# MAPS BETWEEN HIGHER BRUHAT ORDERS AND HIGHER STASHEFF-TAMARI POSETS

HUGH THOMAS

ABSTRACT. We make explicit a description in terms of convex geometry of the higher Bruhat orders $B(n,d)$ sketched by Kapranov and Voevodsky. We give an analogous description of the higher Stasheff-Tamari poset $S_1(n,d)$. We show that the map $f$ sketched by Kapranov and Voevodsky from $B(n,d)$ to $S_1([0,n+1],d+1)$ coincides with the map constructed by Rambau, and is a surjection for $d \leq 2$. We also give geometric descriptions of $\mathrm{lk}_0 \circ f$ and $\mathrm{lk}_{\{0,n+1\}} \circ f$. We construct a map analogous to $f$ from $S_1(n,d)$ to $B(n-1,d)$, and show that it is always a poset embedding. We also give an explicit criterion to determine if an element of $B(n-1,d)$ is in the image of this map.

RÉSUMÉ. Nous décrivons explicitement l'esquisse de Kapranov et Voevodsky des ensembles partiellement ordonnés "higher Bruhat" $B(n,d)$, utilisant la géometrie convexe, et nous démontrons qu'elle coïncide avec la description de Manin et Schechtman. Nous donnons aussi une description analogue des ensembles partiellement ordonnés "higher Stasheff-Tamari" $S_1(n,d)$. Nous démontrons que la fonction $f$ esquissée par Kapranov et Voevodsky de $B(n,d)$ à $S_1([0,n+1],d+1)$ coïncide avec celle construite par Rambau, et qu'elle est sujective pour $d \leq 2$. Nous donnons des descriptions géometriques de $\mathrm{lk}_0 \circ f$ et $\mathrm{lk}_{\{0,n+1\}} \circ f$. Nous construisons une fonction analogue à $f$ de $S_1(n,d)$ à $B(n-1,d)$, et nous démontrons qu'elle est toujours un plongement d'ensembles partiellement ordonnés. Nous donnons un critère explicite pour déterminer si un élément de $B(n-1,d)$ est dans l'image de $S_1(n,d)$.

## 1. Introduction

The higher Bruhat orders $B(n,d)$ were introduced by Manin and Schechtman [9] in connection to discriminental hyperplane arrangements. They give a combinatorial definition of $B(n,d)$ which we shall review in the next section. The choice of name stems from the fact that $B(n,1)$ is isomorphic to weak Bruhat order on the symmetric group.

Shortly following the definition of the higher Bruhat orders, Kapranov and Voevodsky wrote a paper [6] which presented two alternative interpretations for the higher Bruhat orders, in terms of oriented matroids, and in terms of convex geometry. The oriented matroid approach was later taken up by Ziegler [17]. The convex geometric approach has not been significantly written about since. It is the focus of the first part of our paper.

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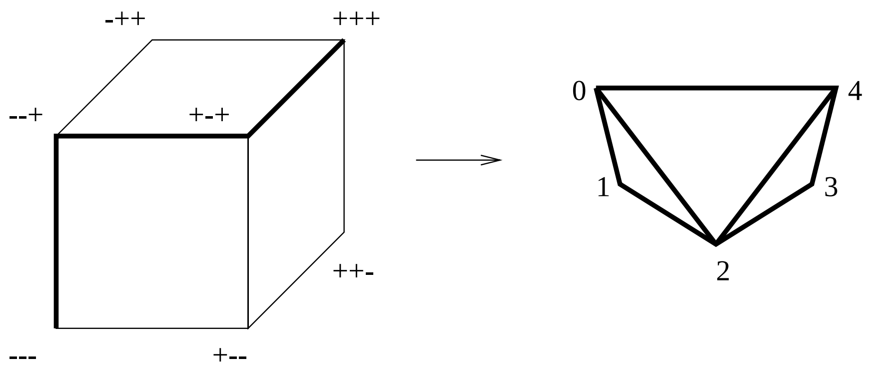
The convex-geometric approach to the higher Bruhat orders is as follows. Consider the $n$-cube $[-1,1]^n$. Let $B(n,0)$ denote the set of vertices of the cube, with the usual Cartesian product order, so that $B(n,0)$ is a Boolean lattice. Its minimum element is $(-1,\dots,-1)$, and its maximum element is $(1,\dots,1)$. Now let $B(n,1)$ be the set of increasing paths along edges of the cube from $(-1,\dots,-1)$ to $(1,\dots,1)$. There are $n!$ of these, and they are naturally in bijection with $S_n$. We put an order on this set by defining covering relations: $\sigma \gtrdot \tau$ if $\sigma$ and $\tau$ coincide except on the boundary of some 2-face, where $\sigma$ uses the top two edges of the face, and $\tau$ uses the bottom two edges. (We will be more explicit about how to understand "bottom" and "top" in the next section.) Under this order, the poset $B(n,1)$ is isomorphic to weak Bruhat order on the symmetric group.

Now consider collections of 2-faces of the cube which form a homotopy from the minimum path to the maximum path, and which are non-backtracking. (This non-backtracking condition generalizes the "increasing" condition in the dimension 1 case. We shall give more precise definitions in the next section.) These homotopies form the elements of $B(n,2)$. As before, the order on $B(n,2)$ is defined by specifying covering relations: $\sigma \gtrdot \tau$ if $\sigma$ and $\tau$ coincide except on the boundary of a 3-face, where $\sigma$ uses the top three facets, and $\tau$ uses the bottom three facets. The other $B(n,d)$ are defined similarly. The first goal of this paper is to write down this description explicitly, and to show that it is equivalent to the combinatorial definition of [9].

In order to describe the second goal of the paper, we must now turn to the higher Stasheff-Tamari posets. In fact, there are two different Stasheff-Tamari posets structures $S_1(n,d)$ and $S_2(n,d)$ defined on the same set of objects $S(n,d)$. We shall only be interested in the first of these posets, so we shall suppress the subscript. $S(n,d)$ is usually viewed as the set of triangulations of the cyclic polytope $C(n,d)$. We will give an equivalent definition, analogous to the one above for higher Bruhat orders where the cube has been replaced by a simplex.

In order to define $S(n,d)$, start with an $n-1$-simplex, with vertices labelled from 1 to $n$. $S(n,0)$ is the set of vertices, with the order given by the labelling. The objects of $S(n,1)$ are the increasing paths from the bottom vertex to the top vertex. The order is by reverse refinement: the bottom path is the path that includes every vertex, and the top path is the one that uses only 1 and $n$. The objects of $S(n,2)$ are the sets of 2-faces of the simplex which form a non-backtracking homotopy from the bottom path to the top path. The order on $S(n,2)$ is defined by specifying covering relations: $S \gtrdot T$ if $S$ and $T$ coincide except on the boundary of a 3-simplex, where $S$ contains the upper faces and $T$ the lower faces. The higher $S(n,d)$ are defined similarly.

In [6], a map called $f$ from $B(n,d)$ to $S([0,n+1],d+1)$ was described as follows. (We write $S([0,n+1],d+1)$ to indicate that the vertices are labelled by the numbers from 0 to $n+1$.) There is a poset isomorphism from vertices of the $n$-cube (i.e. $B(n,0)$) to elements of $S([0,n+1],1)$: namely, the coordinates of the vertex which are negative tell you which vertices belong in the path in addition to 0 and $n+1$. Now, an element of $B(n,1)$, which is a path through the $n$-cube, determines a sequence of vertices of the cube. We apply the map from $B(n,0)$ to $S([0,n+1],1)$ to each vertex in succession, to get a sequence of paths through the $n+1$-simplex. Two successive paths differ in that one vertex which is present in the first path is not present in the second. To each pair of successive paths, we associate the triangle whose vertices are the removed vertex and its two neighbours along the path. These triangles form a homotopy from the bottom path (which contains all the vertices) to the top path (which contains only the end-points), and hence define an element of $S([0, n+1], 2)$. In the example below, the bold path through the cube on the left gives rise to the triangulation shown on the right.




It is claimed in [6] that one can define a similar map $f : B(n, d) \to S([0, n+1], d+1)$ for all $d$, and further that this map is surjective for all $d$. Rambau [10] constructed an explicit map from $B(n, d)$ to $S([0, n+1], d+1)$, but he did not show that it coincided with the map described in [6]. In the second part of the paper, we prove that the map described in [6] does indeed coincide with that defined in [10], and that it is surjective for $d \leq 2$.

In the second part of the paper, we also give geometric interpretations of two maps associated to $f$. For $S \in S([0, n+1], d+1)$, $\mathrm{lk}_0(S)$ is the link of $S$ at zero, which can be viewed in a natural way as lying in $S(n+1, d)$. We can also take $\mathrm{lk}_{\{0,n+1\}}(S)$; this lies in $S(n, d-1)$. We show that, for $\pi \in B(n, d)$, $\mathrm{lk}_{\{0,n+1\}}(f(\pi))$ coincides with the vertex figure of $\pi$ at $(1, \ldots, 1) \in [-1, 1]^n$. We also give a similar interpretation for $\mathrm{lk}_0(f(\pi))$.

The third part of our paper consists of the construction of a map $g : S(n, d) \to B(n-1, d)$. As already explained, $S(n, 0)$ is a chain of $n$ elements which we view as the vertices of an $n-1$-simplex. We map vertex $a$ to the corner of $[-1, 1]^{n-1}$ whose final $a-1$ coordinates are +1, and the others -1. An element of $S(n, 1)$ is mapped to an increasing path through the $n-1$-cube which passes through the vertices corresponding to the vertices on the path through the $n-1$-simplex. For $S \in S(n, 2)$, $g(S)$ is defined as the unique homotopy from the minimal path through the cube to the maximal path through the cube, which passes through all the paths corresponding to paths through the $n-1$-simplex along edges in $S$. An analogous statement holds for $d > 2$.

We give several equivalent definitions for $g$, including two which are explicit and non-inductive. We also show that the map $g$ is a poset embedding, and we give an explicit criterion to determine if an element of $B(n-1, d)$ is in the image of $g$. This amounts to giving a new equivalent definition of $S(n, d)$ without reference to convex geometry.

## 2. Higher Bruhat orders

We begin by recalling the definition of the higher Bruhat order $B(n, d)$ for $d \leq n$ positive integers, given by Manin and Schechtman [9]. We write $\binom{[n]}{d}$ for the set of subsets of $[n]$ of size $d$. A $d$-packet consists of the subsets of size $d$ of a set of $d+1$ integers.

A total order on $\binom{[n]}{d}$ is admissible if every $d$-packet occurs in either lexicographic order or its reverse. The set of admissible orders on $\binom{[n]}{d}$ is called $A(n,d)$.

Two admissible orders are said to be equivalent if they differ by a sequence of transpositions of adjacent elements not both lying simultaneously in any $d$-packet. If $\pi$ is an admissible order, we write $[\pi]$ for its equivalence class.

$B(n,d)$ is a poset whose elements are the equivalence classes of admissible orders on $\binom{[n]}{d}$. The order is given by specifying covering relations. Let $\pi \in A(n,d)$. Suppose there is some $d$-packet which occurs consecutively in $\pi$, and in lexicographic order. Let $\sigma$ denote the (automatically admissible) order obtained by reversing this $d$-packet. Then $[\pi] \lessdot [\sigma]$ in $B(n,d)$. The order on $B(n,d)$ is the transitive closure of these covering relations.

There are two orders on $\binom{[n]}{d}$ which are clearly admissible. Let $\hat{0}_d$, $\hat{1}_d$ denote the class in $B(n,d)$ of the lexicographic order and its reverse respectively. It is clear that these elements are minimal and maximal respectively in $B(n,d)$; in fact, they are its minimum and maximum elements, see [9].

There is a map $I : A(n,d) \to \mathcal{P}(\binom{[n]}{d+1})$ which associates to any $\pi \in A(n,d)$ the set of $d$-packets which occur in reverse order. $I(\pi)$ is called the inversion set of $\pi$. This generalizes the usual notion of inversion set for a permutation. It is clear that $I$ is constant on equivalence classes, and so passes to $B(n,d)$. As a map from $B(n,d)$, $I$ is injective.

A subset of $\binom{[n]}{d+1}$ is said to be consistent if its restriction to any $(d+1)$-packet consists of either an initial or a final subset with respect to lex order. Ziegler showed in [17] that a subset of $\binom{[n]}{d+1}$ is in the image of $I$ iff it is consistent.

It will be convenient to define $B(n,0)$ to be the set of subsets of $[n]$, ordered by inclusion. The inversion set of an element of $B(n,0)$ is just the set itself. $\hat{1}_0 = [n]$; $\hat{0}_0 = \emptyset$.

We now give the convex-geometric definition of the higher Bruhat orders, formalizing ideas from [6]. $[-1,1]^n$ will be our standard $n$-cube. We shall keep track of its faces as maps from $[n]$ to the set $\{-1,*,1\}$, where a $d$-face will have $*$ occurring in $d$ places, these being the dimensions in which the face extends. We will sometimes refer to a set of faces of the cube when what we mean is the union of the set of faces.

For $G$ a set, let $\Xi_G(x) = 1$ if $x \in G$ and $-1$ otherwise. For $X = \{a_1, \ldots, a_d\}_< \subset [n]$, $y \notin X$, define:

$$p(y,X) = \begin{cases} 1 & y < a_1 \\ (-1)^i & a_i < y < a_{i+1} \\ (-1)^d & a_d < y \end{cases}$$

Fix $\alpha \in B(n,d)$. For each $X \in \binom{[n]}{d}$, let

$$F_X^\alpha(i) = \begin{cases} * & i \in X \\ p(i,X)\Xi_{I(\alpha)}(X \cup \{i\}) & i \notin X \end{cases}$$

Let $K(\alpha)$ consist of the $F_X^\alpha$ for all $X$.

We identify linear maps from $\mathbb{R}^n$ to $\mathbb{R}^d$ with $d \times n$ matrices. We say that a map $T : \mathbb{R}^n \to \mathbb{R}^d$ is totally positive if the determinants of all its minors are positive. (Note that there are many totally positive matrices, for example, a Vandermonde matrix

$T_{ij} = c_j^i$ with $0 < c_1 < \cdots < c_n$. The determinant of any minor of this matrix equals a Vandermonde determinant times a Schur function, both of which are positive.)

We say that a collection of convex sets tiles a region if the region is the union of the convex sets and the sets overlap only on boundaries.

We have the following proposition:

**Proposition 2.1.** *For any $\alpha \in B(n,d)$, the set $K(\alpha)$ of $d$-faces of the standard $n$-cube is homeomorphic to a disk, has boundary $K(\hat{0}_{d-1}) \cup K(\hat{1}_{d-1})$, and the image of the $K(\alpha)$ under any totally positive map $T$ from $\mathbb{R}^n$ to $\mathbb{R}^d$ forms a tiling of the image of the standard $n$-cube under $T$.*

We divide the facets (maximal proper faces) of a polytope up into upper facets and lower facets, depending on whether the polytope lies above or below the facet, with respect to the final coordinate. When we refer to the upper or lower facets of $[-1,1]^n$ or its faces, we mean those facets which are upper or lower in the image under a totally positive map. Which facets are upper and which are lower does not depend on the choice of map.

To describe the order relation on $B(n,d)$ in terms of this description, we have the following theorem:

**Theorem 2.1.** *For $\sigma, \tau \in B(n,d)$, $\sigma \gtrdot \tau$ iff $K(\sigma)$ and $K(\tau)$ coincide except on the facets of a $d+1$-cube, where $K(\sigma)$ contains the upper facets and $K(\tau)$ contains the lower facets.*

## 3. The higher Stasheff-Tamari posets

As explained in the introduction, the usual way of thinking of the higher Stasheff-Tamari posets $S(n,d)$ is as a poset on the set of triangulations of a cyclic polytope. To motivate the existence of a connection to the higher Bruhat orders, a different definition, one analogous to the convex-geometric definition of $B(n,d)$ given above, will be more relevant. To avoid confusion, we shall give the poset we define in this manner a new name, $T(n,d)$, and then prove that $T(n,d)$ coincides with $S(n,d)$.

The standard $n-1$-simplex, $\Delta_{n-1}$, is the convex hull of the basis vectors in $\mathbb{R}^n$. Its $d$-faces are indexed by $d+1$-subsets of $[n]$, which designate which vertices lie on the face.

If $W : \mathbb{R}^n \to \mathbb{R}^d$, let $\overline{W}$ be the linear map from $\mathbb{R}^n$ to $\mathbb{R}^{d+1}$ defined by setting $\overline{W}(e_i) = (1, W(e_i))$. In terms of matrices, we can say that the matrix for $\overline{W}$ is obtained from that for $W$ by adding a first row of all ones. We say that $W$ is *affinely positive* if $\overline{W}$ is totally positive. When we speak of upper or lower facets of a face of $\Delta_{n-1}$, we mean facets which are upper or lower in the image of the face under an affinely positive map. Which facets are upper and which are lower does not depend on the choice of affinely positive map.

**Definition of $T(n,d)$.** *An element $S$ of $T(n,d)$ is a set of $d$-faces of $\Delta_{n-1}$, with the property that under some (or equivalently any) affinely positive map $W$, the images under $W$ of the faces in $S$ tile $W(\Delta_{n-1})$. The order on $T(n,d)$ is defined by giving covering relations: $S \gtrdot T$ iff $S$ and $T$ coincide except on the boundary of a $d+1$-simplex, where $S$ contains the upper facets of the simplex and $T$ contains the lower facets.*

We now define two elements of $T(n,d)$, $\hat{1}_d$ and $\hat{0}_d$, as follows. Let $W$ be an affinely positive map from $\mathbb{R}^n$ to $\mathbb{R}^d$. The boundary facets of the image of $\Delta_n$ under $W$ do not depend on $W$, so let $\hat{1}_d$ consist of the faces of $\Delta_{n-1}$ corresponding to upper boundary facets of $W(\Delta_{n-1})$, and let $\hat{0}_d$ consist of its lower boundary facets. We remark that $\hat{1}_d$ is clearly a maximal element of $T(n,d)$, and $\hat{0}_d$ is clearly a minimal element. They are in fact maximum and minimum respectively, which we know from [10] (once we know that $T(n,d)$ coincides with $S(n,d)$).

We have the following proposition:

**Proposition 3.1.** *If $S \in T(n,d)$, the faces in $S$ are homeomorphic to a disk, and their boundary equals $\hat{0}_{d-1} \cup \hat{1}_{d-1}$.*

Finally, we show that $T(n,d)$ coincides with the poset $S(n,d)$ as conventionally defined. We begin by reviewing the definition of $S(n,d)$.

Fix $d$ a positive integer. Let $M(t) = (t, t^2, \ldots, t^d)$. Choose $n$ real numbers $t_1 < \cdots < t_n$. Let $P$ be the convex hull of the $M(t_i)$. $P$ is called a cyclic polytope.

Many combinatorial properties of $P$ depend only on $d$ and $n$, and not on the choice of $t_i$. Let $I$ be a $d$-set contained in $[n]$. Then whether or not the $M(t_i)$ for $i \in I$ form a boundary facet of $P$ does not depend on the choice of $i$. (In fact, the boundary facets are described by the well-known "Gale's Evenness Criterion," see [5].) Further, let $S \subset \binom{[n]}{d+1}$. To each $A \in S$, we can associate a simplex contained in $P$. And again, whether or not this collection of simplices forms a triangulation of $P$ does not depend on the choice of the $t_i$. Thus, we shall usually refer to "the" cyclic polytope in dimension $d$ with $n$ vertices, and denote it $C(n,d)$. When we wish to emphasize the choice of some particular $t_i$, we speak of a geometric realization of $C(n,d)$.

The partial order on $S(n,d)$ is given by describing its covering relations. If one is familiar with the language of bistellar flips, one can say that the covering relations $S \lessdot T$ are given by pairs $S$ and $T$ which are related by a single bistellar flip, where bistellar flips are given a certain natural orientation to determine whether $S$ precedes $T$ or vice versa. The reader interested in a thorough explanation of this can consult [3].

More explicitly, we can define the covering relations as follows, following [10]. Let $M'(t) = (t, t^2, \ldots, t^{d+1}) \in \mathbb{R}^{d+1}$. Pick $t_1 < \cdots < t_n$. This yields geometric realizations of $C(n, d+1)$ and $C(n,d)$, where the map forgetting the last co-ordinate maps $C(n, d+1)$ down to $C(n,d)$. A triangulation $S \in S(n,d)$ defines a section $\Gamma_S \subset \mathbb{R}^{d+1}$ over $C(n,d)$ by lifting its vertices $M(t_i)$ to $M'(t_i)$ and then extending linearly over the simplices of $S$. Now, $S \lessdot T$ precisely if $\Gamma_S$ and $\Gamma_T$ coincide except within the convex hull of $d+2$ vertices, where $\Gamma_S$ forms the bottom facets and $\Gamma_T$ the top facets of a $d+1$-dimensional simplex.

Now we have the following proposition:

**Proposition 3.2.** *The poset $T(n,d)$ and the poset $S(n,d)$ coincide.*

Since $T(n,d)$ and $S(n,d)$ coincide, we shall use the conventional notation of $S(n,d)$, but the reader is advised that we will tacitly use the intuition that the elements of $S(n,d)$ can be considered as sets of $d$-faces of an $n-1$-simplex.

## 4. The map from $B(n,d)$ to $S([0,n+1],d+1)$

We will now proceed to elucidate the poset map sketched in [6] from $B(n,d)$ to $S([0,n+1],d+1)$. The definition is by induction.

**Definition 1.** *If $\alpha \in B(n,0)$, then $f_1(\alpha)$ is the path whose vertices are the elements of $[0,n+1]$ not in $\alpha$, in increasing order.*

*For $d > 0$, let $\pi \in A(n,d)$. Define $\alpha_i \in B(n,d-1)$ by $I(\alpha_i) = \mathrm{In}_i(\pi)$. Let $S_i = f(\alpha_i) \in S([0,n+1],d)$. We claim that for all $i$, either $S_i$ and $S_{i+1}$ coincide, or they differ precisely in that $S_i$ contains the bottom facets of some $d+1$-simplex $A_{i+1}$, while $S_{i+1}$ contains its top facets. Then, $f_1([\pi])$ consists of the collection of the $A_{i+1}$ for all $i$ for which $S_i$ and $S_{i+1}$ are different.*

Because of the reliance on the claim mentioned, this definition doesn't establish the existence of $f_1$. We shall now give an explicit definition of $f_2$, essentially the map called $\mathcal{T}_{\mathrm{dir}}$ in [10]. An induction argument will then show that $f_2$ satisfies Definition 1.

If $X = \{a_1, \ldots, a_d\}_< \in \binom{[n]}{d}$. Let $x^\alpha_X$ be the greatest positive integer less than $a_1$ such that $F^\alpha_X(x^\alpha_X) = -1$, and set $x^\alpha_X = 0$ if there is no such integer. Similarly, set $z^\alpha_X$ to be the least integer less than or equal to $n$ such that $F^\alpha_X(z^\alpha_X) = -1$, and set $z^\alpha_X = n+1$ if there is no such integer.

**Definition 2.** *For $d = 0$, define $f_2$ as in Definition 1.*

*For $d > 0$, let $\alpha \in B(n,d)$. Let $X = \{a_1, \ldots, a_d\}_< \in \binom{[n]}{d}$. If, $\forall y \notin X$ such that $a_1 < y < a_d$, $F^\alpha_X(y) = 1$, then we associate to $X$ a simplex $\{x^\alpha_X, a_1, \ldots, a_d, z^\alpha_X\}$. Define $f_2(\alpha)$ to be the set of simplices associated to some $X$.*

We remark that it is by no means obvious that $f_2(\alpha)$ forms a triangulation; this will follow from the following theorem.

**Theorem 4.1.** *The map $f_2$ satisfies Definition 1.*

We shall denote by $f$ the map defined by these two equivalent definitions. We now have the following proposition:

**Proposition 4.1.** *The map $f : B(n,d) \to S([0,n+1],d+1)$ is order-preserving.*

For completeness we give yet another definition of $f$, also from [10].

**Definition 3.** *Let $\beta \in B(n,d)$. Let $I(\beta) = \{X_1, \ldots, X_r\}$ where the $X_i$ are ordered so that every initial subsequence is also consistent. Set $T_0 = \hat{0}_d \in S([0,n+1],d)$. Define $T_i$ by induction: if $T_{i-1}$ contains the bottom facets of a simplex with vertices $\{x\} \cup X_i \cup \{z\}$ for some $x$ less than any element of $X_i$ and $z$ greater than any element of $X_i$, then let $T_i$ consist of the facets of $T_{i-1}$ with these bottom facets replaced by the simplex's corresponding top facets. Otherwise, let $T_i = T_{i-1}$. Then set $f_3(\beta) = T_r$.*

**Theorem 4.2** ([10])**.** *The map $f_3$ is well-defined and coincides with the map $f$.*

It is claimed (without proof) in [6] that $f : B(n,d) \to S([0,n+1],d+1)$ is surjective for all $n$ and $d$. We cannot prove this in general.

When $d = 1$, the map $f$ has been studied in other contexts: it is essentially a familiar map from permutations to planar binary trees (see [13: 1.3.13, 1, 7, 16]). Let $Y_n$ denote the planar binary trees with $n$ internal vertices. For $(a_1,\ldots,a_p)$ a sequence of $p$ distinct numbers, let $\mathrm{std}(a_1,\ldots,a_p)$, the standardization of $(a_1,\ldots,a_p)$, denote the sequence of numbers from 1 to $p$ arranged in the same order. Define $\psi : B(n,1) \to Y_n$ inductively, as follows: for $n$=0, $\psi$ applied to the empty permutation is an empty tree; and for $n \geq 1$, $\pi \in B(n,1)$, write $\pi = (a_1 \ \ldots \ a_p \ n \ b_1 \ \ldots \ b_q)$, and then let $\psi(\pi)$ be the tree consisting of one parent node with left subtree $\psi(\mathrm{std}(a_1,\ldots,a_p))$, and right subtree $\psi(\mathrm{std}(b_1,\ldots,b_q))$. We recover the map $f : B(n,1) \to S([0,n+1],2)$ by composing $\psi$ with a standard bijection $\theta$ between triangulations of an $n+2$-gon and planar binary trees with $n$ internal vertices.

We recall the following facts from the literature (see [7, 8, 1, 2]):

**Proposition 4.2.** *The map $f$ is surjective, and there are maps* $\mathrm{Min}, \mathrm{Max} : S([0,n+1],2) \to B(n,1)$ *such that the fibre of $f$ over $S \in S([0,n+1],2)$ is the non-empty closed interval* $[\mathrm{Min}(S), \mathrm{Max}(S)]$ *in* $B(n,1)$.

When $d=2$, we have the following proposition, which is new:

**Proposition 4.3.** $f : B(n,2) \to S([0,n+1],3)$ *is surjective.*

However, the fibres of $f$ are not as simple in this case: explicit examples show that the fibre of $f$ over an element of $S([0,n+1],3)$ need not be a closed interval.

For larger $d$, no results have been proven.

## 5. Interpretation of $\mathrm{lk}_0 \circ f$ and $\mathrm{lk}_{\{0,n+1\}} \circ f$

Let $S \in S([0,n+1],d+1)$. Then $\mathrm{lk}_0(S) = \{A \setminus \{0\} \mid 0 \in A \in S\}$, the link of $S$ at 0. As was remarked in [3] and proved in [10], thinking of this link as describing faces of the vertex figure of $\Delta_{n+1}$ at 0, we see that $\mathrm{lk}_0(S) \in S(n+1,d)$. Similarly, we can define $\mathrm{lk}_{n+1}(S) \in S([0,n],d)$. The map $\mathrm{lk}_0$ is order-preserving; $\mathrm{lk}_{n+1}$ is order-reversing. (One might wonder about taking links at other vertices. For $d$ even, these other links are not naturally elements of $S(n+1,d)$; for $d$ odd one can define a link in a suitably labelled $S(n+1,d)$ but this link map does not respect the poset structures.)

In this section, we give geometric interpretations of $\mathrm{lk}_0 \circ f$ and $\mathrm{lk}_{\{0,n+1\}} \circ f$.

**Proposition 5.1.** *Let $\pi \in B(n,d)$. Then a simplex $A \in \mathrm{lk}_{\{0,n+1\}} \circ f$ iff $F^\pi_A$ contains $(1,\ldots,1)$. In other words, $\mathrm{lk}_{\{0,n+1\}}(f(\pi))$ is the vertex figure of $K(\pi)$ at $(1,\ldots,1)$.*

Let us write $[-1,1]^n_\leq$ for the weakly increasing $n$-tuples from $[-1,1]$. This set forms a simplex with $n+1$ vertices, whose coordinates consisting of a string of $-1$s followed by a string of 1s. We identify this simplex with the standard $n$-simplex by labelling the vertex whose first $a-1$ coefficients are $-1$ as vertex $a$.

Let us define a map:

$$W : [-1,1]^n \to [-1,1]^n_\leq$$
$$W(a_1,\ldots,a_n) = (a_1, \max(a_1,a_2), \ldots, \max(a_1,a_2,\ldots,a_n))$$

**Proposition 5.2.** *Let $\pi \in B(n,d)$. Then a simplex $A \in \mathrm{lk}_0(f(\pi))$ iff* $\dim(W(F^\pi_A)) = \dim(F^\pi_A)$. *Consequently,* $\mathrm{lk}_0(f(\pi)) = W(K(\pi))$.

## 6. Combinatorics of $S(n,d)$

For the remainder of the paper, we shall need a combinatorial description of $S(n,d)$ introduced in [15]. We begin with some preliminary definitions.

For $\{a_1,\dots,a_{d+1}\}_<$ a subset of $[n]$, let $r(a_1,\dots,a_{d+1})$ denote the subset of $\binom{[n-1]}{d}$ which consists of those $d$-sets consisting of exactly one element from $[a_i, a_{i+1}-1]$ for $1 \le i \le d$. Subsets of $\binom{[n-1]}{d}$ of this form are called *snug rectangles*. We say that a set of snug rectangles forms a *snug partition* if each $d$-set in $\binom{[n-1]}{d}$ occurs in exactly one of the snug rectangles.

To $S \in S(n,d)$, we associate the collection of snug rectangles $r(S)$ which consists of the rectanges $r(a_1,\dots,a_{d+1})$ for each simplex $\{a_1,\dots,a_{d+1}\}_<$ in $S$. Then we have the following theorem:

**Theorem 6.1** ([15])**.** *The map $r$ defines a bijection from $S(n,d)$ to snug partitions of $\binom{[n-1]}{d}$.*

We now describe an important feature of the combinatorics of $S(n,d)$, namely, the collapse maps, poset maps from $S(n,d)$ to $S(p,d)$ with $p < n$.

Let $I$ be a subset of $[n-1]$. Let $m_I : [n] \to I \cup \{n\}$ be the map defined by $m_I(a) = \min\{i \in I \cup \{n\}, i \ge a\}$.

Consider the map from $\mathbb{R}^n$ to $\mathbb{R}^{I\cup\{n\}}$ which takes $e_i$ to $e_{m_I(i)}$. This defines a map from $\Delta_{n-1}$ to $\Delta_{|I|} \subset \mathbb{R}^{I\cup\{n\}}$. We define a map $c_I$ on faces of $\Delta_{n-1}$, which takes a face to its image in $\Delta_{|I|}$, or to $\emptyset$ if its image is lower dimensional. Explicitly, if $A = \{a_1,\dots,a_{d+1}\}_<$, then

$$c_I(A) = \{m_I(a_1),\dots,m_I(a_{d+1})\}$$

provided the $m_I(a_i)$ are all distinct, and $c_I(A) = \emptyset$ otherwise.

Now, for $S \in S(n,d)$, define $c_I(S)$ to be the collection of non-empty $c_I(A)$ for $A \in S$. Then $c_I(S) \in S(I \cup \{n\}, d)$.

## 7. The map $g : S(n,d) \to B(n-1,d)$

In this section we define a map $g : S(n,d) \to B(n-1,d)$, which is analogous to $f$ in ways which will be made clear later.

Observe that $S(d+2,d)$ consists of 2 elements, which, as usual, we denote $\hat{0}_d$ and $\hat{1}_d$. For $S \in S(n,d)$, let $I(S) = \{X \in \binom{[n-1]}{d+1} \mid c_X(S) = \hat{1}\}$. We wish to define $g(S)$ by setting $I(g(S)) = I(S)$. In order for this to make sense, we must prove the following proposition:

**Proposition 7.1.** *For $S \in S(n,d)$, $I(S) \subset \binom{[n-1]}{d+1}$ is a consistent set*

*Proof.* The proof consists of a reduction to the case $n = d+3$. $S(d+3,d)$ is quite easy to get a handle on. □

We can say more about $I(S)$. Let us say that $I \subset \binom{[n-1]}{d+1}$ is *superconsistent* if its intersection with any $d+1$-packet is either an initial segment of odd length or a final segment of the same parity as $d$ (or empty or full). Then for $S \in S(n,d)$, $I(S)$ is superconsistent. And more is true:

**Theorem 7.1.** *$I \subset \binom{[n-1]}{d+1}$ is superconsistent iff it is the inversion set of some $S \in S(n,d)$. Equivalently, the image of $g : S(n,d) \to B(n-1,d)$ consists of those elements with superconsistent inversion sets.*

*Proof.* The proof is an explicit construction of a triangulation corresponding to the superconsistent set $I$. □

Next, we check certain properties of $g$. First, we have the following theorem:

**Theorem 7.2.** *The map $g : S(n,d) \to B(n-1,d)$ is order-preserving.*

We now recall Rambau's definition in [10] of the *extension* map from $S(n,d)$ to $S([0,n],d+1)$ (with a trivial modification to suit our conventions). Let $S \in S(n,d)$. Then by definition $\hat{S}$, the extension of $S$, is

$$\hat{S} = \{A \cup \{0\} \mid A \in S\} \\ \cup\{(x, x+1, a_2, \ldots, a_{d+1}) \mid \{a_1, \ldots, a_{d+1}\}_< \in A, a_1 \leq x \leq a_2 - 2\}$$

It is a nice application of the theory of snug partitions to check that $\hat{S} \in S([0,n],d+1)$.

There is a simple geometrical idea motivating this definition. Let $S \in S(n,d)$, thought of as triangulations of $C(n,d)$. $S$ defines a hypersurface $\Gamma_S$ in $C(n,d+1)$. Add a new point on the moment curve which precedes all the vertices of $C(n,d)$, and label it 0. All the faces of $\Gamma_S$ are visible from 0. $\hat{S}$ consists of all the simplices formed by joining 0 to simplics of $S$, together with a canonical way to fill in the remainder of $C([0,n],d+1)$. It is clear either from this description, or directly from the definition, that, as is shown in [10], $\mathrm{lk}_0(\hat{S}) = S$.

**Proposition 7.2.** *For $S \in S(n,d)$, $f(g(S)) = \hat{S}$.*

*Proof.* One checks, using Definition 2 of $f$, that each of the simplices of $\hat{S}$ appears in $f(g(S))$. □

Since both $f$ and $g$ are order-preserving, we recover the result from [10] that the map $S \to \hat{S}$ is order-preserving.

**Theorem 7.3.** *The map $g : S(n,d) \to B(n-1,d)$ is a poset embedding*

*Proof.* Suppose that $S, T \in S(n,d)$, and $g(S) > g(T)$ in $B(n-1,d)$. Then, since $f$ is order preserving, $f(g(S)) > f(g(T))$. So $\hat{S} > \hat{T}$, so $S = \mathrm{lk}_0 \hat{S} > \mathrm{lk}_0 \hat{T} = T$. Thus $g$ is a poset embedding. □

## 8. Alternative definitions of $g$

In this section we show that $g$ satisfies three alterative definitions, including analogues of Definitions 1 and 3 of $f$.

We will have occasion to consider a special type of linear order on the elements of a snug rectangle in $\binom{[n-1]}{d}$. We say that such an order is *rectangular* if

$$(x_1, \ldots, x_i, \ldots, x_d) > (x_1, \ldots, x_i + 1, \ldots, x_d) \text{ if } d - i \text{ is even} \\ (x_1, \ldots, x_i, \ldots, x_d) < (x_1, \ldots, x_i + 1, \ldots, x_d) \text{ if } d - i \text{ is odd}$$

For $S \in S(n,d)$, a linear order on its simplices is said to be *ascending* if for any pair of simplices sharing a facet, the simplex lying above the intersection facet follows the simplex below the intersection. It is shown in [10] that there exist ascending orders on the simplices of any triangulation of a cyclic polytope of arbitrary dimension.

The following proposition gives us an alternative definition of $g$:

**Proposition 8.1.** *Let $S \in S(n,d)$. Fix an ascending order on the simplices of $S$, say, $A_1, \ldots, A_r$. Consider the order on $\binom{[n-1]}{d}$ which consists of the element of $r(A_1)$ followed by the elements of $r(A_2)$, etc., where the elements within any $r(A_i)$ are written in a rectangular order. This order is admissible, and the element of $B(n,d)$ which it defines is $g(S)$.*

We now prove the equivalence of a definition of $g$ analogous to the Definition 1 of $f$.

**Proposition 8.2.** *Let $d \geq 2$, and $S \in S(n,d)$. Fix an ascending order on the simplices of $S$. Let $\hat{0} = T_0 \lessdot T_1 \lessdot \cdots \lessdot T_r = \hat{1}$ be the corresponding chain in $S(n,d-1)$. Refine the chain $g(\hat{0}) < g(T_1) < \cdots < g(\hat{1})$ to a maximal chain in $B(n-1,d-1)$. Then $g(S)$ is the element of $B(n-1,d)$ corresponding to that chain.*

Interestingly, this definition fails for $d = 0, 1$. Here, different refinements of the chain of $g(T_i)$ yield different elements of $B(n-1,d)$ (though it is of course easy to specify which refinement to use).

We now give a definition of $g$ analogous to Definition 3 of $f$, in that it relies on the choice of an unrefinable chain from $\hat{0}$ to $S$ in order to define $g(S)$.

**Proposition 8.3.** *Let $S \in S(n,d)$. Choose an unrefinable chain $\hat{0}_d = T_0 \lessdot T_1 \lessdot \cdots \lessdot T_r = S$. Let $R_i$ be the snug rectangle in $\binom{[n-1]}{d+1}$ corresponding to the simplex where $T_{i-1}$ and $T_i$ differ. Then*

$$I(S) = \bigcup_{i=1}^{r} R_i.$$

## 9. Further Directions

We would like to understand the fibres of $f$ better. Perhaps, as a first step, one might study the fibres of $\mathrm{lk}_0 \circ f$, since the fibre of $\mathrm{lk}_0 \circ f$ over $S$ has a distinguished element, namely $g(S)$. (Contrary to what one might hope, $g(S)$ is neither always minimal nor always maximal in the fibre.)

We would also like to see the question of the surjectivity of $f$ settled.

The motivation for [6] was from the still-developing theory of $n$-categories. We hope that our results may have some application in this area. In particular, according to some definitions (see [6], [14]), there is an $n$-category $\Delta_n$ associated to the $n$-simplex, and an $n$-category $I_n$ associated to the $n$-cube. It appears that the map $g$ defines a map of $n$-categories from $\Delta_n$ to $I_n$ (as the map $f$ was shown in [6] to define a map from $I_n$ to $\Delta_{n+1}$).

The order complex of $B(n,d)$ is homotopic to a sphere of dimension $n-d-2$ [11]. The order complex of $S(n,d)$ is homotopic to a sphere of dimension $n-d-3$ [4]. Thus, the maps $g$ and $f \circ g$ induce maps between order complexes which are homotopy equivalent.

It seems likely that these maps are homotopy equivalences. (The map $f$ does not induce a map on order complexes because it takes non-minimal elements to $\hat{0}$.)

We would also like to understand the homotopy type of intervals in these posets, or, more restrictedly, the Möbius functions of these posets. There is an interesting conjectural description for both, see [12]. Perhaps the existence of the new map $g$ will help, at the very least, to connect the questions for the higher Stasheff-Tamari posets and the higher Bruhat orders more closely together.

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Fields Institute, 222 College St., Toronto, ON, M5T 3J1 Canada
*E-mail address*: `hthomas@fields.utoronto.ca`