On the Complexity of Immersed Normal Surfaces

Benjamin A. Burton* Éric Colin de Verdière† Arnaud de Mesmay‡

Abstract

Normal surfaces are a ubiquitous tool in computational 3-manifold theory. In this paper, we investigate a relaxed notion of normal surfaces where we remove the quadrilateral conditions. This yields normal surfaces that are no longer embedded. We prove that it is NP-hard to decide whether a given singular normal surface is immersed. Our proof uses a reduction from boolean constraint satisfaction problems where every variable appears in at most two clauses, using a classification theorem of Feder.

1 Introduction

The field of computational topology aims at providing computational and efficient tools to deal with topological problems. In this theory, the dimension of the problems we consider has a very direct impact on the complexity of the algorithms designed to solve them. Fundamental problems tend to have polynomial time solutions for surfaces [1], while in dimensions larger than three, things easily become undecidable [13, Chapter 9]. In the intermediate case, most of the problems we encounter in 3-dimensional topology are decidable but typically solved with (at least) exponential-time algorithms. The most famous of these is detecting the unknot, whose complexity is in NP [5], and in co-NP assuming the Generalized Riemann Hypothesis [7], but for which no polynomial time algorithm nor hardness proof is known.

A standard way to study 3-manifolds is to investigate which surfaces can be embedded in them. Normal surfaces, used in a wealth of algorithms, are perhaps the most ubiquitous tool for that. First brought to the algorithmic light by Haken [4], normal surfaces provide a compact and structured way to analyze and enumerate the most interesting surfaces embedded in a 3-manifold. Starting with a triangulation T of a 3-manifold M with t tetrahedra, a *normal surface* is a vector in $\mathbb{Z}_+^{7t}$ describing a (possibly disconnected) surface in M. In rough terms, many interesting surfaces, such as for example a Seifert disk for the unknot, are witnessed by a normal surface having coordinates at most exponential in t. This is the starting point of many algorithms based on the enumeration of normal surfaces, which naturally have an exponential complexity. See Hass et al. [5] for a nice exposition.

In addition to providing a succinct representation of embedded surfaces, normal surfaces possess an additional algebraic structure. The natural addition and scalar multiplication of vectors carry over to operations on normal surfaces, and the space of normal surfaces in $\mathbb{Z}_+^{7t}$ is characterized by two sets of constraints: the *matching equations* and the *quadrilateral conditions*. The former are linear equations specifying how pieces of the surface glue together, while the latter are non-linear and ensure that the surface we obtain is embedded. Spaces defined by linear constraints can be studied through linear programming, a powerful framework to deal with decision and optimization problems; in contrast, the quadrilateral conditions are combinatorial in nature, and provide the source of the exponential complexity. This motivates the study of a notion of relaxed normal surfaces, where we remove the quadrilateral conditions to obtain a polyhedral structure on the space of normal surfaces.

Removing the quadrilateral conditions amounts to removing the embeddedness of normal surfaces. Therefore, it amounts to dealing with *singular normal surfaces*. Among these, the *immersed normal surfaces* are well-behaved, in the sense that while they can self intersect, they are still 2-manifolds locally. Moreover, their Euler characteristic is a linear form on the polyhedron of singular normal surfaces—this fact is crucial in algorithms that work with embedded normal surfaces, but does not hold in general for singular normal surfaces. By coupling singular normal surface theory with an algorithm that efficiently separates immersed normal surfaces from the others, we would have powerful tools at our disposal: this could lead to efficient algorithms to find immersed low genus surfaces in 3-manifolds, and through classical topological results like Dehn's Lemma or the Loop Theorem [6] we would obtain embedded surfaces, which are the key behind the unknot problem and many others.

In this paper, we show some inherent limitations of this approach (Theorem 4): it is NP-hard to detect whether a singular normal surface is immersed.

*University of Queensland, Brisbane, Australia. email: bab@maths.uq.edu.au. Portions of this work were done while this author was invited professor at École normale supérieure.

†Département d'informatique, École normale supérieure, CNRS, Paris, France. email: Eric.Colin.de.Verdiere@ens.fr. Portions of this work were done while this author was invited professor at University of Queensland.

‡Département d'informatique, École normale supérieure, Paris, France. email: Arnaud.de.Mesmay@ens.fr

Immersed normal surfaces have been studied from a mathematical point of view in [8] and from a computational perspective in [9, 11], in the particular case of a fixed triangulation of the figure eight knot complement. In these papers, the authors devise and implement an algorithm to test whether a given singular normal surface is immersed. While the complexity of this algorithm is not explicitly computed, it is at least doubly exponential in the input size. Our main result shows that the problem is inherently hard and no polynomial solution is to be expected. The main result of [9, 11] is that in the case of the complement of the figure eight knot, a projective variant of the space of immersed normal surfaces has a nice and low complexity polyhedral structure. Our hardness proof complements this by showing that this simplicity is not to be expected in the general case.

The complexity reduction used in the proof of this theorem relies on an intricate classification theorem in the complexity of boolean constraint satisfaction problems [2, 3]. Hardness results are scarce in 3-dimensional computational topology, and thus our result displays a different intractibility aspect of this theory, which gives another possible justification for the exponential complexity of most known algorithms.

2 Normal surfaces

2.1 3-manifolds and embedded normal surfaces

In this paper, we describe 3-manifolds using ***triangulations***, i.e., tetrahedra glued in pairs along two dimensional faces. We assume that the link of every vertex is a 2-sphere, and no edge is glued to itself in the opposite direction: this is the case if and only if the underlying space is a 3-manifold. Note that such a triangulation is in general *not* a simplicial complex. Conversely, it is known that any 3-manifold M is the underlying space of such a triangulation [10]. Henceforth, T denotes a triangulation of a 3-manifold M.

An ***(embedded) normal surface*** in T is a properly embedded and possibly disconnected surface in T that meets each tetrahedron in a possibly empty collection of triangles and quadrilaterals, called ***normal disks***. In each tetrahedron, there are 4 possible types of triangles and 3 possible types of quadrilaterals, pictured in Figure 1. The intersection of a triangle or quadrilateral with a face of the triangulation gives rise to a ***normal arc***. To each embedded normal surface,

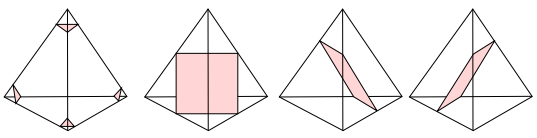

Figure 1: The seven types of normal disks.

one can associate a vector in $(\mathbb{Z}_+)^{7t}$, where t is the number of tetrahedra in T, by listing the number of triangles and quadrilaterals of each type in each tetrahedron. This vector provides a very compact description of that surface, since the bit representation of integers allows for an exponential compression. Reciprocally, to recover an embedded normal surface from a vector in $(\mathbb{Z}_+)^{7t}$, called ***normal coordinates***, the vector must satisfy two types of equations:

- The ***matching equations*** stipulate that at the interface of two tetrahedra, there are as many arcs of each arc type coming from both sides.
- The ***quadrilateral conditions*** stipulate that, within any tetrahedron, at most one of the three quadrilateral coordinates must be non-zero.

Proposition 1 ([5]) *Let T be a triangulation of size t and $v \in (\mathbb{Z}_+)^{7t}$. Then v corresponds to an embedded normal surface if and only if the matching equations and the quadrilateral conditions are fulfilled.*

2.2 Singular and immersed normal surfaces

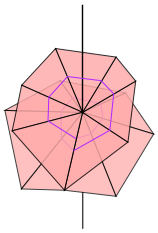

Figure 2: A branch point.

Given normal coordinates satisfying the matching equations and the quadrilateral conditions, one obtains a family of triangles and quadrilaterals that can be glued together to form an embedded surface, which is unique up to isotopy. When one drops the quadrilateral conditions, one can still glue their boundaries pairwise to obtain an abstract surface, since they satisfy the matching equations. However, the resulting surface might not be embedded any more: It may have singularities, and so is called it a ***singular normal surface***. Also, when we allow singular surfaces, different gluings are possible that (in general) give radically different surfaces, and so the singular surface is not uniquely defined. After a small perturbation, the surface we obtain is either an ***immersed normal surface***, i.e., the image of a usual surface by a locally one-to-one map, or it has a branch point, as pictured in Figure 2.

Given the data of normal coordinates satisfying the matching equations, together with a ***gluing***, i.e., a choice of how to glue the normal triangles and quadrilaterals, we can easily determine whether the corresponding singular normal surface is immersed or not. However, if we only have the normal coordinates, then, depending on the choice of the gluings, some of the resulting surfaces may be immersed while some other may have branch points. If there exists a gluing whose corresponding singular surface is immersed, we say that the normal coordinates are ***immersible***. Our main result is about the computational complexity of the following problem.

Problem 2 (Immersibility)
***Input**: A triangulation T and normal coordinates N satisfying the matching equations.*
***Output**: Are the normal coordinates N immersible?*

Two difficulties are at the heart of this problem: Not only do we need to guess a "good" gluing, but this gluing may have an exponential complexity in the input, since the normal coordinates are naturally compressed by the bit representation. Therefore, the naive algorithm (implemented by Matsumoto and Rannard [9]) is doubly exponential.

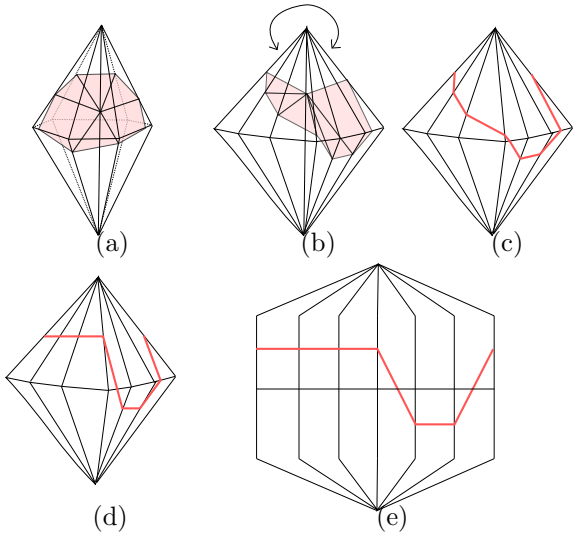

Figure 3: A schematic representation of a *block*.

We introduce a schematic representation of singular normal surfaces, more specifically to draw a family of tetrahedra that all have one edge in common, which we call a ***block*** like in Figure 3(a). In order to picture cleanly what happens on the back of this block, we will unfold it as in Figure 3(b), with the implicit convention that the rightmost face is glued to the leftmost face. Although normal disks can be drawn inside this block, the pictures easily become congested when there are several of them. Instead, we will forget the edge in common in the representation and represent the normal disks by their normal arcs, i.e., by their intersection with the front faces (Figure 3(c)). These normal arcs are glued together and form possibly self-intersecting closed curves, called ***block curves***.

Abstracting a bit more, horizontal lines will represent triangles, while diagonal ones will stand for quadrilaterals (Figure 3(d)). To make these pictures even more readable, we will draw the edges between the tetrahedra vertically, only linking them at the extreme top and bottom parts of the figures (Figure 3(e)). Finally, when we represent normal coordinates without a specific gluing, the normal arcs are drawn so that they connect the midpoints of the corresponding edges of the triangulation (as in Figure 4).

2.3 Boolean Constraint Satisfaction Problems

In this section, we recall a few basic results about boolean constraint satisfaction problems; our presentation is inspired from Dalmau and Ford [2] and we refer to their paper for the notations.

For an r-ary relation $R \subseteq \{0,1\}^r$, we denote by ***SAT(R)*** the corresponding satisfiability problem, i.e., whether a given conjunction of R-clauses (possibly with constants) is satisfiable. A relation R is *Schaefer* if it is Horn, dual-Horn, bijunctive or affine (see [2] for the corresponding definitions). The celebrated classification theorem of Schaefer [12] shows that if a relation R is Schaefer, then $\mathrm{SAT}(R)$ is in P, otherwise it is NP-complete.

For our reduction, we will restrict ourselves to constraint satisfaction problems where every variable occurs at most twice; we denote the corresponding problem by ***SAT(2, R)***. We introduce a last concept to classify this restricted class of satisfaction problems.

Let $R \subseteq \{0,1\}^r$ be a relation. Let $x, y, x' \in \{0,1\}^r$, then x' is a ***step*** from x to y if $d(x,x') = 1$ and $d(x,x') + d(x',y) = d(x,y)$, where d is the Hamming distance. R is a ***Δ-matroid*** (relation) if it satisfies the following *two-step axiom*:

> For all $x, y \in R$ and for all x' a step from x to y, either $x' \in R$ or there exists $x'' \in R$ which is a step from x' to y.

Feder [3] proved that if R is a non-Δ-matroid relation, then $\mathrm{SAT}(2,R)$ is polynomially equivalent to $\mathrm{SAT}(R)$. This result, combined with Schaefer's theorem [12], immediately implies:

Theorem 3 *If R is neither Schaefer nor a Δ-matroid, then $\mathrm{SAT}(2,R)$ is NP-complete.*

3 NP-hardness of detecting immersibility

Our main result is the following theorem.

Theorem 4 *The problem* Immersibility *is NP-hard.*

The relation. We define the following relation:

$$\begin{aligned} R \;=\; \big\{&(0,0,0,0,0,0);(0,0,0,1,0,1);(0,1,0,1,0,0);\\ &(0,1,0,0,0,1);(0,0,1,0,1,0);(0,1,1,0,1,1);\\ &(1,0,1,0,0,0);(1,0,1,1,0,1);(1,0,0,0,1,0);\\ &(1,1,1,1,1,1);(1,1,0,1,1,0)\big\} \end{aligned}$$

One can check that R satisfies the hypotheses of Theorem 3; thus:

Proposition 5 *$\mathrm{SAT}(2,R)$ is NP-complete.*

The proof of Theorem 4 proceeds by a reduction of SAT(R) to Immersibility. We start with a formula Φ that is a conjunction of clauses of the form $R(x_{i_1} \dots x_{i_6})$, where x_i is either a variable or a constant, and every variable appears at most twice in Φ.

Each clause is represented by a ***clause gadget***. For each variable occurring exactly twice in Φ, we connect these two occurrences in the clauses using ***tubes***. Finally, the ***constant gadgets*** are used to represent the constants 0 or 1. The idea for the proof is that a clause is satisfiable if and only if the normal coordinates in the corresponding gadget are immersible; the tubes then enforce consistency between the clauses. Therefore, the whole formula will be satisfiable if and only if the associated normal coordinates are immersible.

The clause gadget. For every clause in Φ, we create one copy of the clause gadget in Figure 4. The

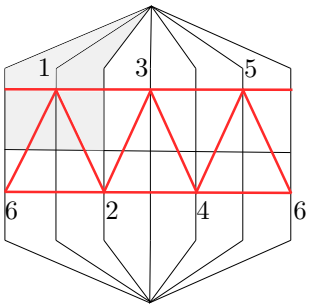

Figure 4: A block representing a clause gadget.

rationale behind this gadget is the following. At the interface of two adjacent tetrahedra, exactly two possibilities exist for the gluings. This choice of gluing can be described by a variable $x_i \in \{0, 1\}$: a value of 1 corresponds to the fact that the two block curves at the specified position cross. Therefore, each variable in a clause corresponds to a couple of faces in the gadget; for example, the two shaded triangles in Figure 4 are the couple of faces associated to the variable x_1. This way, a gluing of the singular normal surface is described by an element $x \in \{0, 1\}^6$. For every possible choice of x, testing whether the corresponding surface is immersed yields the following lemma.

Lemma 6 *The singular normal surface in the gadget G specified by the gluing $x \in \{0, 1\}^6$ is immersed if and only if $x \in R$.*

The tubes. A tube is the block pictured in Figure 5.a. The gluing of the singular normal surface in a tube is specified by two variables. For each variable appearing in exactly two clauses, we put a tube between the corresponding occurences of the variable. The following lemma shows that a tube enforces the values of the variables to be consistent among clauses.

Lemma 7 *The singular normal surface in a tube specified by the gluing is immersed if and only if both variables of the tube are equal.*

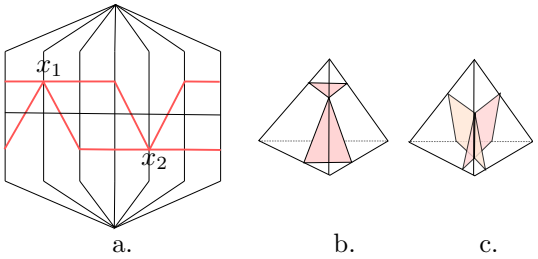

Figure 5: a. A tube gadget. b.c. The constant gadgets CG_0 and CG_1.

The constants. We finally glue the gadgets CG_0 and CG_1 pictured in Figure 5.b,c respectively on the clauses where the constants 0 or 1 appear. Their structure makes them so as to force the corresponding gluing in the clause gadget to be respectively a 0 or a 1 gluing.

References

[1] É. Colin de Verdière, *Topological algorithms for graphs on surfaces*, habilitation thesis, École normale supérieure, 2012. Available at `http://www.di.ens.fr/~colin/`.

[2] V. Dalmau and D. Ford, *Generalized satisfiability with limited occurrences per variable: a study through delta-matroid parity*, in Mathematical Foundations of Computer Science (MFCS), vol. 2747 of Lecture Notes in Computer Science, 2003, pp. 358–367.

[3] T. Feder, *Fanout limitations on constraint systems*, Theoretical Computer Science, 255 (2001), pp. 281–293.

[4] W. Haken, *Theorie der Normalflachen, ein Isotopiekriterium für den Kreisnoten*, Acta Mathematica, 105 (1961), pp. 245–375.

[5] J. Hass, J. C. Lagarias, and N. Pippenger, *The computational complexity of knot and link problems*, Journal of the ACM, 46 (1999), pp. 185–211.

[6] A. Hatcher, *Notes on basic 3-manifold topology.* Notes available on the author's webpage, 2007.

[7] G. Kuperberg, *Knottedness is in NP, modulo GRH.* To appear in Adv. Math. arXiv:1112.0845, 2011.

[8] D. M. Letscher, *Immersed normal surfaces and decision problems for 3-manifolds*, PhD thesis, U. Michigan, 1997.

[9] S. Matsumoto and R. Rannard, *The regular projective solution space of the figure-eight knot complement*, Experimental Mathematics, 9 (2000), pp. 221–234.

[10] E. E. Moise, *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung*, Annals of Mathematics, Second Series, 56 (1952), pp. 96–114.

[11] R. Rannard, *Computing immersed normal surfaces in the figure-eight knot complement*, Experimental Mathematics, 8 (1999), pp. 73–84.

[12] T. J. Schaefer, *The complexity of satisfiability problems*, in Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC), 1978, pp. 216–226.

[13] J. Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag, New York, second ed., 1993.