

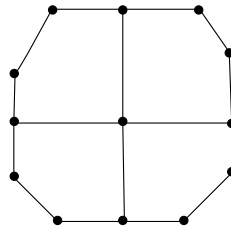
Algorithms and Combinatorics of Embedded Graphs

Homework, due November 20th, 2023

Exercise 1:

Let G be a simple plane graph with a Hamiltonian cycle C , i.e., a cycle going through all the vertices of G . Denote by f_k and g_k the number of k -gonal faces of the embedding that are inside and outside of C , respectively, with respect to a fixed outer face.

1. Prove that $\sum_{k \geq 3} (k-2)(f_k - g_k) = 0$. *Hint: Look separately at the plane graph inside the Hamiltonian cycle, and the plane graph outside the Hamiltonian cycle. In each of them, apply both the Euler formula and double counting of edges.*
2. Deduce from this question that the graph pictured below does not contain a Hamiltonian cycle.



Exercise 2:

The aim of this exercise is to give a direct proof that simple triangulated planar graphs are 3-connected. So do not use Lemma 1.19 from the lecture notes.

Let G be a simple plane graph, which we think of as being embedded on the sphere \mathbb{S}^2 , and that is **triangulated**: all the faces are of degree three.

1. Show that G is 2-connected.

Let us assume that G is not 3-connected.

3. If x and y are vertices such that $G \setminus \{x, y\}$ is not connected, show that there exists a Jordan curve (simple closed curve) γ on \mathbb{S}^2 , intersecting G exactly at x and y and separating at least two components of G .
4. The curve γ is cut into two paths γ_1 and γ_2 by x and y . Show that γ_1 and γ_2 lie in two different faces of G .
5. Show that one of these faces has degree at least four. Conclude by contradiction that any simple planar graph is 3-connected.

Exercise 3:

In a weighted graph on n vertices, encoding the distances between all pairs of vertices takes space $\Omega(n^2)$: there is no better way than storing all the distances. But in a planar graph, one can do better. Let G be a weighted planar graph on n vertices. Design an algorithm computing a table T with the following properties:

- The table T occupies $O(n^{3/2})$ space, and
- For any pair of vertices u and v in G , one can compute the distance between u and v using G and T in time $O(\sqrt{n})$.

Hint: Use balanced separators, taking inspiration from the application viewed in class.