Abstract. Zonal spherical functions of the Gelfand pair \((W(B_n), S_n)\) are expressed in terms of the Krawtchouk polynomials which are a special family of Gauss’ hypergeometric functions. Its generalizations are considered in this abstract. Some class of orthogonal polynomials are discussed in this abstract which are expressed in terms of \((n + 1, m + 1)\)-hypergeometric functions. The orthogonality comes from that of zonal spherical functions of certain Gelfand pairs of wreath product.

Résumé. Des fonctions sphériques zonales de la paire \((W(B_n), S_n)\) de Gelfand sont exprimées en termes de polynômes de Krawtchouk qui sont une famille spéciale des fonctions hypergéométriques des Gauss. Ses généralisations sont considérées dans cet abstrait. Une certaine classe des polynômes orthogonaux sont discutées dans cet abstrait qui sont exprimés en termes de fonctions \((n + 1, m + 1)\)-hypergéométriques. L’orthogonalité vient de celle des fonctions sphériques zonales de certaines paires de Gelfand de produits en couronne.

1. Introduction

Askey-Wilson polynomials and \(q\)-Racah polynomials are fundamental orthogonal polynomials which are described by the basic hypergeometric functions. Roughly speaking there are two points of view of orthogonal polynomials. One is through the Riemannian symmetric spaces which are homogeneous spaces of Lie groups. The other is through the finite groups. In this abstract we discuss some discrete orthogonal polynomials arising from Gelfand pairs [15, 16] of wreath products.

A pair of groups \((G, H)\) is called a Gelfand pair if the induced representation \(1^G_H \cong C(G/H)\), where \(C(G/H)\) is a complex valued functions defined over \(G/H\), is multiplicity free as a \(G\)-module. In this situation there exists a unique \(H\)-invariant element in each irreducible component of \(1^G_H\) which is called the zonal spherical function. There are interesting relations between zonal spherical functions on finite groups [15] and hypergeometric functions [5]. A hypergeometric function of

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one variable is by definition
\[ \ell F_m(a_1, a_2, \ldots, a_\ell; b_1, b_2, \ldots, b_m; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_\ell)_k}{(b_1)_k(b_2)_k \cdots (b_m)_k} \frac{x^k}{k!}. \]

In the following we recall some known results: Zonal spherical functions of the Gelfand pair, \( W(B_n) \), the hyperoctahedral group, and \( S_n \), the symmetric group, are expressed in terms of the Krawtchouk polynomials. The Krawtchouk polynomials is a special family of Gauss’ hypergeometric polynomials \([5, 6, 7, 17, 18]\);
\[ _2F_1(-k, -j; -n; 2). \]
Here \( k \) and \( j \) are nonnegative integers at most \( n \). A generalization is given by the Gelfand pair \((S_q \wr S_n, S_q \wr S_n)\). In this case the zonal spherical functions are realized as follows \([5, 17, 18]\);
\[ _2F_1(-k, -j; -n; q). \]
By considering another Gelfand pair, \( S_k \) and its maximal parabolic subgroup \( S_v \times S_i (1 \leq i \leq \lfloor r/2 \rfloor) \), we have the Hahn polynomials \([5]\);
\[ _3F_2(-i, -j, -v - \ell + j; -k, -v + k; 1). \]
We remark that in this case the classical hypergeometric functions of one variable still occur, since the Gelfand pairs \((S_q \wr S_n, S_q \wr S_n)\) and \((S_k, S_v \times S_i)\) are of “rank one”. One might expect a multivariate version of hypergeometric functions arise naturally as zonal spherical functions for certain types of Gelfand pairs. General hypergeometric functions are known as \((n+1, m+1)\)-hypergeometric functions \([3, 9, 10, 12, 13, 19]\);
\[ F(\alpha, \beta; \gamma, X) = \sum_{(a_{ij}) \in M_{n, m-n-1}(\mathbb{N}_0)} \frac{\prod_{i=1}^{n}(a_i) \prod_{j=1}^{m-n-1} a_{ij} \prod_{i=1}^{n-1}(\beta_i) \prod_{j=1}^{m} a_{ji} \prod x_{ij}^{a_{ij}}}{(\gamma)^{\sum_{i,j} a_{ij}}} \prod a_{ij}!, \]
which are originally due to K. Aomoto and I. M. Gelfand. Here we denote by \( X \) the set of variables \( x_{ij} (1 \leq i \leq n, 1 \leq j \leq m - n - 1) \).

In this abstract we consider another generalization of the Gelfand pair \((S_q \wr S_n, S_q \wr S_n)\). We will see that its zonal spherical functions are realized by means of a discrete orthogonal polynomials coming from \( F(\alpha, \beta; \gamma, X) \).

2. Main Results

We denote the shifted factorial of an indeterminate \( x \) by
\[ (x)_m = x(x+1)(x+2) \cdots (x+m-1) \]
for \( m \in \mathbb{Z}_{>0} \) and
\[ (x)_0 = 1. \]
Now if $-N$ is a negative integer, then we define the finite series called the $(n + 1, m + 1)$-hypergeometric functions\cite{3, 19};

$$F(\alpha, \beta; -N, X) = \sum_{\sum_{i,j} a_{ij} \leq N \atop (a_{ij}) \in \mathbb{M}_{n,m-n-1}(\mathbb{N})} \frac{\prod_{i=1}^{m} (\alpha_i) \prod_{j=1}^{m-n-1} (\beta_j) \prod_{i=1}^{m} a_{ij} \prod_{i,j} \pi_{ij}^{a_{ij}} (-N)^{\sum_{i,j} a_{ij}}}{\prod_{a_{ij}}}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \ldots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$. Our purpose of this paper is to obtain the following orthogonality relations.

**Theorem 2.1.** For a positive integer $r$, we assume that $k = (k_0, \ldots, k_{r-1})$, $k' = (k'_0, \ldots, k'_{r-1})$ and $\ell = (\ell_0, \ldots, \ell_{r-1})$ are elements of $\mathbb{N}_0^r$ such that $\sum_{i=0}^{r-1} k_i = \sum_{i=0}^{r-1} k'_i = \sum_{i=0}^{r-1} \ell_i = n$. We put $\tilde{\ell} = (\ell_0, \ldots, \ell_{r-1})$ for $\ell = (\ell_0, \ell_1, \ldots, \ell_{r-1})$, $\xi = \exp(2\pi\sqrt{-1}/r)$ and $\tilde{\Xi}_r = (1 - \xi^{ij})_{1 \leq i,j \leq r-1}$. Then we have

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_{r-1} = n} \binom{n}{\ell_0, \ldots, \ell_{r-1}} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Xi}) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Xi}_r) = \binom{n}{k_0, \ldots, k_{r-1}}^{-1} \prod \delta_{kk'}.$$ 

**Theorem 2.2.** For a positive integer $m = [r/2]$, we assume that $k = (k_0, \ldots, k_m)$, $k' = (k'_0, \ldots, k'_m)$ and $\ell = (\ell_0, \ldots, \ell_m)$ are elements of $\mathbb{N}_0^{m+1}$ such that $\sum_{i=0}^{m} k_i = \sum_{i=0}^{m} k'_i = \sum_{i=0}^{m} \ell_i = n$. We put $\tilde{\Theta}_r = (1 - \cos(2\pi ij/r))_{1 \leq i,j \leq m}$. Then we have

1. If $r$ is an odd positive integer,

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0} \binom{n}{\ell_0, \ldots, \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r) = 2^{-n+k_0} \binom{n}{k_0, \ldots, k_m}^{-1} \delta_{kk'}.$$ 

2. If $r$ is an even positive integer,

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0-\ell_m} \binom{n}{\ell_0, \ldots, \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r) = 2^{-n+k_0+k_m} \binom{n}{k_0, \ldots, k_m}^{-1} \delta_{kk'}.$$ 

Actually these relations are obtained from orthogonality of zonal spherical functions of the Gelfand pair of finite groups.

3. **Theory of Zonal Spherical Functions on Finite Groups**

Let $G$ be a finite group and $H$ be its subgroup.

**Definition 3.1.** If the induced representation $1^G_H$ is multiplicity free, then the pair $(G, H)$ is called a **Gelfand pair**.
Assume from now that \((G, H)\) is a Gelfand pair and the induced representation is decomposed as \(G\)-module:

\[
V = 1^G_H = \bigoplus_{i=1}^{s} V_i, \; V_i \not\cong V_j \; (i \neq j).
\]

It is a well known fact that \(s = \lvert H \backslash G / H \rvert\). We denote by \(\{g_i ; 1 \leq i \leq s\}\) the set of representatives of the double coset \(H \backslash G / H\). Put \(D_i = Hg_iH\). Let \(V_i^H\) be an \(H\)-invariant subspace of \(V_i\). Using the Frobenius reciprocity we have:

\[
\dim V_i^H = \langle V_i, 1^H_H \rangle = \langle V_i, 1^G_H \rangle_G = 1.
\]

Here \(\langle V, W \rangle_G\) denotes the intertwining number. Let \([*]*\) be a \(G\)-invariant Hermitian scalar product on \(V_i\). We assume that \(\dim V_i = n\).

Now we can choose \(\{v_i^1, \ldots, v_i^n\}\) as an orthonormal basis of \(V_i\) and \(v_i^1 \in V_i^H\). Let \((\rho_{k,l}^i)_{1 \leq k,l \leq n}\) be a matrix representation of \(G\) afforded by \(V_i\). We denote by \(C(G/H)\) the set of functions which have constant value on each right coset, i.e.,

\[
C(G/H) := \{f : G \to \mathbb{C}; \; f(xh) = f(x) \; \forall x \in G, \; \forall h \in H\}.
\]

It is clear that \(\dim C(G/H) = [G : H]\). Define a linear map

\[
\varphi_i : V_i \longrightarrow C(G/H)
\]

by

\[
\varphi_i(v)(g) = [v|gv_i^1]
\]

for \(g, h \in G\) and \(v \in V_i\). Since

\[
\varphi_i(gv)(k) = [gv|kv_i^1] = [v|g^{-1}kv_i^1] = \varphi_i(v)(g^{-1}k) = (g\varphi_i(v))(k)
\]

and \(\varphi \neq 0\), \(\varphi\) is an injective \(G\)-linear map. Now we obtain the following

\[
C(G/H) = \bigoplus_{i=1}^{s} \varphi_i(V_i).
\]

We define \(\omega_i \in \varphi_i(V_i)\) to be a function such that \(\omega_i(g) = [v_i^1|gv_i^1] = \rho_{11}^i(g)\) for any element \(g \in G\). As can be seen in the argument above we see

\[
\varphi_i(V_i)^H = \mathbb{C}\omega_i.
\]

**Definition 3.2.** The functions \(\omega_i\) are called zonal spherical functions of Gelfand pair \((G, H)\).

We list some easy cosequences from definition of zonal spherical functions.

**Proposition 3.3.** (1) \(\omega_i(h_1gh_2) = \omega_i(g)\) for any \(g \in G\) and \(h_1, h_2 \in H\).
(2) \(\omega_i(1) = 1\) and \(\omega_i(g^{-1}) = \omega_i(g)\) for any \(g \in G\).
Proposition 3.4. If we write $\omega_i(D_k) = \omega_i(g_i)$ for $g \in D_k$, then
\[
\frac{1}{|G|} \sum_{k=1}^{s} |D_k| \omega_i(D_k)\overline{\omega_j(D_k)} = \delta_{ij} \text{dim } V_i^{-1}.
\]

4. Zonal Spherical Functions of $(G(r, 1, n), S_n)$

Fix $r \in \mathbb{Z}_+$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Put $\xi = \exp(2\pi\sqrt{-1}/r)$. Let $C^n = \langle \xi \rangle \times \cdots \times \langle \xi \rangle$ denote the $n$-fold direct product of the cyclic group $C = \langle \xi \rangle$. The symmetric group $S_n$ acts on $C^n$ by:
\[
\sigma(\xi_1, \xi_2, \ldots, \xi_n) = (\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \ldots, \xi_{\sigma^{-1}(n)}), \quad (\xi_1, \xi_2, \ldots, \xi_n) \in C^n, \quad \sigma \in S_n.
\]

The wreath product $C \wr S_n$ is the semidirect product of $C^n$ with $S_n$ defined by this action [11, 15]. Let denote $G(r, 1, n) = C \wr S_n$. The conjugacy classes and the irreducible representations of $G(r, 1, n)$ are determined by the $r$-tuples of partitions $(\nu^0, \ldots, \nu^{r-1})$ such that $|\nu^0| + \cdots + |\nu^{r-1}| = n$. In this section we consider the pair of groups $G = G(r, 1, n)$ and its subgroup $H = G(1, 1, n) = S_n$.

Proposition 4.1. (1) The representatives of double coset $H \backslash G / H$ are given by
\[
\left\{ (1, \ldots, 1, \xi_1, \ldots, \xi_1, \ldots, \xi_{r-1}, \ldots, \xi_{r-1}; e) \in G; \sum_{i=0}^{r-1} \ell_i = n \right\},
\]
where $e$ is a unit element of $S_n$.

(2)
\[
|H (1, \ldots, 1, \xi_1, \ldots, \xi_1, \xi_{r-1}, \ldots, \xi_{r-1}; e) H| = \binom{n}{\ell_0, \ell_1, \ldots, \ell_{r-1}} n!.
\]

The group $G$ acts on the ring of polynomials of $n$-variables as
\[
(\xi_1, \xi_2, \ldots, \xi_n; \sigma) f(x_1, \ldots, x_n) = f(\xi_{\sigma^{-1}(1)} x_{\sigma(1)}, \xi_{\sigma^{-1}(2)} x_{\sigma(2)}, \ldots, \xi_{\sigma^{-1}(n)} x_{\sigma(n)}).
\]

We define the map from $\mathbb{N}_0^r$ to the set of partitions $Par$ as follows.
\[
\psi : \mathbb{N}_0^r \ni (k_0, k_1, \ldots, k_{r-1}) \mapsto (0^{k_0} 1^{k_1} \cdots (r-1)^{k_{r-1}}) \in Par.
\]

Proposition 4.2. The induced representation $1_{S_n}^{G(r, 1, n)}$ is decomposed as the following.
\[
1_{S_n}^{G(r, 1, n)} \cong \bigoplus_{\sum_{i=0}^{r-1} k_i = n} V^{(k_0, k_1, \ldots, k_{r-1})}.
\]

Each $V^{(k_0, k_1, \ldots, k_{r-1})}$ is an irreducible $G(r, 1, n)$-module which is realized as follows;
\[
V^{(k_0, k_1, \ldots, k_{r-1})} = \bigoplus_{f \in \mathcal{M}_n(\psi^{(k_0, k_1, \ldots, k_{r-1})})} \mathbb{C}f.
\]

Here $M_n(\lambda) = \{ x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(n)}^{\lambda_n}; \sigma \in S_n \}$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$.
Since this decomposition is multiplicity free [4], we have the following proposition.

**Proposition 4.3.** $(G, H)$ is a Gelfand pair.

**Example 4.4.** We take $G = G(3, 1, 4)$ and $H = S_4$. Then the induced representation $1^G_H$ is decomposed as follows:

$$1^G_H = V(4,0,0) \oplus V(0,4,0) \oplus V^{(3,1,0)} \oplus V^{(3,0,1)} \oplus V^{(1,3,0)} \oplus V^{(1,0,3)} \oplus V^{(0,3,1)} \oplus V^{(0,1,3)} \oplus V^{(2,1,1)} \oplus V^{(1,2,1)} \oplus V^{(1,1,2)} \oplus V^{(2,2,0)} \oplus V^{(2,0,2)} \oplus V^{(0,2,2)}.$$  

We write down a basis of some irreducible components.

$$V^{(1,1,2)} = \bigoplus_{\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4\}} \mathbb{C}x_{i_1}^2 x_{i_2}^2 x_{i_3}, \quad V^{(0,4,0)} = \mathbb{C}x_1 x_2 x_3 x_4.$$  

The $S_4$-invariant element of $V^{(1,2,2)}$ is a monomial symmetric function 

$$m_{(2,2,1)}(x_1, x_2, x_3, x_4).$$  

We define the inner product on $1^G_H$ as follows

$$[\alpha x^\lambda | \beta x^\mu] = \alpha \overline{\beta} \delta_{\lambda, \mu} \frac{1}{n! \{k_0, k_1, \ldots, k_{r-1}\}}.$$  

Here $\alpha$ and $\beta$ are complex numbers, $k_i$ is the number of parts of $\lambda$ which is equal to $i$, and $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. It is easy to see that this inner product is $G(r, 1, n)$-invariant, i.e.,

$$[(gf_1)(x)| (gf_2)(x)] = [f_1(x)| f_2(x)]$$  

for $g \in G(r, 1, n)$, $f_1(x), f_2(x) \in V^{(k_0, k_1, \ldots, k_{r-1})}$. We recall the monomial symmetric functions for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (0^{k_0}1^{k_1}2^{k_2} \cdots (r - 1)^{k_{r-1}})$. Clearly the monomial symmetric functions satisfy

$$[m_\lambda(x)| m_\mu(x)] = \delta_{\lambda, \mu}.$$  

**Theorem 4.5.** The zonal spherical functions of Gelfand pair $(G, H)$ are

$$\omega^{(k_0, k_1, \ldots, k_{r-1})}(\xi_1, \xi_2, \ldots, \xi_n; \sigma) = m_\lambda(\xi_1, \xi_2, \ldots, \xi_n)/m_\lambda(1, \ldots, 1)$$  

where $\lambda = (0^{k_0}1^{k_1}2^{k_2} \cdots (r - 1)^{k_{r-1}})$ and $\sum_{i=0}^{r-1} k_i = n.$

For $\lambda = (0^{k_0}1^{k_1}2^{k_2} \cdots (r - 1)^{k_{r-1}})$, we define

$$m_{(k_0, k_1, \ldots, k_{r-1})}^{(0^\ell_0,1^\ell_1, \ldots, r^\ell_{r-1})} = m_\lambda(1, \ldots, \underbrace{1, \ldots, 1}_{\ell_0}, \xi, \ldots, \underbrace{\xi, \ldots, 1}_{\ell_1}, \ldots, \underbrace{\xi, \ldots, 1}_{\ell_{r-1}}, \ldots, \underbrace{\xi, \ldots, 1}_{\ell_{r-1}}).$$

**Proposition 4.6.** We assume that $\sum_{i=0}^{r-1} \ell_i = \sum_{i=0}^{r-1} k_i = n.$
We define the subgroup $H$ be the dihedral group of order $2$. Therefore we obtain

$$\text{Theorem 4.8.}$$

The zonal spherical functions of Gelfand pair $(G, \mathcal{S})$ have $(n+1, m+1)$-hypergeometric expressions

$$\omega^{(k_0, k_1, \ldots, k_{r-1})}_{(\ell_0, \ell_1, \ldots, \ell_{r-1})} = F((-\ell_1, \ldots, -\ell_{r-1}), (-k_1, \ldots, -k_{r-1}); -n; \tilde{\Xi}_r).$$

Here $\tilde{\Xi}_r = (1 - \xi^{ij})_{1 \leq i, j \leq r-1}$ with $\xi = \exp(2\pi \sqrt{-1}/r)$.

5. Zonal Spherical Functions of $(D(r, n), D(1, n))$

Let

$$D_r = \langle a, b; a^2 = b^r = (ab)^2 = 1 \rangle$$

be the dihedral group of order $2r$. We denote by $G = D(r, n) = D_r \wr S_n$. We define the subgroup $H$ of $G$ by

$$H = \langle a \rangle \wr S_n \cong D(1, n).$$
We consider the pair of groups \((G, H)\).

We remark that \(D(1, n) \cong W(B_n)\), where \(W(B_n)\) is the Weyl group of type \(B\) and that \(D(2, n) \cong V_4 \wr S_n\), where \(V_4\) denotes by Kleinsche Vierergruppe. We define another subgroup \(K\) of \(G\) by

\[ K = \langle b \rangle \wr S_n \cong G(r, 1, n), \]

where \(G(r, 1, n)\) is the imprimitive complex reflection group.

**Proposition 5.1.** (1) The representatives of double coset \(H \backslash G / H\) are given by

\[ \{ (\ldots, 1, b, \ldots, b, \ldots, b^m, \ldots, b^m; e) \in G; \sum_{i=0}^m \ell_i = n \}, \]

where \(m = \frac{r-1}{2}\) if \(r\) is odd, \(m = \frac{r}{2}\) if \(r\) is even, and \(e\) is a unit element of \(S_n\).

(2)

\[ |H(\ldots, 1, b, \ldots, b, \ldots, b^m, \ldots, b^m; e)H| = \begin{cases} 2^{2n-\ell_0}(\ell_0, \ldots, \ell_m)!, & \text{if } r = 2m + 1 \\ 2^{2n-\ell_0-\ell_m}(\ell_0, \ldots, \ell_m)!, & \text{if } r = 2m. \end{cases} \]

**Proposition 5.2.** The induced representation \(1^G_H\) is decomposed as follows.

\[ 1^G_H \cong \bigoplus_{\sum_{i=0}^m k_i=n} W^{(k_0, k_1, \ldots, k_m)}. \]

Each \(W^{(k_0, k_1, \ldots, k_m)}\) is an irreducible \(G\)-module which is realized as follows;

\[ W^{(k_0, k_1, \ldots, k_m)} = \bigoplus_{f \in M_n(\psi_{(k_0, k_1, \ldots, k_m)})} \mathbb{C}f. \]

Here, in the case that \(r = 2m + 1\),

\[ M_n(\lambda) = \left\{ x_{\sigma(1)}^{\epsilon_1}x_{\sigma(2)}^{\epsilon_2}\ldots x_{\sigma(n)}^{\epsilon_n}; \epsilon_i \in \{\pm 1\}, \sigma \in S_n \right\}, \]

and if \(r = 2m\),

\[ M_n(\lambda) = \left\{ x_{\sigma(1)}^{\lambda_1} + x_{\sigma(1)}^{-\lambda_1}, x_{\sigma(2)}^{\lambda_2} + x_{\sigma(2)}^{-\lambda_2}, \ldots, x_{\sigma(n)}^{\lambda_n} + x_{\sigma(n)}^{-\lambda_n} \right\} \]

for \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i \geq \lambda_{i+1} \geq 0\).

By this proposition we can say that \((G, H) = (D(r, n), D(1, n))\) is a Gelfand pair. We define the inner product on each \(W^{(k_0, \ldots, k_m)}\) as follows;

\[ [\alpha x_1^{\epsilon_1\lambda_1}\ldots x_n^{\epsilon_n\lambda_n} | \beta x_1^{\eta_1\mu_1}\ldots x_n^{\eta_n\mu_n}] = \frac{\alpha \beta}{(k_0, k_1, \ldots, k_m)^{2n-k_0}} \prod_{i=1}^n \delta_{\epsilon_i, \eta_i; \mu_i}. \]
Here $\alpha, \beta \in \mathbb{C}, \epsilon_i, \eta_i \in \{\pm 1\}$, and $(\lambda_1, \lambda_2, \ldots, \lambda_n) = (0^{k_0}1^{k_1} \ldots m^{k_m})$. It is easy to see that this inner product is $G$-invariant on $W^{(k_0, \ldots, k_m)}$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) = (0^{k_0}1^{k_1} \ldots m^{k_m})$, we define the polynomial of $n$ variables by

$$f_\lambda(x) = \frac{2^{-k_0}}{k_0!k_1! \cdots k_m!} \sum_{\sigma \in S_n} (x_{\sigma(1)}^{\lambda_1} + x_{\sigma(2)}^{-\lambda_1})(x_{\sigma(2)}^{\lambda_2} + x_{\sigma(2)}^{-\lambda_2}) \cdots (x_{\sigma(n)}^{\lambda_n} + x_{\sigma(n)}^{-\lambda_n}).$$

Note that $f_\lambda$ satisfies

$$[f_\lambda(x)|f_\mu(x)] = \delta_{\lambda\mu}.$$

Let $g = (a^{s_1}b^{t_1}, \ldots, a^{s_n}b^{t_n}; \sigma) \in G$.

**Theorem 5.3.** The zonal spherical function of the Gelfand pair $(G, H)$

$$\omega^{(k_0, k_1, \ldots, k_m)}(a^{s_1}b^{t_1}, a^{s_2}b^{t_2}, \ldots, a^{s_n}b^{t_n}; \sigma) = f_\lambda(\xi^{t_1}, \xi^{t_2}, \ldots, \xi^{t_n})/f_\lambda(1, \ldots, 1),$$

where $\lambda = (0^{k_0}1^{k_1}2^{k_2} \ldots m^{k_m})$ and $\sum_{i=0}^m k_i = n$.

For $\lambda = (0^{k_0}1^{k_1}2^{k_2} \ldots m^{k_m})$, we define

$$f^{(k_0, k_1, \ldots, k_m)}_{(t_0, \ell_1, \ldots, \ell_m)} = \frac{1}{2^{n-k_0}} f_\lambda(1, \ldots, 1, \xi_{\ell_0}, \xi_{\ell_1}, \ldots, \xi_{\ell_m}) / (\xi_{\ell_0}, \xi_{\ell_1}, \ldots, \xi_{\ell_m}).$$

**Proposition 5.4.**

$$f^{(k_0, k_1, \ldots, k_m)}_{(t_0, \ell_1, \ldots, \ell_m)} = \sum_{a \in \mathcal{A}} \prod_{i=0}^m \left( a_{i_0}, a_{i_1}, \ldots, a_{i_m} \right) \prod_{0 \leq i, j \leq m} \left( \cos \left( \frac{2\pi i j}{r} \right) \right)^{a_{ij}},$$

where

$$\mathcal{A} = \mathcal{A}^{(k_0, k_1, \ldots, k_m)}_{(t_0, \ell_1, \ldots, \ell_m)} = \left\{ a = (a_{ij}) \in M(m+1, \mathbb{N}_0); \sum_{i=0}^m a_{ij} = k_j, \sum_{j=0}^m a_{ij} = \ell_i \right\}.$$

**Theorem 5.5.**

$$\omega^{(k_0, k_1, \ldots, k_m)}_{(t_0, \ell_1, \ldots, \ell_m)} = F(\ell_1, \ldots, \ell_m, -k_1, \ldots, -k_m; -n; \tilde{\Theta}_r)$$

Here $\tilde{\Theta}_r = (1 - \cos(2\pi ij/r))_{1 \leq i, j \leq m}$.

6. **General Result**

In this section we consider a generalization of our main results. We remark that, in Theorem 2.1,

$$\Xi_r = J_{r-1} - (\xi_{ij})_{1 \leq i, j \leq r-1}.$$  

Here $\Xi_r = (\xi_{ij})_{0 \leq i, j \leq r-1}$ is a table of zonal spherical functions of Gelfand pair $(\mathbb{Z}/r\mathbb{Z}, 1)$ and, in Theorem 2.2,

$$\tilde{\Theta}_r = J_m - (\cos 2\pi ij/r)_{1 \leq i, j \leq m}.$$  

Here $\Theta_r = (\cos 2\pi ij/r)_{0 \leq i, j \leq m}$ is a table of zonal spherical functions of Gelfand pair $(D_r, \langle a \rangle)$.
We assume that \((G, H)\) is a Gelfand pair and the induced representation \(1^G_H\) is decomposed as follows:

\[
1^G_H \cong \bigoplus_{i=0}^{s-1} V_i, \quad \dim V_i = d_i.
\]

Let \(Z(G, H)\) be a table of zonal spherical functions of \((G, H)\). Then we have the Gelfand pair \((G \wr S_n, H \wr S_n)\). We obtain next theorem.

**Theorem 6.1.** The zonal spherical functions of Gelfand pair \((G \wr S_n, H \wr S_n)\) have \((n+1, m+1)\)-hypergeometric expressions

\[
\omega^{(k_0, k_1, \ldots, k_{s-1})}_{(l_0, l_1, \ldots, l_{s-1})} = F((-\ell_1, \ldots, -\ell_{s-1}), (-k_1, \ldots, -k_{s-1}); -n; J_{s-1} - \tilde{Z}(G, H)).
\]

Here \(J_{s-1}\) is a \(s-1 \times s-1\) all-one-matrix and \(\tilde{Z}(G, H)\) is a matrix which is obtained by removing 0th row and 0th column of \(Z(G, H)\).

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