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SOME PROPERTIES OF T-NORMS WITH THRESHOLD

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Some properties of t-norms with threshold

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Abstract.

T-norm with threshold firstly is introduced by Dubois and Prade [4] and more considered by Inacu [5].Then some new classes of t-norms with threshold, t-conorms with threshold and fuzzy implications with threshold are discussed in [6,7]. This paper is devoted to the definitions and some new properties of these operators.

1.Introduction

T-norms ,t-conorms and fuzzy implications are basis connectives in fuzzy logic. Threshold is also an important natural concept in many real-world problems. A combination of these concepts should give a new approach to new problems of the intelligent systems.
T-norm with threshold firstly is introduced by Dubois and Prade [4] and more considered by Inacu [5]. Then some new claseses of t-norms with threshold, t-conorms with threshold and fuzzy implications with threshold are discussed in [6,7].
In this paper we will give some new properties of these connectives . Some new operators have been added in Fuzzy ToolBox of the MatLab.

2. T-norm with threshold

Definition 2.1. T-norms are functions t: [0,1] x [0,1] $\rightarrow$ [0,1] which satisfy the following conditions :
i/ $t(1,x) = x$, for any x
ii/ $t(x,y) = t(y,x)$
iii/ $t(x_1,y_1) \le t(x_2,y_2)$, if $x_1 \le x_2$, $y_1 \le y_2$
iv/ $t(t(x,y),z) = t(x, t(y,z))$, for any $0 \le x,y,z \le 1$
(see [1.p.30], [3, p.82]).

Some t-norms are the followings:

- min (Zadeh) $t(x,y) = \min(x,y)$
- production $t(x,y) = x.y$
- Lukasiewicz: $t(x,y) = \max\{x+y-1,0\}$
-

$$t(x, y) = min_0 \{x, y\} = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{if } x + y \le 1 \end{cases}$$

- drastic product t-norm

$$Z(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max(x,y) = 1 \\ 0 & \text{if } \max(x,y) < 1 \end{cases}$$

Let α be a threshold, i.e. $\alpha = (\alpha_1, \alpha_2)$, $0 \le \alpha_1,\alpha_2 < 1$.
Let $t_1(x,y)$, $t_2(x,y)$ be t-norms such that $t_2(x,y) \le t_1((x,y)$ for $0 \le x,y \le 1$.

Definition 2.2. A t-norm with threshold $T(x,y,\alpha)$ is defined on [0,1] x [0,1] by

$$T(x,y,\alpha) = \begin{cases} t_1(x,y) & \text{if } \alpha_1 \le x \text{ and } \alpha_2 \le y \\ t_2(x,y) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

$T(x,y,\alpha)$ is a t-norm with threshold of first type if

$$T(x,y,\alpha) = \begin{cases} \min(x,y) & \text{if } \alpha_1 \le x \text{ and } \alpha_2 \le y \\ t_2(x,y) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

Let $t_2(x,y)$ be a t-norm such that $t_2(x,y) \le x.y$ for all x,y

$T(x,y,\alpha)$ is a Larsen' t-norm with threshold if

$$T(x,y,\alpha) = \begin{cases} x.y & \text{if } \alpha_1 \le x \text{ and } \alpha_2 \le y \\ t_2(x,y) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

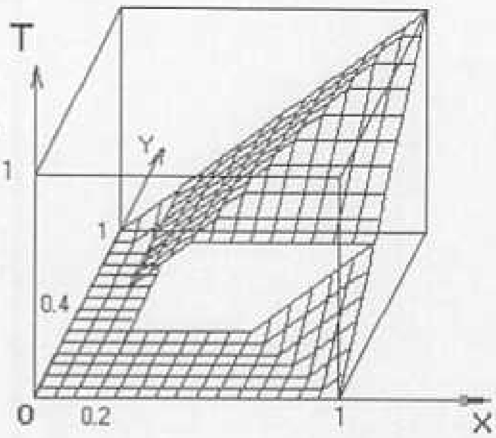
Proposition 2.3. For any threshold α, $T(x,y,\alpha)$ have the following properties:
i) $Z(x,y) \le T(x,y,\alpha) \le \min(x,y)$ for any x,y,
ii) $T(x,y,\alpha)$ is monotone nondecreasing in x,y and monotone nonincreasing in α,
iii) $T(x,1,\alpha) = T(1,x,\alpha) = x$, for any x,
iv) $T(x,0,\alpha) = T(0,x,\alpha) = 0$, for any x.

The commutativity and the associativity of the t-norm with threshold are not allways hold. We shall consider the following example.
Assume that $\alpha_2 < \alpha_1$. Let $x=(x_1, x_2)$ be a point such that $\alpha_1 < x_1 < 1$, $\alpha_2 < x_2 < \alpha_1$, Since $x_1 \ge \alpha_1$ and $x_2 \ge \alpha_2$, $T(x_1,x_2,\alpha) = t_1(x_1,x_2) = \min(x_1,x_2) = x_2$. But $x_2 < \alpha_1$, the t-norm with threshold $T(x_2,x_1,\alpha) = t_2(x_2,x_1)$. If we choose $t_2(x,y) = xy$, we have $T(x_2,x_1,\alpha) = x_1x_2 < x_2$. It means $T(x_1,x_2,\alpha) \ne T(x_2, x_1,\alpha)$.
For the associativity we consider the following example.
Assume $\alpha_1 = 0.5$, $\alpha_2 = 0.32$. $(x,y,z) = (0.6, 0.4, 0.3)$. Then we choose $t_1(x_1,x_2) = \min(x_1,x_2)$, $t_2(x,y) = x.y$. $T(x,y,\alpha) = t_1(x_1,x_2) = \min(0.6,0.4) = 0.4$. Thus $T(T(x,y,\alpha),z,\alpha) = t_2(0.4,0.3) = 0.12$. But $T(y,z,\alpha) = T(0.4, 0.3, \alpha) = 0.12$ and therefore
$T(x, T(y,z,\alpha),\alpha) = t(0.6, 0.12) = 0.6 \times 0.12 = 0.072 \ne T(T(x,y,\alpha),z,\alpha)$.

Now we denote $a = \min(\alpha_1,\alpha_2)$, $b = \max(\alpha_1, \alpha_2)$. Denote
$D^*(\alpha) = \{(x,y):\ b \le x,y \le 1\}$,
$D_*(\alpha) = \{(x,y):\ 0 \le x < a \text{ or } 0 \le y < a \text{ or } a \le x,y \le b\}$

Proposition 2.4. The t-norm with threshold $T(x,y,\alpha)$ is commutative, i.e. $T(x,y,\alpha) = T(y,x,\alpha)$, if (x,y) belongs one of the following sets : $D^*(\alpha)$, $D_*(\alpha)$.
If $\alpha_1 = \alpha_2$, then $T(x,y,\alpha)$ is commutative on $[0,1] \times [0,1]$.

Example 2.5.



$$T(x,y,\alpha) = \begin{cases} \min(x,y) & \text{if } 0.2 \le x \text{ and } 0.4 \le y \\ \max\{x+y-1, 0\} & \text{if } x < 0.2 \text{ or } y < 0.4 \end{cases}$$

3. t-conorm with threshold

Definition 3.1. t-conorms are functions $s: [0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following conditions :
i/ $s(o,x) = x$, for any $x \in [0,1]$
ii/ $s(x,y) = s(y,x)$
iii/ $s(x_1,y_1) \le s(x_2,y_2)$, if $x_1 \le x_2$, $y_1 \le y_2$
iv/ $s(s(x,y),z) = s(x,s(y,z))$, for any $0 \le x,y,z \le 1$.
(see [1,p.31] , [3, p.82])

Let β be a threshold, i.e. $\beta = (\beta_1, \beta_2)$, $0 < \beta_1, \beta_2 \le 1$.
Let $s_1(x,y)$, $s_2(x,y)$ be t-conorms such that $s_1(x,y) \le s_2(x,y)$ for all x,y.

Definition 3.2. T-conorm with threshold $S(x,y,\beta)$ is defined by

$$S(x,y,\beta) = \begin{cases} s_1(x,y) & \text{if } x \le \beta_1 \text{ and } y \le \beta_2 \\ s_2(x,y) & \text{if } \beta_1 < x \text{ or } \beta_2 < y \end{cases}$$

A t-conorm with threshold of first type $S(x,y,\beta)$ is defined on $[0,1] \times [0,1]$ by

$$S(x,y,\beta) = \begin{cases} \max(x,y) & \text{if } x \le \beta_1 \text{ and } y \le \beta_2 \\ s_2(x,y) & \text{if } \beta_1 < x \text{ or } \beta_2 < y \end{cases}$$

We denote $s_*(x,y)$ is the t-conorm, such that $s_*(x,y) = \max(x,y)$, if $\min(x,y) = 0$, and $s_*(x,y) = 1$, otherwise.

Proposition 3.3. For any threshold β, S(x,y,β) have the following properties :

i) max(x,y) ≤ S(x,y,β) ≤ s•(x,y) for any x,y

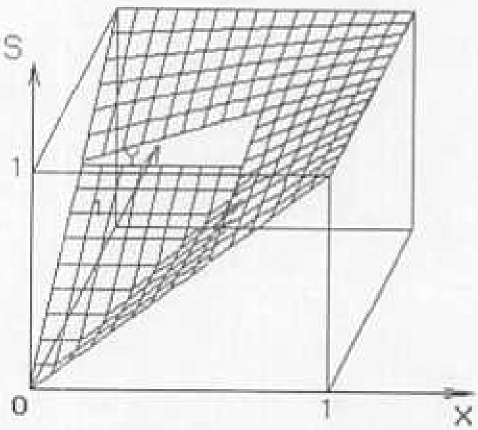
ii) S(x,y,β) is monotone non-decreasing in x,y and monotone non-increasing in β

iii) S(0,x,β) = x = S(x,0,β), for any x,

iv) S(1,x,β) = 1 = S(x,1,β), for any x.

Analogously to the t-norm with threshold , the commutativity and the associativity of the t-conorm with threshold are not allways hold.

Example 3.4.



$$S(x,y,\beta)=\begin{cases} max(x,y) & if\, x\le 0.6 \text{ and } y\le 0.6 \\ x+y-xy & if\, 0.6< x \text{ or } 0.6< y \end{cases}$$

4. De Morgan triples

Definition 4.1. A function n : [0,1] → [0,1] satisfying conditions : n(0) =1, n(1) =0 , nonincreasing is called a negation.. If n(n(x)) = x for all x, the negation is a strong negation. [2, p.100].

Let t be a t-norm , let s be a t-conorm and let n be a strong negation.

Definition 4.2. The triple (t,s,n) is called a De Morgan triple if
n (s(x,y)) = t(n(x), n(y)) for all x,y.

Let T(x,y,α) be a t-norm with threshold . let S(x,y,β) be a t-conorm with threshold .

Definition 4.3. The triple (T,S, n) is called a De Morgan triple with threshold if
n(S(x,y,β)) = T(x,y,α) for all x,y.

Theorem 4.4. Let (t_1,s_1,n), (t_2,s_2,n) be a De Morgan triple, n be a strong negation. Let T be a t-norm with threshold α and let S be a t-conorm with threshold β and β = n (α), then the triple (T,S,n) is a De Morgan triple with threshold .

5. t-norm with threshold and genertors

Let t be a t-norm, let f be an order isomorphism, $f \in Aut(I)$ (see [2,p.87]).
Denote $t^2(a) = t(a,a)$, $t^3(a)= t(a,t^2(a,a))$, and so on. A t-norm is nilpotent if for $a \neq 1$, $t^n(a) = 0$ for some positive integer n, the n depending on a.. A t-norm is strict if for $a \neq 0$, $t^n(a) > 0$ for every positive integer n. (see [2,p.90]).

Theorem 5.1. Define T_f by

$$T_f(x,y) = f^{-1}(t(f(x),f(y))) \text{ for } 0 \le x,y \le 1$$

The function T_f is an t-norm.
Moreover if t is a continuous, Archimedean, then T_f is also continuous and Archimedean.

Let $\alpha = (\alpha_1, \alpha_2)$ be a threshold. Define $\alpha' = (f^{-1}(\alpha_1), f^{-1}(\alpha_2))$.

Theorem 5.2. Let $T(x,y,\alpha)$ be a t-norm with threshold α .The function $T_f : [0,1]\times[0,1] \to [0,1]$ defined by

$$T_f(x,y,\alpha') = f^{-1}(T(f(x),f(y),\alpha)$$

for $0 \le x,y \le 1$

is a t-norm with threshold α'.

Proof.
First, we suppose $x \ge f^{-1}(\alpha_1)$ and $y \ge f^{-1}(\alpha_2)$
Since f is an order isomorphism, we have

$$f(x) \ge ff^{-1}(\alpha_1), \quad f(y) \ge ff^{-1}(\alpha_2),$$

i.e $f(x) \ge \alpha_1$, $f(y) \ge \alpha_2$, then

$$T_f(f(x),f(y),\alpha') = f^{-1}(t(f(x),f(y),\alpha)$$
$$= f^{-1}(t_1(f(x),f(y)))$$

Moreover, using theorem 5.1, the function $f^{1}(t_1(f(x),f(y)))$ is an t-norm.

If $x < f^{-1}(\alpha_1)$ or $y < f^{-1}(\alpha_2)$, then

$f(x) < ff^{-1}(\alpha_1)$ or $f(y) < ff^{1}(\alpha_2)$, i.e.

$x < \alpha_1$ or $y < \alpha_2$

Therefore

$$T_f(x,y,\alpha') = f^{-1}(T(f(x),f(y),\alpha)) = f^{-1}(t_2(f(x),f(y)))$$

Moreover the function $f^{-1}(t_2(f(x),f(y)))$ is a t-norm.

Finally, we have

$$T_f(x,y,\alpha') = \begin{cases} f^{-1}(t_1(f(x),f(y))) & \text{if } x \geq f^{-1}(\alpha_1) \text{ an } \\ & y \geq f^{-1}(\alpha_2) \\ f^{-1}(t_2(f(x),f(y))) & \text{if } x < f^{-1}(\alpha_1) \text{ or } \\ & y < f^{-1}(\alpha_2) \end{cases}$$

Since for any $0 \leq x,y \leq 1$,

$t_2(x,y) \leq t_1(x,y)$ .Then

$t_2(f(x),f(y)) \leq t_1(f(x),f(y))$

Since f is an order isomorphism

$f^{-1}(t_2(f(x),f(y))) \leq f^{-1}(t_1(f(x),f(y)))$

It means that $T_f(x,y,\alpha')$ is a t-norm with threshold $\alpha' = (f^{-1}(\alpha_1), f^{-1}(\alpha_2))$.

Corollary 5.3.

If

$$t(x,y,\alpha') = \begin{cases} \min(x,y) & \text{if } x \geq \alpha_1 \text{ and } y \geq \alpha_2 \\ t_2(x,y) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

then $T_f(x,y,\alpha')$ is a t-norm with threshold α' of type 1.

Indeed, if $t_1(x,y) = \min(x,y)$, then

$f^{-1}(\min(f(x),f(y))) = \min(x,y)$

Corollary 5.4. If $T(x,y,\alpha)$ ia a Larsen's type t-norm with threshold, then $T_f(x,y,\alpha')$ is also a Larsen's type t-norm with threshold α' if and only if $f(x,y) = f(x)\cdot f(y)$ for all $x \geq f^{-1}(\alpha_1)$, $y \geq f^{-1}(\alpha_2)$

Indeed, for each order isomorphism

$f(x\cdot y) = f(x)\cdot f(y)$

is equivalence to $f^{-1}(f(x)\cdot f(y)) = x\cdot y$

This case is hold, for example for $f(x) = x^r$ for $r > 0$.

Now we generalize theorem 5.2 to Archimedean t-norms.

Definition 5.5. Let t be a t-norm such that t is continuous in each variable. t is called to be an Archimedean if $t(x,x) < x$ for all $x \in (0,1)$.

Let $0 \leq a < 1$ and let f be an order isomorphism from [0,1] to [a,1]. This means that f is one-to-one and onto and $x \leq y$ if and only if $f(x) \leq f(y)$.

Denote $z_1 \vee z_2 = max(z_1, z_2)$, for $z_1, z_2 \in R^1$.

Theorem 5.6 . Let t be an Archimedean t-norm. The function T_f is defined by

$T_f(x,y) = f^{-1}(t(f(x),f(y)) \vee a)$

T_f is an Archimedean t-norm.

See [2 , p.87, 88]

Let $t_1(x,y), t_2(x,y)$ be t-norms such that $t_2(x,y) \leq t_1(x,y)$ for all $0 \leq x,y \leq 1$.

Let f be an order isomorphism $f:[0,1] \to [a,1]$.

Theorem 5.7. We define

$$T_f(x,y) = \begin{cases} f^{-1}(t_1(f(x),f(y)) \vee a & \text{if } f(x) \geq \alpha_1 \\ & \text{and } f(y) \geq \alpha_2 \\ f^{-1}(t_2(f(x),f(y)) \vee a & \text{if } f(x) < \alpha_1 \\ & \text{or } f(y) < \alpha_2 \end{cases}$$

If t_1, t_2 are Archimedean then T_f is a t-norm with threshold

$\alpha' = (f^{-1}(\alpha_1 \vee f(0)),\ f^{-1}(\alpha_2 \vee f(0)))$.

Proof.

Since theorem 5.6 the function on [0,1] x [0,1]

$f^{-1}(t_1(f(x),f(y) \vee a)),\quad f^{-1}(t_2(f(x),f(y) \vee a))$

, are Archimedean t-norms.

Using the order isomorphism f, we have

$t_2(f(x),f(y)) \leq t_1(f(x),f(y))$ for all $0 \leq x,y \leq 1$

$f^{-1}(t_2(f(x),f(y) \vee a) \leq f^{-1}(t_1(f(x),f(y) \vee a)$

Moreover,

If $z_1 \geq f^{-1}(\alpha_1 \vee f(0))$

$f(z_1) \geq ff^{-1}(\alpha_1 \vee f(0) = \alpha_1 \vee f(0) = \alpha_1 \vee a$

Inverse $f(z_1) \geq \alpha_1$, and it is obvious $f(z_1) \geq a$
It implies $f(z_1) \geq z_1 \vee a$ then
$f^{-1}(f(z_1)) \geq f^{-1}(\alpha_1 \vee a)$.
It means that $z_1 \geq f^{-1}(\alpha_1 \vee a)$
Analogously $z_2 \geq f^{-1}(\alpha_2 \vee f(0))$ and it is equivalent to $f(z_2) \geq \alpha_2$.

Definition 5.8. The t-norm with threshold is defined by

$$T(x,y,\alpha) = \begin{cases} t_1(x,y) & \text{if} \quad x \geq \alpha_1 \quad \text{and} \quad y \geq \alpha_2 \\ t_2(x,y) & \text{if} \quad x < \alpha_1 \quad \text{or} \quad y < \alpha_2 \end{cases}$$

$T(x,y,\alpha)$ is called Archimedean if $t_1(x,y)$, $t_2(x,y)$ are Archimedean t-norm.

Corollary 5.9. If t_1,t_2 are Archimedean t-norm, the function $T_f(x,y)$ defined in Theorem 5.7 is an Archimedean t-norm.

Corollary 5.10. Let $T(x,y,\alpha)$ be a t-norm with threshold of type 1. Let $f:[0,1] \to [a,1]$ be an order isomorphism. If $a \leq \min(\alpha_1,\alpha_2)$ then T_f given in Theorem 5.7. is a t-norm with threshold of type 1 with threshold $\alpha' = (f^{-1}(\alpha_1), f^{-1}(\alpha_2))$.

Corollary 5.11. Let $T(x,y,\alpha)$ be a Larsen's t-norm with threshold α. Let $f:[0,1] \to [a,1]$ be an order isomorphism. If $a \leq \min(\alpha_1,\alpha_2)$ then T_f given in Theorem 5.7. is a Larsen's t-norm with threshold $\alpha' = (f^{-1}(\alpha_1), f^{-1}(\alpha_2))$ if and only if $f(x,y) = f(x) \cdot f(y)$ for all $x \geq \alpha_1$, $y \geq \alpha_2$.

Corollary 5.12. Let $T(x,y,\alpha)$ be a t-norm with threshold α. Let $f:[0,1] \to [a,1]$ be an order isomorphism. If $a \geq \max(\alpha_1,\alpha_2)$ then T_f defined in the Theorem 5.7 has the form
$T_f(x,y) = f^{-1}(t_1(f(x), f(y) \vee a))$,

Let $f \in Aut(I)$, t_P=x.y, t_L=max{x+y-1,0}. We consider the following t-norms with threshold of first type:

$$t_P(x,y,\alpha) = \begin{cases} \min(x,y) & \text{if } x \geq \alpha_1 \text{ and } y \geq \alpha_2 \\ x.y & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

$$t_L(x,y,\alpha) = \begin{cases} \min(x,y) & \text{if } x \geq \alpha_1 \text{ and } y \geq \alpha_2 \\ \max\{x+y-1,0\} & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

Theorem 5.13. Let $T(x,y,\alpha)$ be a t-norm with threshold of first type. If $t_2(x,y)$ is a strict t-norm, then there is an isomorphism $f \in Aut(I)$ such that

$$T(x,y,\alpha) = t_{Pf}(x,y,\alpha) = \begin{cases} f^{-1}(\min(f(x), f(y)) & \text{if } x \geq \alpha_1 \text{ and } y \geq \alpha_2 \\ f^{-1}(f(x), f(y)) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

If $t_2(x,y)$ is a nilpotent t-norn then there is an isomorphism $f \in Aut(I)$ such that

$$T(x,y,\alpha) = t_{Lf}(x,y,\alpha) = \begin{cases} f^{-1}(\min(f(x), f(y))) & \text{if } x \geq \alpha_1 \text{ and } y \geq \alpha_2 \\ f^{-1}(\max(f(x)+f(y)-1,0)) & \text{if } x < \alpha_1 \text{ or } y < \alpha_2 \end{cases}$$

6. Fuzzy implication with threshold

Definition 6.1. A fuzzy implication I is a function I: [0,1] × [0,1]→ [0,1] satisfying condition I_0: I(0,0) =1, I(0,1)=1, I(1.0)=0. I(1,1)=1.
([2].p.144)

For fuzzy implication one can consider following conditions:
I_1. I(0,y) =1 for all y
I_2. I(x,1) =1 for all x
I_3. If $x_1 \leq x_2$ then $I(x_1,y) \geq I(x_2,y)$ for all y
I_4. If $y_1 \leq y_2$ then $I(x,y_1) \leq I(x,y_2)$ for all x .
(see [3],p.86).
These properties are required in different papres and they could be important also in some applications.

Let β be a threshold, i.e. β = (β_1, β_2),
$0 < \beta_1, \beta_2 \leq 1$.
Let S be a t-conorm with threshold and let n be a negation.

Definition 6.2. An S-implication is a function I_S: [0,1] x [0,1] → [0,1] of the form
$I_S(x,y,\beta) = S(n(x),y,\beta)$.

Proposition 6.3. An S-implication I_S is a fuzzy implication. Moreover, I_S satisfies conditions I_1, I_2, I_3, I_4 .

Definition 6.4. Let T be a t-norm with threshold α. An T-implication is a function
I_T: $[0,1] \times [0,1] \to [0,1]$ of the form

$$I_T(x,y,\alpha) = \sup\{u: T(x,y,\alpha) \leq y\}.$$

Proposition 6.5. An I-implication I_T is a fuzzy implication. Moreover, I_T satisfies conditions I_1, I_2, I_3, I_4 .

A generalization of these implications is the following.
Let $I_1(x,y)$, $I_2(x,y)$ be implications such that $I_1(x,y) \leq I_2(x,y)$ for all x,y .

Definition 6.6. An implication with threshold $I(x,y,\beta)$ is defined by

$$I(x,y,\beta) = \begin{cases} I_1(x,y) & \text{if } (1-\beta_1) \leq x \text{ and } y \leq \beta_2 \\ I_2(x,y) & \text{if } x < (1-\beta_1) \text{ or } \beta_2 < y \end{cases}$$

Theorem 6.7.
An implication with threshold $I(x,y,\beta)$ is a fuzzy implication. Moreover, if $I_1(x,y)$, $I_2(x,y)$ satisfy conditions I_1, I_2, I_3, I_4 , then $I(x,y,\beta)$ also satisfies these conditions.
S-implication I_S and I-implication I_T are implications with threshold, satisfying conditions I_1, I_2, I_3, I_4 .

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