{\operatorname{pow}_2} \prod_{j=0}^{s-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j - 1, w_{j+1} \end{bmatrix}_q + \\ & \sum_{a=0}^{s-1} q^{\operatorname{pow}_3} t^{\operatorname{pow}_2} \begin{bmatrix} w_a + w_{a+1} \\ w_a, w_{a+1} \end{bmatrix}_q \prod_{j=0}^{a-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j, w_{j+1} - 1 \end{bmatrix}_q \prod_{j=a+1}^{s-1} \begin{bmatrix} w_j + w_{j+1} - 1 \\ w_j - 1, w_{j+1} \end{bmatrix}_q \Bigg), \end{aligned} \tag{8}$$

where $\operatorname{pow}_1 = \sum_{j=0}^s \binom{w_j}{2}$, $\operatorname{pow}_2 = \sum_{j=0}^s j w_j$, *and* $\operatorname{pow}_3 = \operatorname{pow}_1 + \sum_{0 \le j < a} w_j$.

For $0 < k < n$, there is a similar fermionic formula for $R_{n,k}(q,t)$. We simply use the second line of (8), summing over all $(w_0, \ldots, w_s)$ and all a such that $w_0 + \cdots + w_s = n$, $w_i > 0$, $s_i \ge 1$, $0 \le a \le s-1$, *and* fixing $w_0 = k$.

2.6. **Specialization at $t = 1/q$.** Our next goal is to derive explicit formulas for $R_{n,k}(q,1/q)$ and $S_n(q,1/q)$ similar to the formulas for $C_{n,k}(q,1/q)$ and $C_n(q,1/q)$ from §1.1.

Theorem 9. *For $1 \le k \le n$,*

$$q^{\binom{n-k+1}{2}-\binom{k}{2}} R_{n,k}(q,1/q) = \begin{bmatrix} 2n-k-1 \\ n-1, n-k \end{bmatrix}_q + q^k \begin{bmatrix} 2n-k-1 \\ n, n-k-1 \end{bmatrix}_q. \tag{9}$$

For $k = 0$, $q^{\binom{n}{2}} R_{n,0}(q,1/q) = q^{\binom{n}{2}} C_n(q,1/q)$ is given by formula (5).

Theorem 10. *For all $n \ge 1$,*

$$q^{\binom{n}{2}} S_n(q,1/q) = 2\begin{bmatrix} 2n-1 \\ n, n-1 \end{bmatrix}_q = \frac{2}{1+q^n}\begin{bmatrix} 2n \\ n, n \end{bmatrix}_q.$$

We remark that the methods in [24] can be used to mechanically translate the preceding algebraic manipulations into bijective proofs of the same results. However, because of all the subtractions involved, the bijections will be extremely complicated.

3. Algebraic Conjectures for Square Paths

We now give some conjectures connecting square q,t-lattice paths to Macdonald polynomials and the nabla operator. These conjectures closely resemble the corresponding results for the q,t-Catalan numbers from §1.2.

3.1. **Unlabelled Paths.** *Master Conjecture for Square q,t-Lattice Paths:* For all $n \ge 1$, the following elements of $\mathbb{Q}(q,t)$ are all equal (and are, therefore, elements of $\mathbb{N}[q,t]$):

(a) $\sum_{S \in \mathcal{SQ}_n^N} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)}$
(b) $\sum_{S \in \mathcal{SQ}_n^E} q^{\mathrm{area}(S)} t^{\mathrm{bounce}(S)}$
(c) $\sum_{S \in {}^N\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}$
(d) $\sum_{S \in {}^E\mathcal{SQ}_n} q^{\mathrm{dinv}(S)} t^{\mathrm{area}(S)}$
(e) $\sum_{S \in \mathcal{SQ}_n^E} q^{\mathrm{dinv}_0(S)} t^{\mathrm{area}(S)}$
(f) $(-1)^{n-1}\langle \nabla(p_n), s_{1^n}\rangle$
(g) $\sum_{\mu \vdash n}(1-t^n)(1-q^n)\Pi_\mu T_\mu^2/w_\mu = \sum_{\mu\vdash n} MB_{(n^n)}\Pi_\mu T_\mu^2/w_\mu$

We have already seen that (a) through (e) are equal, using the bijections ψ, ϕ, and cyc^{-1}. To see that (f) equals (g), we use the expansion of p_n in terms of the basis $\{\tilde{H}_\mu\}$, namely

$$p_n = \sum_{\mu\vdash n}((-1)^{n-1}(1-t^n)(1-q^n)\Pi_\mu/w_\mu)\tilde{H}_\mu. \tag{10}$$

This expansion follows immediately from Corollary 2.4 in [7] and the definition of plethysm. Applying ∇ replaces each $\tilde{H}_\mu$ by $T_\mu\tilde{H}_\mu$, and taking the scalar product with s_{1^n} turns $\tilde{H}_\mu$ into another factor T_μ. Hence, (f) equals (g). The main conjecture, asserting that (a) equals (f), has been tested for $1 \le n \le 8$.

3.2. **Labelled Paths.** Fix n and N with $n \le N \le \infty$. Let $\mathcal{SQF}_n$ denote the set of all pairs (S,r), where: S is a path in $\mathcal{SQ}_n^E$ (so that S ends with an east step); and $r = r_0 \ldots r_{n-1}$ is a *label vector* with $r_i \in \{1,2,\ldots,N\}$ such that $g_{i+1}(S) = g_i(S)+1$ implies $r_i < r_{i+1}$. If we attach the labels r_i to the vertical steps of S as we did for Dyck paths, then the last condition means that labels in each column must strictly increase from bottom to top. Let $\mathcal{SQH}_n$ denote the subset of $\mathcal{SQF}_n$ such that $r_0 \dots r_{n-1}$ is a permutation of $\{1, 2, \dots, n\}$. Given $(S, r) \in \mathcal{SQF}_n$, define $\operatorname{area}(S, r) = \operatorname{area}(S)$ and

$$\operatorname{dinv}_0(S,r) = \sum_{i<j} \chi((g_i(S) - g_j(S) = 0 \text{ and } r_i < r_j) \text{ or }$$
$$(g_i(S) - g_j(S) = 1 \text{ and } r_i > r_j)) + \sum_i \chi(g_i(S) < 0).$$

(It is equivalent to use all labelled paths *beginning* with an east step, replacing $\chi(g_i(S) < 0)$ by $\chi(g_i(S) < -1)$ in the definition of dinv_0.)

Hilbert series conjecture for square q,t-lattice paths: For all $n \geq 1$,

$$(-1)^{n-1}\langle \nabla(p_n), h_{1^n} \rangle = \sum_{(S,r) \in \mathcal{SQH}_n} q^{\operatorname{area}(S,r)} t^{\operatorname{dinv}_0(S,r)}.$$

Frobenius series conjecture for square q,t-lattice paths: For all $n \geq 1$,

$$(-1)^{n-1}\nabla(p_n[z_1, \dots, z_N]) = \sum_{(S,r) \in \mathcal{SQF}_n} q^{\operatorname{area}(S,r)} t^{\operatorname{dinv}_0(S,r)} \prod_{i=0}^{n-1} z_{r_i}.$$

Applying ∇ to (10), we see that

$$(-1)^{n-1}\nabla(p_n) = \sum_{\mu \vdash n} ((1-t^n)(1-q^n)\Pi_\mu T_\mu / w_\mu) \tilde{H}_\mu.$$

We remark that the same arguments used in [13] show that the Frobenius series conjecture implies both of the preceding conjectures, along with shuffle-type formulas for any scalar product of the form $\langle \nabla(p_n), h_\mu e_\eta \rangle$. It is an open problem to find a naturally occurring doubly-graded S_n-module M_n that has $(-1)^{n-1}\nabla(p_n)$ as its Frobenius series. Since elements of $\mathcal{SQH}_n$ encode functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ in the obvious way ($f^{-1}(\{i\})$ are the labels in column i), we should have $\dim(M_n) = |\mathcal{SQH}_n| = n^n$.

3.3. **More Nabla Conjectures.** So far, we have seen combinatorial formulas that are conjectured to give the monomial expansions of $\nabla(e_n)$ and $(-1)^{n-1}\nabla(p_n)$. Since $e_n = m_{(1^n)}$ and $p_n = m_{(n)}$, these results suggest that $\nabla(m_\mu)$ may have a nice monomial expansion for any $\mu \vdash n$. In fact, an even stronger statement appears to be true.

Conjecture 11. For all $n \geq 1$ and $\mu, \nu \vdash n$,

$$(-1)^{n-\ell(\mu)}\langle \nabla(m_\mu), s_\nu \rangle \in \mathbb{N}[q,t].$$

The conjecture has been tested for $1 \leq n \leq 8$. If the conjecture is true, it readily follows that $\nabla(m_\mu)|_{m_\nu} \in \mathbb{N}[q,t]$ for all μ and ν. In [3], Bergeron, Garsia, Haiman, and Tesler made the analogous conjecture

$$\iota(\mu')\langle \nabla(s_\mu), s_\nu \rangle \in \mathbb{N}[q,t],$$

where $\iota(\mu) = \binom{\ell(\mu)}{2} + \sum_{\mu_i < (i-1)} (i - 1 - \mu_i)$. This second conjecture implies that $\nabla(s_\mu)|_{m_\nu} \in \mathbb{N}[q,t]$ for all μ and ν. Because of the signs, it is not clear whether either conjecture easily implies the other one.

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Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104
E-mail address: `nloehr@math.upenn.edu`

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109
E-mail address: `warrings@wfu.edu`